

Pseudodifferential operators on manifolds: a coordinate-free approach

P. McKeag and Y. Safarov

Abstract. The main aim of the paper is to demonstrate the advantage of a coordinate-free approach to the theory of pseudodifferential operators. We explain how one can define symbols and construct a symbolic calculus without using local coordinates, briefly review some known definitions and results, and discuss possible applications and further developments.

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1. Introduction

The theory of pseudodifferential operators (PDOs) is a powerful technique, which has many applications in analysis and mathematical physics. In the framework of this theory, one can effectively construct the inverse of an elliptic differential operator L on a closed manifold, its non-integer powers and even some more general functions of L . For operators with constant coefficients in \mathbb{R}^n , this can be easily done by applying the Fourier transform. In a sense, the theory of PDOs

extends the Fourier transform method to operators with variable coefficients and operators on manifolds at the expense of losing infinitely smooth contributions. This is normally acceptable for theoretical purposes and is useful for numerical analysis, since numerical methods for the determination of the smooth part are usually more stable.

Traditionally, PDOs on manifolds are defined with the use of local coordinates. This leads to certain restrictions on operators under consideration, as all the definitions and results must be invariant with respect to transformations of coordinates. The main aim of this paper is to introduce the reader to a little known approach to the theory of PDOs that allows one to avoid this problem.

The paper is constructed as follows. In Section 2 we recall some basic definitions and results of the classical theory of PDOs. Their detailed proofs (as well as other relevant statements and definitions) can be found, for instance, in [H2, Shu, Ta, Tr]. Section 3 gives a brief overview of some elementary concepts of differential geometry (see [KN] or any other textbook for details). In Sections 4 and 5 we explain how to define PDOs without using local coordinates and quote some results from the paper [Sa1] and the conference article [Sa2]. Section 6 contains new results on approximate spectral projections of the Laplacian obtained in the Ph.D. thesis [McK]. Finally, in Section 7 we give a review of other related results and discuss possible developments in the field.

Throughout the paper C_0^∞ denotes the space of infinitely differentiable functions with compact supports, and \mathcal{D}' is the dual space of Schwartz distributions. Recall that, by the Schwartz theorem, for each operator $A : C_0^\infty \mapsto \mathcal{D}'$ there exists a distribution $\mathcal{A}(x, y) \in \mathcal{D}'$ such that $\langle Au, v \rangle = \langle \mathcal{A}(x, y), u(y)v(x) \rangle$ for all $u, v \in C_0^\infty$. The distribution $\mathcal{A}(x, y)$ is called the *Schwartz kernel* of A .

2. PDOs: local definition and basic properties

Let $a(x, y, \xi)$ be a C^∞ -function defined on $U \times U \times \mathbb{R}^n$, where U is an open subset of \mathbb{R}^n .

Definition 2.1. The function a belongs to the class $S_{\rho, \delta}^m$ with $\rho, \delta \in [0, 1]$ and $m \in \mathbb{R}$ if

$$\sup_{(x, y) \in K} |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{K; \alpha, \beta, \gamma} (1 + |\xi|)^{m + \delta(|\alpha| + |\beta|) - \rho|\gamma|} \quad (2.1)$$

for each compact set $K \subset U \times U$ and all multi-indices α, β, γ , where $C_{K; \alpha, \beta, \gamma}$ are some positive constants.

Definition 2.2. An operator $A : C_0^\infty(U) \mapsto \mathcal{D}'(U)$ is said to be a *pseudodifferential* operator of class $\Psi_{\rho, \delta}^m$ if

- (c) its Schwartz kernel $\mathcal{A}(x, y)$ is infinitely differentiable outside the diagonal $\{x = y\}$,
- (c₁) $\mathcal{A}(x, y) = (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} a(x, y, \xi) d\xi$ with some $a \in S_{\rho, \delta}^m$ in a neighbourhood of the diagonal.

The function a in (\mathbf{c}_1) is called an *amplitude*, and the number m is said to be the *order* of the amplitude a and the corresponding PDO A . Note that for amplitudes of order $m > -n$ the integral in (\mathbf{c}_1) does not converge in the usual sense. However, it is well defined as a distribution in x and y .

Let $S^{-\infty} := \bigcap_{m \in \mathbb{R}} S_{\rho, \delta}^m$, and let $\Psi^{-\infty}$ be the class of operators with infinitely differentiable Schwartz kernels. If $a \in S^{-\infty}$ (that is, if a and all its derivatives vanish faster than any power of $|\xi|$ as $|\xi| \rightarrow \infty$) then the corresponding PDO A belongs to $\Psi^{-\infty}$. The classical theory of PDOs is used to study singularities. Therefore one usually assumes that a is defined modulo $S^{-\infty}$ and that x is close to y .

Let $a \in S_{\rho, \delta}^m$ and $a_j \in S_{\rho, \delta}^{m_j}$ for some $\rho, \delta \in [0, 1]$, where $m_j \rightarrow -\infty$ as $j \rightarrow \infty$. We shall write

$$a \sim \sum_j a_j, \quad |\xi| \rightarrow \infty, \quad (2.2)$$

if $a - \sum_{j < k} a_j \in S_{\rho, \delta}^{n_k}$ where $n_k \rightarrow -\infty$ as $k \rightarrow \infty$. Such series $\sum_j a_j$ are called asymptotic. If $m_j \rightarrow -\infty$ then for every collection of amplitudes $a_j \in S_{\rho, \delta}^{m_j}$ there exists an amplitude a satisfying (2.2). Obviously, if a' is another amplitude satisfying (2.2) then $a - a' \in S^{-\infty}$ (or, in other words, (2.2) defines a modulo $S^{-\infty}$).

Lemma 2.3. *Let $z_\tau := x + \tau(y - x)$ where $\tau \in [0, 1]$. If $\delta < \rho$ and $a \in S_{\rho, \delta}^m$ then*

$$\int e^{i(x-y) \cdot \xi} a(x, y, \xi) \, d\xi = \int e^{i(x-y) \cdot \xi} \sigma_{A, \tau}(z_\tau, \xi) \, d\xi$$

modulo an infinitely differentiable function, where $\sigma_{A, \tau}(z, \xi)$ is an amplitude of class $S_{\rho, \delta}^m$ given by the asymptotic expansion

$$\begin{aligned} & \sigma_{A, \tau}(z, \xi) \\ & \sim \sum_{\alpha, \beta} \frac{(-i)^{|\alpha|+|\beta|} \tau^{|\alpha|} (1-\tau)^{|\beta|}}{\alpha! \beta!} \partial_x^\alpha \partial_y^\beta \partial_\xi^{\alpha+\beta} a(x, y, \xi) \Big|_{y=x=z}, \quad |\xi| \rightarrow \infty. \end{aligned} \quad (2.3)$$

Sketch of proof. Expand the amplitude a by Taylor's formula at the point $(x, y) = (z_\tau, z_\tau)$, replace $(y-x)e^{i(x-y) \cdot \xi}$ with $i \nabla_\xi e^{i(x-y) \cdot \xi}$ and integrate by parts with respect to ξ . \square

The amplitude $\sigma_{A, \tau}(x, \xi)$ is called the τ -symbol of the PDO A . It is uniquely defined by the operator A modulo $S^{-\infty}$. The 0-symbol is usually called just the symbol and is denoted σ_A . The $\frac{1}{2}$ -symbol and 1-symbol are said to be the Weyl and the dual symbol respectively.

In the theory of PDOs, properties of operators are usually described and results are stated in terms of their symbols. The following composition formula plays a key role in the symbolic calculus.

Theorem 2.4. *Let $A \in \Psi_{\rho,\delta}^{m_1}$ and $B \in \Psi_{\rho,\delta}^{m_2}$. If $\delta < \rho$ then the composition AB is a PDO of class $\Psi_{\rho,\delta}^{m_1+m_2}$ whose symbol admits the asymptotic expansion*

$$\sigma_{AB}(x, \xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} \sigma_A(x, \xi) \partial_x^{\alpha} \sigma_B(x, \xi), \quad |\xi| \rightarrow \infty. \quad (2.4)$$

Sketch of proof. From the inversion formula for the Fourier transform it follows that the Schwartz kernel of AB is given by (\mathbf{c}_1) with the amplitude $a(x, y, \xi) = \sigma_A(x, \xi) \sigma_{B,1}(y, \xi)$. Applying Lemma 2.3 with $\tau = 0$ to a , we obtain (2.4). \square

Remark 2.5. Theorem 2.4 implies, in particular, that the resolvent of an elliptic differential operator is a PDO. Using (2.4), one can also show that a PDO of order m maps $W_2^s \cap C_0^{\infty}$ into W_2^{s-m} , where W_2^r are the Sobolev spaces.

Note that in the above lemmas the condition $\delta < \rho$ is of crucial importance; if it is not fulfilled then the terms in the right hand sides of (2.3) and (2.4) do not form asymptotic series.

Clearly, the phase function $(x - y) \cdot \xi$ in (\mathbf{c}_1) depends on the choice of coordinates on U . Passing to new coordinates \tilde{x} and \tilde{y} , we obtain

$$\mathcal{A}(\tilde{x}, \tilde{y}) = (2\pi)^{-n} \int e^{i(x(\tilde{x}) - y(\tilde{y})) \cdot \xi} a(x(\tilde{x}), y(\tilde{y}), \xi) d\xi.$$

In a sufficiently small neighbourhood of the diagonal $\{\tilde{x} = \tilde{y}\}$, the new phase function $\varphi(\tilde{x}, \tilde{y}, \xi) = (x(\tilde{x}) - y(\tilde{y})) \cdot \xi$ can be written in the form

$$\varphi(\tilde{x}, \tilde{y}, \xi) = (\tilde{x} - \tilde{y}) \cdot \Phi(\tilde{x}, \tilde{y}) \xi,$$

where $\Phi(\tilde{x}, \tilde{y})$ is a smooth $n \times n$ -matrix function such that $\det \Phi(\tilde{x}, \tilde{y}) \neq 0$. Changing variables $\eta = \Phi(\tilde{x}, \tilde{y}) \xi$, we see that

$$\mathcal{A}(\tilde{x}, \tilde{y}) = (2\pi)^{-n} \int e^{i(\tilde{x} - \tilde{y}) \cdot \eta} \tilde{a}(\tilde{x}, \tilde{y}, \eta) d\eta,$$

where

$$\tilde{a}(\tilde{x}, \tilde{y}, \eta) = |\det \Phi(\tilde{x}, \tilde{y})|^{-1} a(x(\tilde{x}), y(\tilde{y}), \Phi^{-1}(\tilde{x}, \tilde{y})\eta)$$

is a new amplitude. Thus Definition 2.2 does not depend on the choice of coordinates. However, there are two obvious problems.

Problem 2.6. If $a \in S_{\rho,\delta}^m$ then, generally speaking, the new amplitude \tilde{a} belongs only to the class $S_{\rho,\delta'}^m$ with $\delta' := \max\{\delta, 1 - \rho\}$. If $\rho < \frac{1}{2}$ then $\delta' > \rho$ and the above lemmas fail. Thus for $\delta < \rho < \frac{1}{2}$ it is impossible to define PDOs of class $S_{\rho,\delta}^m$ on a manifold and to develop a symbolic calculus using local coordinates.

Problem 2.7. If $\max\{\delta, 1 - \rho\} < \rho$ then the ‘‘main part’’ of the symbol σ_A (called the *principal* symbol of A) behaves as a function on the cotangent bundle under change of coordinates. However, lower order terms in (2.3) do not have a clear geometric meaning. Therefore, the coordinate approach does not allow one to study the subtle properties of PDOs, which depend on the lower order terms.

3. Linear connections

The above problems do not arise if we define the phase function $(x - y) \cdot \xi$ in an invariant way, without using local coordinates. It is possible, in particular, when the manifold is equipped with a linear connection. In this section we shall briefly recall some relevant definitions and results from differential geometry.

Let M be an n -dimensional C^∞ -manifold. Further on we shall denote the points of M by x, y or z . The same letters will be used for local coordinates on M . Similarly, ξ, η and ζ will denote points of (or the dual coordinates) on the fibres T_x^*M, T_y^*M and T_z^*M of the cotangent bundle T^*M .

We are going to consider operators acting in the spaces of κ -densities on M , $\kappa \in \mathbb{R}$. Recall that a complex-valued “function” u on M is said to be a κ -density if it behaves under change of coordinates in the following way

$$u(y) = |\det\{\partial x^i / \partial y^j\}|^\kappa u(x(y)).$$

The usual functions on M are 0-densities. The κ -densities are sections of some complex linear bundle Ω^κ over M . We denote by $C^\infty(M; \Omega^\kappa)$ and $C_0^\infty(M; \Omega^\kappa)$ the spaces of smooth κ -densities and smooth κ -densities with compact supports respectively. If $u \in C_0^\infty(M; \Omega^\kappa)$ and $v \in C^\infty(M; \Omega^{1-\kappa})$ then the product uv is a density and the integral $\int_M uv dx$ is independent of the choice of coordinates. This allows one to define the inner product $(u, v) = \int_M u \bar{v} dx$ on the space of half-densities $C_0^\infty(M; \Omega^{1/2})$ and to introduce the Hilbert space $L_2(M; \Omega^{1/2})$ in the standard way.

In this and the next sections we shall be assuming that the manifold M is provided with a linear connection Γ (which may be non-complete). This means that, for each local coordinate system, we have fixed a set of smooth “functions” $\Gamma_{jk}^i(x)$, $i, j, k = 1, \dots, n$, which behave under change of coordinates in the following way,

$$\sum_l \frac{\partial y^i}{\partial x^l} \Gamma_{pq}^l(x) = \sum_{p,q} \frac{\partial y^j}{\partial x^p} \Gamma_{jk}^i(y(x)) \frac{\partial y^k}{\partial x^q} + \frac{\partial^2 y^i}{\partial x^p \partial x^q}. \quad (3.1)$$

The “functions” $\Gamma_{jk}^i(x)$ are called the *Christoffel symbols*. They can be chosen in an arbitrary way (provided that (3.1) holds), and every set of Christoffel symbols determines a linear connection of M .

A linear connection Γ is uniquely characterized by the torsion tensor $T_{jk}^i := \Gamma_{jk}^i - \Gamma_{kj}^i$ and the curvature tensor

$$R_{jkl}^i := \partial_{y^k} \Gamma_{lj}^i - \partial_{y^l} \Gamma_{kj}^i + \sum_p \Gamma_{kp}^i \Gamma_{lj}^p - \sum_p \Gamma_{lp}^i \Gamma_{kj}^p.$$

If both these tensors vanish on an open set $U \subset M$ then one can choose local coordinates on a neighbourhood of each point $x \in U$ in such a way that $\Gamma_{jk}^i = 0$. Such connections are called *flat*. A connection Γ is called *symmetric* if $T_{jk}^i = 0$.

Let $\nu = \sum \nu^k(y) \partial_{y^k}$ be a vector field on M . The equality (3.1) implies that

$$\nabla_\nu := \sum_k \nu^k(y) \partial_{y^k} + \sum_{i,j,k} \Gamma_{kj}^i(y) \nu^k(y) \eta_i \partial_{\eta_j} \quad (3.2)$$

is a correctly defined vector field on T^*M . The vector field (3.2) is called the *horizontal lift* of ν . The horizontal lifts generate a n -dimensional subbundle HT^*M of the tangent bundle TT^*M over T^*M , which is called the *horizontal distribution*. The vertical vector fields $\partial_{\eta_1}, \dots, \partial_{\eta_n}$ generate another n -dimensional subbundle $VT^*M \subset TT^*M$ which is called the *vertical distribution*. Since $HT^*M \cap VT^*M = \{0\}$, the tangent space $T_{(y,\eta)}T^*M$ at each point $(y, \eta) \in T^*M$ coincides with the sum of its horizontal and vertical subspaces. Obviously, the horizontal subspaces depend on the choice of Γ whereas the vertical subspaces do not.

A curve in the cotangent bundle T^*M is said to be horizontal (or vertical) if its tangent vectors belong to HT^*M (or VT^*M). For any given curve $y(t) \subset M$ and covector $\eta_0 \in T_{y(0)}^*M$ there exists a unique horizontal curve $(y(t), \eta(t)) \subset T^*M$ starting at the point $(y(0), \eta_0)$, whose projection onto M coincides with $y(t)$. It is defined in local coordinates y by the equations

$$\frac{d}{dt} \eta_j(t) - \sum_{i,k} \Gamma_{kj}^i(y(t)) \dot{y}^k(t) \eta_i(t) = 0, \quad \forall j = 1, \dots, n,$$

and is called the *horizontal lift* of $y(t)$. The corresponding linear transformation $\eta_0 \rightarrow \eta(t)$ is said to be the *parallel displacement* along the curve $y(t)$. By duality, horizontal curves and parallel displacements are defined in the tangent bundle TM (and then in all the tensor bundles over M).

A curve $y(t) \subset M$ is said to be a geodesic if the curve $(y(t), \dot{y}(t)) \subset TM$ is horizontal or, equivalently, if

$$\ddot{y}^k(t) + \sum_{i,j} \Gamma_{ij}^k(y(t)) \dot{y}^i(t) \dot{y}^j(t) = 0, \quad \forall k = 1, \dots, n.$$

in any local coordinate system. If U_x is a sufficiently small neighbourhood of x then for every $y \in U_x$ there exists a unique geodesic $\gamma_{y,x}(t)$ such that $\gamma_{y,x}(0) = x$ and $\gamma_{y,x}(1) = y$. The mapping $U_x \ni y \mapsto \dot{\gamma}_{y,x}(0) \in T_xM$ is a bijection between U_x and a neighbourhood of the origin in T_xM , and the corresponding coordinates on U_x are called the *normal coordinates*. In the normal coordinates y centred at x we have $\gamma_{y,x}(t) = x + t(y - x)$, so that $\dot{\gamma}_{y,x}(t) = y - x$ for all $t \in [0, 1]$.

Let $\Phi_{y,x} : T_x^*M \rightarrow T_y^*M$ be the parallel displacement along the geodesic $\gamma_{y,x}$, and let $\Upsilon_{y,x} = |\det \Phi_{y,x}|$. One can easily check that $\Upsilon_{y,x}$ is a density in y and a (-1) -density in x (the map $w_x \rightarrow \Upsilon_{y,x} w_x$ is the parallel displacement along $\gamma_{y,x}$ between the fibres of the bundle Ω). Note that $\Phi_{y,x}$ and $\Upsilon_{y,x}$ depend on the torsion tensor, whereas the geodesics are determined only by the symmetric part of Γ .

Given local coordinates $x = \{x^1, \dots, x^n\}$, let us denote by ∇_i the horizontal lifts of vector fields ∂_{x^i} . For a multi-index α with $|\alpha| = q$, let $\nabla_x^\alpha = \frac{1}{q!} \sum \nabla_{i_1} \dots \nabla_{i_q}$

where the sum is taken over all ordered collections of indices i_1, \dots, i_q corresponding to the multi-index α . The following simple lemma can be found, for instance, in [Sa2, Section 3].

Lemma 3.1. *If $a \in C^\infty(T^*M)$ then $a(y, \Phi_{y,x}\xi)$ admits the following asymptotic expansion,*

$$a(y, \Phi_{y,x}\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \hat{\gamma}_{y,x}^{\alpha} \nabla_x^{\alpha} a(x, \xi), \quad y \rightarrow x. \quad (3.3)$$

Sketch of proof. Write down the left hand side in normal coordinates y centred at x and apply Taylor's formula. \square

Remark 3.2. If a is a function on T^*M then the ‘‘hypermatrix’’ $\{\partial_{\xi}^{\beta} \nabla_x^{\alpha} a(x, \xi)\}_{\substack{|\alpha|=q \\ |\beta|=p}}$ behaves as a (p, q) -tensor under change of coordinates. Therefore all the formulae in the next section have an invariant meaning and do not depend on the choice of coordinates.

4. PDOs: a coordinate-free approach

Definition 4.1. We shall say that an amplitude a defined on $M \times T^*M$ belongs to the class $S_{\rho, \delta}^m(\Gamma)$ with $\rho, \delta \in [0, 1]$ and $m \in \mathbb{R}$ if

$$\sup_{(x,z) \in K} |\partial_x^{\alpha} \partial_{\zeta}^{\beta} \nabla_{i_1} \nabla_{i_2} \dots \nabla_{i_q} a(x, z, \zeta)| \leq C_{K; \alpha, \beta, q} (1 + |\xi|)^{m + \delta(|\alpha| + q) - \rho|\beta|} \quad (4.1)$$

for each compact set $K \subset M \times M$, all multi-indices α, β and all sets of indices i_1, \dots, i_q , where ∇_k are horizontal lifts of the vector fields ∂_{z_k} and $C_{K; \alpha, \beta, \gamma}$ are some positive constants.

From the definition of the horizontal lifts it follows that $a \in S_{\rho, \delta}^m(\Gamma)$ with $\delta \geq 1 - \rho$ if and only if a satisfies (2.1) in any local coordinate system. In this case the class $S_{\rho, \delta}^m(\Gamma)$ is the same for all linear connections Γ . If $\delta < 1 - \rho$ then $S_{\rho, \delta}^m(\Gamma)$ depends on the choice of Γ . Note that (2.1) is a particular case of (4.1), in which the connection Γ is flat.

Let us fix a sufficiently small neighbourhood V of the diagonal in $M \times M$ and define $z_{\tau} = z_{\tau}(x, y) = \gamma_{y,x}(\tau)$, where $\tau \in [0, 1]$ is regarded as a parameter. Consider the phase function

$$\varphi_{\tau}(x, \zeta, y) = -\langle \dot{\gamma}_{y,x}(\tau), \zeta \rangle, \quad (x, y) \in V, \quad \zeta \in T_{z_{\tau}}^*M, \quad (4.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard pairing between vectors and covectors. The function φ_{τ} is invariantly defined and, by the above, coincides with $(x - y) \cdot \zeta$ in normal coordinates y centred at x .

Definition 4.2. An operator A acting in the space of κ -densities on M is said to be a PDO of class $\Psi_{\rho, \delta}^m(\Omega^{\kappa}, \Gamma)$ if

- (c) its Schwartz kernel $\mathcal{A}(x, y)$ is infinitely differentiable outside the diagonal $\{x = y\}$,

(c₂) $\mathcal{A}(x, y) = (2\pi)^{-n} p_{\kappa, \tau}(x, y) \int_{T_{z_\tau}^* M} e^{i\varphi_\tau(x, \zeta, y)} a(z_s; z_\tau, \zeta) d\zeta$ in a neighbourhood of the diagonal, where $a \in S_{\rho, \delta}^m(\Gamma)$, $p_{\kappa, \tau} := \Upsilon_{y, z_\tau}^{1-\kappa} \Upsilon_{z_\tau, x}^{-\kappa}$ and $s, \tau \in [0, 1]$ are some fixed numbers.

Remark 4.3. If y are normal coordinates centred at x then $\varphi_\tau(x, \zeta, y) = (x - y) \cdot \zeta$ and the integral (c₂) takes the form (c₁). However, Definition 2.2 assumes that x and y are the same local coordinates on U , whereas the above identity holds if we choose coordinates y depending on the point x .

Remark 4.4. The weight factor $p_{\kappa, \tau}$ is introduced for the following two reasons.

- (1) It makes the definition independent of the choice of coordinates ζ in the cotangent space $T_{z_\tau}^* M$.
- (2) Because of this factor, the Schwartz kernel behaves as a $(1 - \kappa)$ -density in y and κ -density in x , that is, (c₂) defines an operator in the space of κ -densities for all $\kappa \in \mathbb{R}$ and all $s, \tau \in [0, 1]$. In particular, this allows us to consider PDOs in the Hilbert space $L_2(M, \Omega^{1/2})$ and to introduce Weyl symbols (corresponding to $\tau = \frac{1}{2}$).

One can replace $p_{\kappa, \tau}$ in Definition 4.2 with any other smooth weight factor $p(x, y)$ which behaves in a similar way under change of coordinates. The precise choice of the weight factor seems to be of little importance, since all formulae in the symbolic calculi corresponding to different weight factors p and \tilde{p} can easily be deduced from each other by expanding the function $p^{-1}\tilde{p}$ into an asymptotic series of the form (3.3), replacing $\hat{\gamma}_{y, x}(z_\tau) e^{i\varphi_\tau(x, \zeta, y)}$ with $i\nabla_\zeta e^{i\varphi_\tau(x, \zeta, y)}$ and integrating by parts with respect to ζ .

Lemma 4.5. *If $\delta < \rho$ and $a \in S_{\rho, \delta}^m(\Gamma)$ then for all $s, \tau \in [0, 1]$*

$$p_{\kappa, \tau} \int_{T_{z_\tau}^* M} e^{i\varphi_\tau(x, \zeta, y)} a(z_s; z_\tau, \zeta) d\zeta = p_{\kappa, \tau} \int_{T_{z_\tau}^* M} e^{i\varphi_\tau(x, \zeta, y)} \sigma_{A, \tau}(z_\tau, \zeta) d\zeta$$

and

$$p_{\kappa, \tau} \int_{T_{z_\tau}^* M} e^{i\varphi_\tau(x, \zeta, y)} \sigma_{A, \tau}(z_\tau, \zeta) d\zeta = p_{\kappa, s} \int_{T_{z_s}^* M} e^{i\varphi_s(x, \zeta, y)} \sigma_{A, s}(z_s, \zeta) d\zeta$$

modulo C^∞ -densities, where $\sigma_{A, \tau}$ and $\sigma_{A, s}$ are amplitudes of class $S_{\rho, \delta}^m(\Gamma)$ given by the asymptotic expansions

$$\sigma_{A, \tau}(x, \xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|} (s - \tau)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \nabla_y^\alpha a(y; x, \xi) \Big|_{y=x}, \quad |\xi| \rightarrow \infty,$$

$$\sigma_{A, s}(x, \xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|} (\tau - s)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \nabla_x^\alpha \sigma_{A, \tau}(x, \xi), \quad |\xi| \rightarrow \infty.$$

Sketch of proof. The first identity is proved by applying (3.3) with $x = z_\tau$ and $y = z_s$ to the function $a(\cdot; z_\tau, \zeta)$ with fixed (z_τ, ζ) , substituting $\hat{\gamma}_{z_s, z_\tau} e^{i\varphi_\tau} = (\tau - s) \nabla_\zeta e^{i\varphi_\tau}$ and integrating by parts. The second is obtained in a similar way, after changing variables $\zeta = \Phi_{z_s, z_\tau} \zeta'$. \square

Lemma 4.5 shows that Definition 4.2 does not depend on the choice of τ and s , and that every PDO A is defined modulo $\Psi^{-\infty}$ by its τ -symbol $\sigma_{A,\tau}$. The other way round, for each linear connection Γ , the τ -symbol $\sigma_{A,\tau}$ is determined by the operator A modulo $S^{-\infty}$.

If $A \in \Psi_{\rho,\delta}^m(\Omega^\kappa, \Gamma)$ then, in a similar way, one can show that

$$\sigma_{A^*,\tau}(x, \xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|} (1-2\tau)^{|\alpha|}}{\alpha!} \partial_{\xi}^{\alpha} \overline{\nabla_x^{\alpha} \sigma_{A,\tau}(x, \xi)}, \quad |\xi| \rightarrow \infty, \quad (4.3)$$

where A^* is the adjoint operator acting in the space of $(1-\kappa)$ -densities. In particular, for the Weyl symbols we have $\sigma_{A^*,1/2} - \overline{\sigma_{A,1/2}} \in S^{\infty}$ for all $\kappa \in \mathbb{R}$.

Remark 4.6. The full τ -symbol $\sigma_{A,\tau}$ depends on Γ and τ . If $\max\{\delta, 1-\rho\} < \rho$ then all the τ -symbols $\sigma_{A,\tau}$ corresponding to different connections Γ coincide with the principal symbol of A modulo a lower order term. However, in the general case it seems to be impossible to define a principal symbol of A without introducing an additional structure on the manifold M or a global phase function (see Subsection 7.8).

Let $\Upsilon_{\kappa}(x, y, z) := \Upsilon_{y,z}^{1-\kappa} \Upsilon_{z,x}^{2-\kappa} \Upsilon_{x,y}^{1-\kappa}$, $\psi(x, \xi; y, z) := \langle \dot{\gamma}_{y,x}, \xi \rangle - \langle \dot{\gamma}_{z,x}, \xi \rangle - \langle \dot{\gamma}_{y,z}, \Phi_{z,x}\xi \rangle$ and

$$P_{\beta,\gamma}^{(\kappa)}(x, \xi) = \left((\partial_y + \partial_z)^{\beta} \partial_y^{\gamma} \sum_{|\beta'| \leq |\beta|} \frac{1}{\beta'!} D_{\xi}^{\beta'} \partial_y^{\beta'} (e^{i\psi} \Upsilon_{\kappa}) \right) \Big|_{y=z=x},$$

where y and z are normal coordinates centred at x . The functions $P_{\beta,\gamma}^{(\kappa)} \in C^{\infty}(T^*M)$ are polynomials in ξ ; we shall denote their degrees by $d_{\beta,\gamma}^{(\kappa)}$.

One can easily show that $P_{0,\gamma}^{(\kappa)} \equiv 0$, $P_{\beta,0}^{(\kappa)} \equiv 0$ and $d_{\beta,\gamma}^{(\kappa)} \leq \min\{|\beta|, |\gamma|\}$ for any connection Γ . Moreover, if Γ is symmetric then $d_{\beta,\gamma}^{(\kappa)} \leq \min\{|\beta|, |\gamma|, (|\beta| + |\gamma|)/3\}$ [Sa2, Lemma 8.1].

Theorem 4.7. *Let $A \in \Psi_{\rho,\delta}^{m_1}(\Omega^\kappa, \Gamma)$ and $B \in \Psi_{\rho,\delta}^{m_2}(\Omega^\kappa, \Gamma)$, where $\rho > \delta$. Assume, in addition, that*

- (1) either $\rho > 1/2$,
- (2) or the connection Γ is symmetric and $\rho > 1/3$,
- (3) or at least one of the PDOs A and B belongs to $\Psi_{1,0}^m(\Omega^\kappa, \Gamma)$.

Then $AB \in \Psi_{\rho,\delta}^{m_1+m_2}(\Omega^\kappa, \Gamma)$ and

$$\begin{aligned} & \sigma_{AB}(x, \xi) \\ & \sim \sum_{\alpha, \beta, \gamma} \frac{1}{\alpha!} \frac{1}{\beta!} \frac{1}{\gamma!} P_{\beta,\gamma}^{(\kappa)}(x, \xi) D_{\xi}^{\alpha+\beta} \sigma_A(x, \xi) D_{\xi}^{\gamma} \nabla_x^{\alpha} \sigma_B(x, \xi), \quad |\xi| \rightarrow \infty. \end{aligned} \quad (4.4)$$

The proof of Theorem 4.7 is similar to that of Theorem 2.4 (see [Sa2, Sec. 8]). In particular, if the connection Γ is flat then $P_{\beta,\gamma}^{(\kappa)} \equiv 0$ as $|\beta| + |\gamma| \geq 1$ and (4.4) turns into (2.4).

Remark 4.8. The conditions on ρ and the estimates for $d_{\beta,\gamma}^{(\kappa)}$ imply that the terms in the right hand side of (4.4) form an asymptotic series. It is plausible that the composition formula (4.4) holds whenever the orders of the terms in the right hand side tend to $-\infty$ as $|\alpha| + |\beta| + |\gamma| \rightarrow \infty$. However, it is not clear how this can be proved.

Remark 4.9. Coefficients of the polynomials $P_{\beta,\gamma}^{(\kappa)}$ are components of some tensors, which are polynomials in the curvature and torsion tensors and their symmetric covariant differentials.

In the same way as in the local theory of PDOs, Theorem 4.7 implies standard results on the boundedness of PDOs in the Sobolev spaces and allows one to construct the resolvent of an elliptic operator in the form of a PDO.

5. Functions of the Laplacian

In this section we assume that M is a compact Riemannian manifold without boundary and denote $|\xi|_x := \sqrt{\sum_{i,j} g^{ij}(x) \xi_i \xi_j}$ where $\xi \in T_x^*M$ and $\{g^{ij}\}$ is the metric tensor. It is well known that there exist a unique symmetric connection $\Gamma_{\mathbf{g}}$ on M , called the Levi–Civita connection, such that the function $|\xi|_x$ is constant along every horizontal curve in T^*M .

Denote by Δ the Laplace operator acting in the space of half-densities; in local coordinates

$$\Delta u(x) = g^{\kappa-1}(x) \sum_{i,j} \partial_{x^i} (g(x) g^{ij}(x) \partial_{x^j} (g^{-\kappa}(x) u(x))),$$

where $g := |\det g^{ij}|^{-1/2}$ is the canonical Riemannian density. Let ν be a self-adjoint first order PDO such that $-\Delta + \nu > 0$, and let $A_\nu := \sqrt{-\Delta + \nu}$. The operator A_ν is a PDO of class $\Psi_{1,0}^1$ whose symbol coincides with $|\xi|_x$ modulo S^0 in any local coordinate system. Thus we have $A_\nu \in \Psi_{1,0}^1(\Omega^{1/2}, \Gamma)$ for any linear connection Γ .

Definition 5.1. If $\rho \in (0, 1]$, let S_ρ^m be the class of infinitely differentiable functions ω on \mathbb{R} such that

$$|\partial_s^j \omega(s)| \leq C_j (1 + |s|)^{m-j\rho}, \quad \forall j = 0, 1, \dots, \quad (5.1)$$

where C_k are some constants.

A natural conjecture is that the operator $\omega(A_\nu)$ is a PDO whenever $\omega \in S_\rho^m$. If it is true then the symbol of $\omega(A_\nu)$ should coincide with $\omega(|\xi|_x)$ modulo lower order terms. If $\rho < 1/2$ then, generally speaking, this function does not belong to the class $S_{\rho,\delta}^m$ with $\rho > \delta$ in any local coordinate system. However, since its horizontal derivatives corresponding to the Levi–Civita connection are equal to zero, we have $\omega(|\xi|_x) \in S_{\rho,\delta}^m(\Gamma_{\mathbf{g}})$ for all $\omega \in S_\rho^m$. In particular, this implies the following

Lemma 5.2. *Let $\tau \in [0, 1)$ and $U_\tau(t) := \exp(itA_\nu^\tau)$. Then $U_\tau(t) \in \Psi_{1-\tau,0}^m(\Omega^{1/2}, \Gamma_{\mathbf{g}})$ for all $t \in \mathbb{R}$ and $\sigma_{U_\tau(t)}(x, \xi) = e^{it|\xi|_x^\tau} b^{(\tau)}(t, x, \xi)$, where $b^{(\tau)} \in C^\infty(\mathbb{R} \times T^*M)$ and $\partial_t^k b^{(\tau)} \in S_{1,0}^0$ for all $k = 0, 1, \dots$ and each fixed t .*

Sketch of proof. Write down $U_\tau(t)$ formally as an integral (\mathbf{c}_2) with an unknown symbol of the form $e^{it|\xi|_x^\tau} b^{(\tau)}(t, x, \xi)$, substitute the integral into the equation $\partial_t U_\tau(t) = iA_\nu^\tau U_\tau(t)$, apply the composition formula (4.4) to $A_\nu^\tau U_\tau(t)$ and equate terms of the same order in the right and left hand sides. \square

Using Lemma 5.2, one can construct other functions of the operator A_ν .

Theorem 5.3. *If $\omega \in S_\rho^m$ then $\omega(A_\nu) \in \Psi_{\rho,0}^m(\Omega^{1/2}, \Gamma_{\mathbf{g}})$ and*

$$\sigma_{\omega(A_\nu)} \sim \omega(|\xi|_x) + \sum_{j=1}^{\infty} c_{j,\nu}(x, \xi) \omega^{(j)}(|\xi|_x), \quad |\xi| \rightarrow \infty, \quad (5.2)$$

where $\omega^{(j)} := \partial_s^j \omega$ and $c_{j,\nu}(x, \xi) \in S_{1,0}^0$. The functions $c_{j,\nu}$ are determined recursively by the equations

$$\sigma_{A_\nu^k}(x, \xi) = |\xi|_x^k + \sum_{j=1}^k \frac{k!}{(k-j)!} |\xi|_x^{k-j} c_{j,\nu}(x, \xi). \quad (5.3)$$

Sketch of proof. Define $\omega_\tau(s) = \omega(s^{1/\tau})$, and let $\widehat{\omega}_\tau(t)$ be the Fourier transform of ω_τ . Then

$$\omega(A_\nu) = (2\pi)^{-1} \int \widehat{\omega}_\tau(t) e^{itA_\nu^\tau} dt.$$

Let $\varsigma \in C_0^\infty(\mathbb{R})$ be equal to 1 in a neighbourhood of the origin and have support contained in a small neighbourhood of the origin. Consider the operators

$$\omega_1(A_\nu) = (2\pi)^{-1} \int \varsigma(t) \widehat{\omega}_\tau(t) e^{itA_\nu^\tau} dt,$$

$$\omega_2(A_\nu) = (2\pi)^{-1} \int (1 - \varsigma(t)) \widehat{\omega}_\tau(t) e^{itA_\nu^\tau} dt.$$

By integration by parts, the operator $\omega_2(A_\nu)$ can be written as

$$\omega_2(A_\nu) = (2\pi)^{-1} A_\nu^{-k} \int D_t^k ((1 - \varsigma(t)) \widehat{\omega}_\tau(t)) e^{itA_\nu^\tau} dt.$$

Since k may be chosen arbitrarily large, this shows that $\omega_2(A_\nu)$ has an infinitely smooth kernel. By Lemma 5.2, the operator $\omega_1(A_\nu)$ is a PDO whose symbol coincides with

$$(2\pi)^{-1} \int_{-\infty}^{\infty} \varsigma(t) \widehat{\omega}_\tau(t) e^{it|\xi|_x^\tau} b^{(\tau)}(t, x, \xi) dt.$$

Expanding $\varsigma(t)b^{(\tau)}(t, x, \xi)$ by Taylor's formula at $t = 0$, we see that the symbol of $\omega_1(A_\nu)$ admits an asymptotic expansion of the form (5.2) with some functions $c_{j,\nu}$. These functions do not depend on ω and can be found by substituting $\omega(s) = s^k$ with $k = 1, 2, \dots$. This leads to (5.3). \square

Definition 5.4. If $\rho \in (0, 1]$, let $S_\rho^m(\mathbf{g})$ be the class of C^∞ -functions on T^*M which admit asymptotic expansions of the form

$$a(x, \xi) \sim \sum_{j=0}^{\infty} c_j(x, \xi) \omega_j(|\xi|_x), \quad |\xi| \rightarrow \infty, \quad (5.4)$$

where $c_j \in S_{1,0}^0$, $\omega_j \in S_\rho^{m_j}$ with $m_0 = m$ and $m_j \rightarrow -\infty$. Denote by $\Psi_\rho^m(\Omega^{1/2}, \mathbf{g})$ the class of PDOs acting in the space of half-densities whose $\Gamma_{\mathbf{g}}$ -symbols belong to $S_\rho^m(\mathbf{g})$.

Theorem 5.3 immediately implies that $\omega(A_\nu) \in \Psi_\rho^m(\Omega^{1/2}, \mathbf{g})$ whenever $\omega \in S_\rho^m$. The other way round, any PDOs of class $\Psi_\rho^m(\Omega^{1/2}, \mathbf{g})$ can be represented in terms of functions of the operator A_ν .

Lemma 5.5. For each $A \in \Psi_\rho^m(\Omega^{1/2}, \mathbf{g})$ there exist PDOs $C_{j,\nu} \in \Psi_{1,0}^0$ and functions $\tilde{\omega}_j \in S_\rho^{l_j}$ such that $l_0 = m$, $l_j \rightarrow -\infty$ and

$$A \sim \sum_{j=0}^{\infty} C_{j,\nu} \tilde{\omega}_j(A_\nu), \quad (5.5)$$

where \sim means that the Schwartz kernel of the difference $A - \sum_{j=0}^k C_{j,\nu} \tilde{\omega}_j(A_\nu)$ becomes smoother and smoother as $k \rightarrow \infty$.

Sketch of proof. Assume that (5.4) holds and denote by C_0 the PDO with symbol $c_0(x, \xi)$. Theorems 4.7 and 5.3 imply that $A = C_0 \omega_0(A_\nu) + A_\nu^{(1)}$ where $A_\nu^{(1)} \in \Psi_\rho^{l_1}(\Omega^{1/2}, \mathbf{g})$ with $l_1 \leq \max\{m_1, m_0 - \rho\}$. The same arguments show that $A_\nu^{(1)} = C_{1,\nu} \tilde{\omega}_1(A_\nu) + A_\nu^{(2)}$ where $C_{1,\nu} \in \Psi_{1,0}^0$, $\tilde{\omega}_1 \in S_\rho^{l_1}$ and $A_\nu^{(2)} \in \Psi_\rho^{l_2}(\Omega^{1/2}, \mathbf{g})$ where $l_2 \leq \max\{m_2, l_1 - \rho\}$. Repeatedly applying this procedure, we obtain a sequence of operators $A_\nu^{(k)} \in \Psi_\rho^{l_k}(\Omega^{1/2}, \mathbf{g})$ such that $A - A_\nu^{(k)} = \sum_{j=0}^{k-1} C_{j,\nu} \tilde{\omega}_j(A_\nu)$, where $C_{j,\nu}$ and $\tilde{\omega}_j$ satisfy the required conditions and $l_k \rightarrow -\infty$ as $k \rightarrow \infty$. \square

Since $\omega_1(A_\nu) \omega_2(A_\nu) = \omega_1 \omega_2(A_\nu)$ for any two functions ω_1 and ω_2 , combining Theorem 4.7 and Lemma 5.5, we obtain

Corollary 5.6. If $A \in \Psi_\rho^{m_1}(\Omega^{1/2}, \mathbf{g})$ and $B \in \Psi_\rho^{m_2}(\Omega^{1/2}, \mathbf{g})$ then the composition AB is a PDO of class $\Psi_\rho^{m_1+m_2}(\Omega^{1/2}, \mathbf{g})$ whose symbol admits the asymptotic expansion (4.4).

Remark 5.7. Under the conditions of Corollary 5.6, the estimates on $d_{\beta,\gamma}^{(\kappa)}$ obtained in Section 4 do not directly imply that (4.4) is an asymptotic series, as it seems to contain terms of growing orders. However, these ‘‘bad’’ terms cancel out due to the symmetries of the curvature tensor. It would be interesting to find a direct proof of Corollary 5.6, which does not use Lemma 5.5 (a relevant problem was mentioned in Remark 4.8).

From the above results it follows that the restriction of the operator $\omega(A_\nu)$ to an open subset of M is determined modulo $\Psi^{-\infty}$ by the restrictions of the metric \mathbf{g} and the operator ν to this subset. More precisely, we have the following

Corollary 5.8. *Let $v \in C_0^\infty(M)$, and let $\{v\}$ be the corresponding multiplication operator. Consider the operator $\tilde{A}_{\tilde{\nu}}$ generated by another metric $\tilde{\mathbf{g}}$ and another first order PDO $\tilde{\nu}$. If $\tilde{\mathbf{g}} = \mathbf{g}$ on the support of the function v and $\tilde{\nu}\{v\} = \nu\{v\}$ then*

$$\{v\}(\omega(A_\nu) - \omega(\tilde{A}_{\tilde{\nu}})) \in \Psi^{-\infty} \quad \text{and} \quad (\omega(A_\nu) - \omega(\tilde{A}_{\tilde{\nu}}))\{v\} \in \Psi^{-\infty}$$

for every $\omega \in S_\rho^m$.

Sketch of proof. The multiplication operator $\{v\}$ is a PDO with symbol $v(x)$, which belongs to $\Psi_\rho^0(\Omega^{1/2}, \mathbf{g})$. Applying Lemma 5.5 and Corollary 5.6, we see that $\{v\}(\omega(A_\nu) - \omega(\tilde{A}_{\tilde{\nu}}))$ and $(\omega(A_\nu) - \omega(\tilde{A}_{\tilde{\nu}}))\{v\}$ are PDOs whose full symbols are identically equal to zero. \square

Remark 5.9. In a similar way, it is possible to define the classes $\Psi_\rho^m(\Omega^\kappa, \mathbf{g})$ which consist of PDOs acting in the space of κ -densities. Theorem 4.7 implies that $A \in \Psi_\rho^m(\Omega^\kappa, \mathbf{g})$ if and only if $g^{1/2-\kappa}Ag^{\kappa-1/2} \in \Psi_\rho^m(\Omega^{1/2}, \mathbf{g})$. Using this observation, one can easily reformulate all results of this section for operators $A \in \Psi_\rho^m(\Omega^\kappa, \mathbf{g})$.

6. An approximate spectral projection

In applications, one often has to deal with functions of an operator which depend on additional parameters. It is more or less clear that the results of the previous section can be extended to parameter-dependent functions ω under the assumption that the estimates (5.1) hold uniformly with respect to the parameters. Therefore, instead of formulating general statements, we shall consider an example which is of particular interest for spectral theory.

Further on we assume that $\lambda > 0$ and denote by $\Psi^{-\infty}(\lambda)$ the class of parameter-dependent operators with infinitely smooth Schwartz kernels $\mathcal{A}_\lambda(x, y)$ such that

$$\lim_{\lambda \rightarrow \infty} \lambda^p |\partial_x^\alpha \partial_y^\beta \mathcal{A}_\lambda(x, y)| = 0$$

for all multi-indices α, β and all $p = 1, 2, \dots$. Similarly, let $S^{-\infty}(\lambda)$ be the class of parameter-dependent amplitudes a_λ such that

$$(\lambda + |\xi|)^p \sup_{(x, y) \in M} |\partial_\xi^\alpha \nabla_x^\beta \nabla_y^\gamma a_\lambda(y; x, \xi)| \rightarrow 0 \quad \text{as } \lambda + |\xi| \rightarrow \infty$$

for all multi-indices α, β, γ and all $p = 1, 2, \dots$.

Let us fix a small $\varepsilon > 0$ and a nonincreasing function $f \in C^\infty(\mathbb{R})$ such that

$$f(s) = \begin{cases} 1 & \text{if } s \leq 0; \\ 0 & \text{if } s \geq \varepsilon, \end{cases}$$

and $0 \leq f(s) \leq 1$ for all $s \in \mathbb{R}$. If $\rho \in (0, 1]$ and $\lambda > 0$, let

$$\chi_\rho(\lambda, s) := f(\lambda^{-\rho}(s - \lambda)).$$

For each fixed $\lambda > 0$, the function χ_ρ vanishes on the interval $[\lambda + \varepsilon\lambda^\rho, \infty)$, is identically equal to 1 on the interval $(-\infty, \lambda]$ and smoothly descends from 1 to 0 on the interval $[\lambda, \lambda + \varepsilon\lambda^\rho]$. Since this function differs from the characteristic function of the interval $(-\infty, \lambda]$ only on the relatively small interval $(\lambda, \lambda + \varepsilon\lambda^\rho)$, the operator $\chi(\lambda, A_\nu)$ can be thought of as an approximate spectral projection of A_ν corresponding to $(-\infty, \lambda]$. The standard elliptic regularity theorem implies that the operator $\chi(\lambda, A_\nu)$ has an infinitely differentiable Schwartz kernel for each fixed λ .

The derivatives $\partial_s^j \chi_\rho(\lambda, s)$ are equal to zero outside the interval $(\lambda, \lambda + \varepsilon\lambda^\rho)$. Therefore

$$|\partial_s^j \chi_\rho(\lambda, s)| \leq \tilde{C}_j (|s| + \lambda)^{-j\rho}, \quad \forall j = 0, 1, \dots, \quad (6.1)$$

for all $s \in \mathbb{R}$ and all $\lambda > 1$, where \tilde{C}_j are some constants independent of λ and s . The same arguments as in the proof of Theorem 5.3 show that $\chi(\lambda, A_\nu)$ is a parameter-dependent PDO whose symbol admits the asymptotic expansion

$$\sigma_{\chi(\lambda, A_\nu)} \sim \chi(\lambda, |\xi|_x) + \sum_{j=1}^{\infty} c_{j,\nu}(x, \xi) \chi^{(j)}(\lambda, |\xi|_x), \quad \lambda + |\xi| \rightarrow \infty, \quad (6.2)$$

where $c_{j,\nu}(x, \xi)$ are the same functions as in (5.3) and $\chi^{(j)}$ denotes j th s -derivative of the function χ .

Note that the functions $\chi^{(j)}(\lambda, |\xi|_x)$ belong to $S^{-\infty}$ for each fixed λ . However, their rate of decay depends on λ . The asymptotic expansion (6.2) is uniform with respect to λ ; it defines $\sigma_{\chi(\lambda, A_\nu)}$ modulo $S^{-\infty}(\lambda)$. Substituting the terms from (6.2) into the integral (\mathbf{c}_2) , we obtain an asymptotic expansion of the Schwartz kernel of $\chi(\lambda, A_\nu)$ into a series of infinitely smooth half-densities, which decay more and more rapidly as $\lambda \rightarrow \infty$. This expansion defines $\chi(\lambda, A_\nu)$ modulo $\Psi^{-\infty}(\lambda)$.

Straightforward analysis of the proof of Theorem 4.7 shows that it remains valid in the case where one of the operators belongs to $\Psi_{1,0}^0$ and the other is a parameter-dependent PDO whose symbol admits an asymptotic expansion of the form (6.2). In this case (4.4) gives an expansion of σ_{AB} as $\lambda + |\xi| \rightarrow \infty$ and defines the symbol modulo $S^{-\infty}(\lambda)$.

Now, in the same way as in Section 5, one can show that the composition of parameter-dependent PDOs whose symbols admit asymptotic expansions of the form

$$\sigma(x, \xi) \sim \sum_{j=0}^{\infty} c_j(x, \xi) \chi^{(j)}(\lambda, |\xi|_x), \quad \lambda + |\xi| \rightarrow \infty, \quad c_j \in \Psi_{1,0}^0 \quad (6.3)$$

is also a parameter-dependent PDO whose symbol is given by (4.4) modulo $S^{-\infty}(\lambda)$.

Let $\Pi_\nu(\lambda)$ be the spectral projection of the operator A_ν corresponding to the interval $(-\infty, \lambda)$. The above results imply the following

Theorem 6.1. *Let $v \in C_0^\infty(M)$, and let $\{v\}$ be the corresponding multiplication operator. Consider the spectral projections $\Pi_\nu(\lambda)$ and $\tilde{\Pi}_{\tilde{\nu}}(\lambda)$ generated by different metrics $\mathbf{g}, \tilde{\mathbf{g}}$ and different first order PDOs $\nu, \tilde{\nu}$ satisfying the conditions of Section 5. If $\tilde{\mathbf{g}} = \mathbf{g}$ on the support of the function v then*

$$\Pi_\nu(\lambda) \{v\} (I - \tilde{\Pi}_{\tilde{\nu}}(\lambda + c\lambda^\rho)) \in \Psi^{-\infty}(\lambda), \quad \forall c, \rho > 0. \quad (6.4)$$

Sketch of proof. Assume that $\rho \in (0, 1]$, and let $\chi(\lambda, s)$ be defined as above with some $\varepsilon < c/3$. Then $\chi(\lambda, s) \equiv 1$ for $s \geq \lambda$ and $\chi(\lambda, s - \varepsilon\lambda^\rho) \equiv 0$ for $s \geq \lambda + c\lambda^\rho$. It follows that

$$\Pi_\nu(\lambda) \chi(\lambda, A_\nu) = \Pi_\nu(\lambda) \quad \text{and} \quad \chi(\lambda, A_{\tilde{\nu}} - \varepsilon\lambda^\rho I) (I - \Pi_{\tilde{\nu}}(\lambda + c\lambda^\rho)) = 0.$$

Consequently, we have

$$\begin{aligned} & \Pi_\nu(\lambda) \{v\} (I - \Pi_{\tilde{\nu}}(\lambda + c\lambda^\rho)) \\ &= \Pi_\nu(\lambda) \chi(\lambda, A_\nu) \{v\} (I - \chi(\lambda, A_{\tilde{\nu}} - \varepsilon\lambda^\rho I)) (I - \Pi_{\tilde{\nu}}(\lambda + c\lambda^\rho)) \end{aligned} \quad (6.5)$$

Since $\chi(\lambda, s) = \chi(\lambda, s) \chi(\lambda, s - \varepsilon\lambda^\rho)$, the composition formula implies that

$$\chi(\lambda, A_\nu) \{v\} (I - \chi(\lambda, A_{\tilde{\nu}} - \varepsilon\lambda^\rho I)) \in \Psi^{-\infty}(\lambda).$$

Now the required result follows from (6.5) and the fact that the Schwartz kernel of the spectral projection is polynomially bounded in λ with all its derivatives (see, for instance, [SV, Section 1.8]). \square

Remark 6.2. It is not surprising that the operator in the left hand side of (6.4) has a lower order than the spectral projections themselves as $\lambda \rightarrow \infty$. However, one would expect its norm to decay as a fixed negative power of λ , since the perturbation $A_\nu - A_{\tilde{\nu}}$ is a more or less arbitrary PDO of order zero. We do not know whether (6.4) can be obtained by other techniques (including that of Fourier integral operators).

Remark 6.3. All results of this section can easily be extended to a noncompact closed manifold M . In this case all the asymptotic expansions are uniform on compact subsets of M and $M \times M$.

7. Other known results and possible developments

7.1. Other definitions for scalar PDOs

If Γ is a linear connection, then the corresponding symbol of a PDO A can easily be recovered from the asymptotic expansion of $A(e^{i\varphi_\tau(x, \zeta, y)} \chi(x, y))$ as $\zeta \rightarrow \infty$, where φ_τ is defined by (4.2) and χ is a smooth cut-off function or κ -density (we suppose that x is fixed and that the operator acts in the variable y). After that, all the standard formulae of the local theory of PDOs can be rewritten in terms of their Γ -symbols. Moreover, making appropriate assumptions about the asymptotic behaviour of $A(e^{i\varphi_\tau(x, \zeta, \cdot)} \chi(x, y))$, one can try to define various classes of PDOs associated with the linear connection Γ .

This approach was introduced and developed by Harold Widom and Lance Drager (see [Wi1], [Wi2] and [Dr]). Its main disadvantage is the absence of an explicit formula representing the Schwartz kernel of a PDO via its symbol. As a consequence, one has to assume that PDOs and the corresponding classes of amplitudes are defined in local coordinates, which makes it impossible to extend the definition to $\rho < \max\{\delta, 1 - \rho\}$.

In [Pf1], Markus Pflaum defined a PDO in the space of functions by the formula

$$Au(x) = (2\pi)^{-n} \int_{T_x^*M} \int_{T_x M} \chi(x, y) e^{i\varphi_0(x, \xi, y)} a(x, \xi) u(y) dy d\xi, \quad (7.1)$$

where $a(x, \xi)$ is a function on T^*M of class $S_{\rho, \delta}^m$, y are normal coordinates centred at x and χ is a smooth cut-off function vanishing outside a neighbourhood of the diagonal. He obtained asymptotic expansions for the symbols of the adjoint operator and the composition of PDOs and, in the later paper [Pf2], extended them to τ -symbols. However, the results in [Pf1, Pf2] are stated and proved with the use of local coordinates and, therefore, the author had to assume that $\max\{\delta, 1 - \rho\} < \rho$.

Recall that under this condition the standard results of the local theory of PDOs hold, and the only advantage of a coordinate-free calculus is that it helps to fight Problem 2.7. A typical example, considered in [Pf1], is the PDO with a symbol of the form $(1 + |\xi|^2)^{b(x)}$ where $b(x)$ is a smooth function on M . Formally speaking, this PDO belongs only to the class $S_{1, \delta}^m$ with $m = \sup_x b(x)$ and any $\delta \in (0, 1)$. But its properties are determined by the values of the function b at all points $x \in M$; in a sense, this operator has a variable order depending on $x \in M$. In such a situation, it is not sufficient to consider only the principal symbol. One has to define a full symbol which can be done with the use of a linear connection.

It is clear that (7.1) differs from Definition 4.2 only by the choice of the weight factor $p_{\kappa, \tau}$. Applying the procedure described in Remark 4.4, one can easily show that

$$\sigma_A(x, \xi) \sim \sum_{\alpha} P_{\alpha}(x) \partial_{\xi}^{\alpha} a(x, \xi), \quad |\xi| \rightarrow \infty, \quad (7.2)$$

where $a(x, \xi)$ is the symbol appearing in (7.1) and P_{α} are components of some tensor fields. Using (7.2), one can rewrite all the results obtained in [Sa2] in terms of symbols defined by (7.1). This shows that Pflaum's formulae can be reformulated in terms of the horizontal derivatives ∇_x^{α} and thus extended to the classes $\Psi_{\rho, \delta}^m(\Omega^{\kappa}, \Gamma)$ and τ -symbols.

In particular, Pflaum's composition formula can be written in the form (4.4) with some other polynomials $\tilde{P}_{\beta, \gamma}^{(\kappa)}$. For operators acting in the space of functions and $\tau = 0$, this result was established by Vladimir Sharafutdinov in [S1]. He chose to give a direct proof instead of deducing the formula from (4.4) and (7.2) and, for some reason, considered only the classes $\Psi_{1, 0}^m$. Sharafutdinov gave an alternative description of the polynomials $\tilde{P}_{\beta, \gamma}^{(0)}$ which may be useful for obtaining more explicit

composition formulae (this investigation was continued in [Ga]). He also proved an analogue of (4.3) in the case $\kappa = 1/2$ and $\tau = 0$ [S1, Theorem 6.1].

Remark 7.1. From (7.2) it easily follows that the degrees of the polynomials $\tilde{P}_{\beta,\gamma}^{(\kappa)}$ admit the same estimates as $d_{\beta,\gamma}^{(\kappa)}$ (see Section 4).

7.2. Operators on sections of vector bundles

In [FK, Pf2, S2, Wi2] the authors considered PDOs acting between spaces of sections of vector bundles over M . In this case, in order to construct a global symbolic calculus, it is sufficient to define parallel displacement and horizontal curves in the induced bundles over T^*M . This can be achieved by introducing linear connections on M and the vector bundles over M . After that the results are stated and proved in the same way as in the scalar case (further details and references can be found in the above papers).

A more radical approach was proposed by Cyril Levy in [Le]. He noticed that in order to develop an intrinsic calculus of PDOs one actually needs only an exponential map, which does not have to be associated with a linear connection. In his paper Levy assumed that the manifold M is noncompact and is provided with a global exponential map (that is, M is a manifold with linearization in the sense of [Bo]). He then defined associated maps in the induced vectors bundles and constructed a global coordinate-free symbolic calculus.

Remark 7.2. All the papers mentioned in this subsection dealt only with symbols whose restriction to compact subsets of M belong to $S_{\rho,\delta}^m$ with $\rho > \max\{\delta, 1-\rho\}$. It should be possible to extend their results to $\rho < 1/2$, using the technique outlined in Section 4.

7.3. Noncompact manifolds

In order to study global properties of PDOs on a noncompact manifold M , one has to assume that all estimates for symbols and their derivatives hold uniformly for all $x \in M$ (rather than only on compact subsets of M , as in Definitions 2.1 and 4.1). In [Ba], Frank Baldus defined classes of symbols and developed an intrinsic calculus of PDOs on a noncompact manifold M under the assumption that M has an atlas satisfying certain global conditions. The statements and proofs in [Ba] were given in terms of local coordinates, and global results were obtained by considering the transition maps between coordinates charts. It is quite possible that these results can be simplified or/and improved under the assumption that M has a global exponential map (as in [Le]).

7.4. Other symbol classes

The paper [Ba] dealt with the more general classes of symbols $S(m, g)$ instead of $S_{\rho,\delta}^m$. The classes $S(m, g)$ were introduced by L. Hörmander in [H1] (see also [H2]). They are defined with the use of coordinates, and in each coordinate system $S_{\rho,\delta}^m$ is a particular case of $S(m, g)$. It would be interesting to construct similar classes $S(m, g)$ associated with a linear connection (or an exponential map) and to study the corresponding classes of symbols and PDOs.

Remark 7.3. Note that the introduction of “coordinate” classes $S(m, g)$ does not help to resolve Problem 2.6. The relation between these “coordinate” classes and the classes $S_{\rho, \delta}^m(\Gamma)$ was discussed in [Sa2, Remark 3.5].

7.5. Operators generated by vector fields

Let $\nu := \{\nu_1, \nu_2, \dots, \nu_n\}$ be a family of smooth vector fields ν_j on M which span $T_x M$ at every point $x \in M$. Consider the corresponding first order differential operators ∂_{ν_j} and denote

$$\partial_\nu^\alpha := \frac{1}{q!} \sum_{j_1, \dots, j_q} \partial_{\nu_{j_1}} \partial_{\nu_{j_2}} \dots \partial_{\nu_{j_q}}$$

where $q = |\alpha|$ and the sum is taken over all ordered sets of indices j_1, \dots, j_q corresponding to the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$. In other words, ∂_ν^α can be thought of as the symmetrized composition of $\partial_{\nu_{j_k}}$.

The family ν generates a unique curvature-free connection Γ_ν , with respect to which all covariant derivatives of the vector fields ν_j are identically equal to zero. The Γ_ν -symbol of ∂_ν^α coincides with $\sigma_1^{\alpha_1} \dots \sigma_n^{\alpha_n}$, where $\sigma_k = \sigma_k(x, \xi) := \langle \nu_k, \xi \rangle$ (see [Sa2, Example 5.4]). Since the functions σ_k are constant along horizontal curves in T^*M generated by the connection Γ_ν , the operators ∂_ν^α and their linear combinations can be regarded as constant coefficient operators relative to the connection Γ_ν (or to the family of the vector fields ν).

This observation was used by Eugene Shargorodsky in [Sha], where he developed a complete theory of pseudodifferential operators generated by a family of vector fields ν . He introduced anisotropic analogues of classes $S_{\rho, \delta}^m$, proved the composition formula for the corresponding classes of PDOs, defined semi-elliptic operators associated with the family ν , and constructed their resolvents. All the results in [Sha] were obtained for operators acting on sections of vector bundles equipped with linear connections (see Section 7.2).

7.6. Operators on Lie groups

In [RT], the authors defined full symbols of scalar PDOs on a compact Lie group M in terms of its irreducible representations and developed a calculus for such symbols. It would be interesting to compare their formulae with those obtained by introducing an invariant linear connection Γ on M and applying the methods of [Sa2] or [Sha].

7.7. Geometric aspects and physical applications

The importance of intrinsic approach in the theory of PDOs for quantum mechanics is explained in the excellent review [Fu] by Stephen Fulling. Further discussions can be found in the Ph.D. thesis [Gu]. Various geometric applications are considered in [BNPW] and [Vo]. We refer the interested reader to the above papers and references therein.

7.8. Global phase functions

It is worth noticing that one does not need a linear connection or even an exponential map to define PDOs on a manifold in a coordinate-free manner. It is sufficient to fix a globally defined phase function satisfying certain conditions.

Namely, let $\varphi(x; y, \eta)$ be an infinitely differentiable function on $M \times T^*M$ such that

$$\operatorname{Im} \varphi(x; y, \eta) \geq 0, \quad \varphi(x; y, \lambda \eta) = \lambda \varphi(x; y, \eta)$$

for all $x \in M$, $(y, \eta) \in T^*M$ and $\lambda > 0$, and

$$\varphi(x; y, \eta) = (x - y) \cdot \eta + O(|x - y|^2 |\eta|), \quad x \rightarrow y,$$

in any local coordinate system. If $a(x; y, \eta)$ is a smooth function on $M \times T^*M$ such that $a \in S_{1,0}^m$ in any local coordinate system then

$$\mathcal{A}(x, y) := \int e^{i\varphi(x; y, \eta)} a(x; y, \eta) d\eta$$

is the Schwartz kernel of a PDO $A \in \Psi_{1,0}^m$ acting in the space of functions. Moreover, there exists an amplitude $a_\varphi(y, \eta)$ independent of x such that

$$\mathcal{A}(x, y) - \int e^{i\varphi(x; y, \eta)} a_\varphi(y, \eta) d\eta \in C^\infty(M \times M),$$

and this amplitude a_φ is uniquely defined by A modulo $S^{-\infty}$. The operator A belongs to $\Psi_{1,0}^m$ if and only if $a_\varphi \in S_{1,0}^m$ in any local coordinate system.

Remark 7.4. For a real-valued phase function φ these are standard results of the theory of Fourier integral operators (see, for instance, [Shu, Section 19]). Complex-valued phase functions were considered in [LSV].

It is natural to call a_φ the φ -symbol of the operator A . Clearly, all the standard results of the classical theory of PDOs can be rewritten in terms of their φ -symbols. In particular, if $A, B \in \Psi_{1,0}^m$ then the φ -symbol of the composition AB is determined modulo $S^{-\infty}$ by an asymptotic series which involves φ -symbols of A and B and their derivatives. Similarly, the φ -symbol of the adjoint operator A^* is given by a series involving the derivatives of φ -symbol of A .

Obviously, the same formulae remain valid under milder assumptions about the symbols. Thus it should be possible to introduce symbol classes associated with the phase function φ and develop a symbolic calculus in these classes (as was done in [Sa2] for the special phase function φ_τ generated by a linear connection).

Such a general approach may allow one to extend results of Section 6 to other elliptic operators. It may also be useful for the study of solutions of hypoelliptic equations and operators on noncompact manifolds.

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P. McKeag
e-mail: petermckeag@gmail.com

Y. Safarov
Department of Mathematics
King's College London
Strand
London, UK
e-mail: yuri.safarov@kcl.ac.uk