On Global Representation of Lagrangian Distributions and Solutions of Hyperbolic Equations

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Abstract

In this paper we develop a new approach to the theory of Fourier integral operators. It allows us to represent the Schwartz kernel of a Fourier integral operator by one oscillatory integral with a complex phase function. We consider Fourier integral operators associated with canonical transformations, having in mind applications to hyperbolic equations. As a by-product we obtain yet another formula for the Maslov index. © 1994 John Wiley & Sons, Inc.

0. Introduction

Let M be a C^∞-manifold without boundary, $\dim M = n$, and $T^*M\backslash 0$ be the cotangent bundle without the zero section. We consider the Lagrangian manifold

$$\Lambda \subset (T^*M\backslash 0) \times (T^*M\backslash 0)$$

generated by a smooth homogeneous canonical transformation $G : T^*M\backslash 0 \to T^*M\backslash 0$. Let

$$\varphi(x,y,\zeta) \in C^\infty\big(M \times M \times (\mathbf{R}^N\backslash 0)\big)$$

be a complex-valued smooth homogeneous function of degree 1 with non-negative imaginary part, and

$$\Sigma_\varphi \;=\; \{\,(x,y,\zeta)\,:\,\varphi_\zeta(x,y,\zeta)\,=\,0\,\}\;.$$

We say that φ locally parametrizes the Lagrangian manifold Λ if in some neighborhood of a given point of Λ and in some local coordinates x, y we have

$$\Lambda \;=\; \{\;(x,\varphi_x(x,y,\zeta))\,,\;(y,\varphi_y(x,y,\zeta))\;:\;(x,y,\zeta)\in\Sigma_\varphi\;\}\;.$$

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Functions parametrizing Lagrangian manifolds are usually called phase functions. A phase function is said to be non-degenerate if the differentials $d(\varphi_{\zeta_1}),\dots,d(\varphi_{\zeta_N})$ are linearly independent on Σ_φ.

Denote by $S^m\big(M\times M\times(\mathbf{R}^N\backslash 0)\big)$ the class of smooth functions $p(x,y,\zeta)$ defined on $M\times M\times(\mathbf{R}^N\backslash 0)$ which admit the asymptotic expansion

$$p(x,y,\zeta)\ \sim\ \sum_{j=0}^{\infty}p_{m-j}(x,y,\zeta)$$

with $p_{m-j}(x,y,\zeta)$ homogeneous in ζ of degree $m-j$. The integral

$$\int e^{i\varphi(x,y,\zeta)}p(x,y,\zeta)\,d\zeta \tag{0.1}$$

with a non-degenerate phase function φ and $p\in S^m\big(M\times M\times(\mathbf{R}^N\backslash 0)\big)$ is called an oscillatory integral with amplitude p. This integral does not converge absolutely but it is interpreted as a distribution on $M\times M$; see, for example, L. Hörmander, [5], [6], and F. Treves, [14]. A distribution which can be represented locally as a finite sum of oscillatory integrals (0.1) with real phase functions locally parametrizing the Lagrangian manifold Λ is called a Lagrangian distribution associated with Λ (or with the corresponding canonical transformation G). An operator, whose Schwartz kernel is a Lagrangian distribution, is said to be a Fourier integral operator.

Analogous notions are introduced when G depends on an additional "time" parameter t, for example, when G is a Hamiltonian flow. Then

$$\Lambda\ \subset\ T^*\mathbf{R}^1\times(T^*M\backslash 0)\times(T^*M\backslash 0)$$

and all the functions and distributions depend also on t. The most common example of a Fourier integral operator depending on t is the operator $\exp(-itA)$ where A is a first-order elliptic pseudodifferential operator on M. In this case G is the Hamiltonian flow generated by the principal symbol of the pseudodifferential operator A.

It was observed in [10] that Lagrangian manifolds in general do not allow a global parametrization by one real phase function φ. One of the obstacles of its global parametrization is the non-triviality of some cohomology class which is usually called the Maslov class. Besides, in the non-stationary case, this fact is motivated by the presence of the so-called caustics. It might be one of the reasons why the classical global theory of Fourier integral operators, developed in [5], was based on the study of local oscillatory integrals (0.1). It leads, nevertheless, to some global objects such as the Maslov index, the Keller-Maslov bundle, etc.

The main purpose of this paper is to propose another approach to Fourier integral operators. We find it to be simpler, and develop it when Λ is generated by a homogeneous canonical transformation. We apply this approach (instead of the classical one) to the study of the asymptotic distribution of eigenvalues in [13].

In this case we prove that the Lagrangian manifold can be parametrized by a *global complex* phase function. This allows us to represent a Lagrangian distribution by only one oscillatory integral with a global complex phase function φ; see Sections 1 and 3. We prove its invariance, with the phase variables $\eta \in T^*_y M\backslash 0$ instead of $\zeta \in \mathbf{R}^N$ with some N. (Note that here the number of phase variables $N = n$ is the least possible.) In Section 1 we first introduce a class of non-degenerate complex phase functions φ globally parametrizing the Lagrangian manifold, and then study how the amplitude p depends on the choice of φ. It leads us to some new definitions of well-known global geometric objects. In particular, in Section 2 we obtain a definition of the Maslov class and the Maslov index by means of de Rham cohomology; see [4] for various definitions of the Maslov index.

In Proposition 2.3 we introduce an integer-valued function Θ related to caustics. It is also an invariant of the Lagrangian manifold Λ, which allows us to give another definition of the Maslov index. This definition has the advantage of being suitable for arbitrary (not necessarily closed) curves. An analogous approach has been developed by V. Arnol'd (see [1]) for generic Lagrangian manifolds.

In Section 3 we prove two theorems on the asymptotics of Fourier transforms clarifying the connection between Θ and the properties of the Lagrangian distribution. In fact, for a Riemannian manifold and the geodesic flow G the value of $-\Theta$ along a geodesic curve coincides with its Morse index; see Section 4.2.

As a corollary, we obtain that the Schwartz kernel of the operator $\exp(-itA)$ can be represented by only one oscillatory integral. In Section 4 we give an independent proof of this result for those readers who are not familiar with the theory of Lagrangian distributions.

This paper is a recast and extended version of the preprint [9]. Note that complex phase functions have been applied to different problems earlier; see, for example, [11], [12], and [3]. The main idea of this paper is based on the use of such phase functions for the study of global properties of Lagrangian distributions.

1. Time-Independent Distributions

1.1. Global Phase Functions

Let G be a smooth homogeneous canonical transformation in the cotangent bundle $T^*M\backslash 0$. For $(y,\eta) \in T^*M\backslash 0$ let us denote

$$G(y,\eta) \;=\; (x^*(y,\eta), \xi^*(y,\eta)) \;.$$

Then $G(y,\lambda\eta) = (x^*(y,\eta), \lambda\xi^*(y,\eta))$ for all $\lambda > 0$. We consider the Lagrangian manifold

$$\Lambda \;=\; \{(x,\xi),(y,-\eta) : (x,\xi) = G(y,\eta)\} \subset (T^*M\backslash 0) \times (T^*M\backslash 0) \;.$$

It is clear that Λ is naturally parametrized by $(y,\eta)\in T^*M\backslash 0$, and this allows us to identify all objects (functions, cohomology classes, etc.) defined on Λ with those on $T^*M\backslash 0$.

A complex homogeneous function of degree 1

$$\varphi(x;y,\eta)\in C^\infty(M\times(T^*M\backslash 0))$$

such that $\operatorname{Im}\varphi\geqq 0$ is said to be a *phase function.* We shall always assume that $\operatorname{Im}\varphi(x;y,\eta)>0$ for x lying outside a small neighborhood of the point $x^*(y,\eta)$.

Denote by $\mathcal{F}$ the class of phase functions φ satisfying the following three conditions:

$$\varphi\big(x^*(y,\eta);y,\eta\big) \;=\; 0\;, \tag{1.1}$$

$$\varphi_x\big(x^*(y,\eta);y,\eta\big) \;=\; \xi^*(y,\eta)\;, \tag{1.2}$$

$$\det\partial_{x\eta}\varphi\big(x^*(y,\eta);y,\eta\big)\neq\;0\;. \tag{1.3}$$

Remark 1.1. The condition (1.3) is invariant. Indeed, when we change the coordinates $x\to\tilde{x}$ and $y\to\tilde{y}$ we obtain

$$\partial_{\tilde{\eta}\tilde{x}}\varphi \;=\; (\partial\tilde{y}/\partial y)\cdot\partial_{\eta x}\varphi\cdot(\partial\tilde{x}/\partial x)^{-1}$$

and therefore $\det\partial_{\tilde{\eta}\tilde{x}}\varphi$ is not equal to zero for $\tilde{x}=\tilde{x}^*$.

LEMMA 1.2. *Any phase function φ satisfying the conditions* (1.1)–(1.3) *gives a global parametrization of the Lagrangian manifold Λ.*

Proof: Let us differentiate with respect to η the identity (1.1). In view of (1.2) we obtain

$$\varphi_{\eta_k}\big(x^*(y,\eta);y,\eta\big) \;+\; \xi^*(y,\eta)\cdot x^*_{\eta_k}(y,\eta) \;=\; 0\;.$$

Since the transformation G preserves the canonical 1-form $\xi\cdot dx$ we have

$$\xi^*\cdot x^*_{\eta_k} \;=\; 0\;, \qquad \xi^*\cdot x^*_{y_k} \;=\; \eta_k\;. \tag{1.4}$$

Therefore

$$\varphi_\eta(x;y,\eta) \;=\; 0 \tag{1.5}$$

for $x=x^*(y,\eta)$. On the other hand, the Euler identity

$$\eta\cdot\varphi_\eta(x;y,\eta) \;=\; \varphi(x;y,\eta)$$

implies that $\varphi(x;y,\eta)=0$ if $\varphi_\eta(x;y,\eta)=0$. So $\varphi_\eta(x;y,\eta)$ can be equal to zero only if x is sufficiently close to $x^*(y,\eta)$. In view of (1.3) in a small neighborhood of the point $x^*(y,\eta)$ the equation (1.5) may have only one solution with respect to x. Therefore the equation (1.5) has the only one global solution $x=x^*(y,\eta)$. By analogy, differentiating (1.1) with respect to y and taking into account (1.2), (1.4), we obtain

$$\varphi_y\big(x^*(y,\eta);y,\eta\big) \;=\; -\eta\;.$$

This completes the proof.

Denote

$$\begin{aligned}
\Phi_{\eta\eta} &= \Phi_{\eta\eta}(y,\eta) = \partial_{\eta\eta}\varphi\big|_{x=x^*}\;,\\
\Phi_{xx} &= \Phi_{xx}(y,\eta) = \partial_{xx}\varphi\big|_{x=x^*}\;,\\
\Phi_{x\eta} &= \Phi_{x\eta}(y,\eta) = \partial_{x\eta}\varphi\big|_{x=x^*}\;,
\end{aligned}$$

$$\Phi_{\eta x} \;=\; \Phi_{\eta x}(y,\eta) \;=\; \Phi^T_{x\eta}(y,\eta)\;. \tag{1.6}$$

The condition (1.3) is equivalent to the fact that the matrix $\Phi_{x\eta}(y,\eta)$ (or $\Phi_{\eta x}(y,\eta)$) is non-degenerate for all (y,η).

Remark 1.3. In view of (1.1) and (1.5) the symmetric matrix $\Phi_{\eta\eta}$ behaves as a tensor. Changing the coordinates $y\to\tilde y$ we obtain

$$\Phi_{\tilde\eta\tilde\eta} \;=\; (\partial\tilde y/\partial y)\cdot\Phi_{\eta\eta}\cdot(\partial\tilde y/\partial y)^T\big|_{x=x^*}\;. \tag{1.7}$$

By analogy, since the function $\operatorname{Im}\varphi(x;y,\eta)$ of the variables x has a second-order zero at the point $x=x^*(y,\eta)$, the imaginary part $\operatorname{Im}\Phi_{xx}$ of the matrix Φ_{xx} is a tensor over the point x^*. This fact together with $\operatorname{Im}\varphi(x;y,\eta)\geqq 0$ implies also

$$\operatorname{Im}\Phi_{xx} \;\geqq\; 0\;. \tag{1.8}$$

On account of the conditions (1.1) and (1.2), in any coordinate system

$$\begin{aligned}
\varphi(x;y,\eta) \;=\; (x-x^*)\cdot\xi^* \;+\; \frac{1}{2}\,\Phi_{xx}(x-x^*)\cdot(x-x^*)&\\
+\;O(|x-x^*|^3|\eta|)\;,\qquad x\to x^*\;,\;|\eta|\to\infty\;,&
\end{aligned} \tag{1.9}$$

where $O(|x-x^*|^3|\eta|)$ is homogeneous with respect to η of degree one. Differentiating the identity (1.9) with respect to x and η and taking into account (1.4), we obtain

$$\Phi_{x\eta}=\xi^*_\eta \;-\; \Phi_{xx}\cdot x^*_\eta\;,\qquad \Phi_{\eta\eta} \;=\; -(x^*_\eta)^T\cdot\xi^*_\eta \;+\; (x^*_\eta)^T\cdot\Phi_{xx}\cdot x^*_\eta\;. \tag{1.10}$$

As the matrix $\Phi_{\eta\eta}$ is symmetric we also have

$$\Phi_{\eta\eta} \;=\; -(\xi^*_\eta)^T \cdot x^*_\eta \;+\; (x^*_\eta)^T \cdot \Phi_{xx} \cdot x^*_\eta \;.$$

Consequently

$$\Phi_{\eta\eta} \;=\; -(x^*_\eta)^T \cdot \Phi_{x\eta} \;=\; -\Phi_{\eta x} \cdot x^*_\eta \;. \tag{1.10'}$$

With account of (1.3) this gives $\ker \Phi_{\eta\eta} = \ker x^*_\eta$. Moreover, (1.8) and (1.10) yield $\operatorname{Im} \Phi_{\eta\eta} \geqq 0$.

The matrices x^*_η and ξ^*_η will be used very often later on. Since the transformation G preserves the canonical 2-form $dx \wedge d\xi$ we have

$$(\xi^*_\eta)^T \cdot x^*_\eta \;-\; (x^*_\eta)^T \cdot \xi^*_\eta \;=\; 0 \tag{1.11}$$

(in fact this also follows from (1.10) and the symmetry of $\Phi_{\eta\eta}$). Changing the coordinates $x \to \tilde{x}$ and $y \to \tilde{y}$ we have

$$\tilde{x}^*_{\tilde{\eta}} \;=\; (\partial \tilde{x}/\partial x) \cdot x^*_\eta \cdot (\partial \tilde{y}/\partial y)^T \;,$$

i.e., x^*_η behaves as a tensor. The matrix ξ^*_η also behaves as a tensor with respect to y. This is not true, however, with respect to x. Indeed, passing from coordinates x to $\tilde{x}$ we obtain

$$\tilde{\xi}^*(y,\eta) \;=\; (\partial x/\partial \tilde{x})^T \,\Big|_{\tilde{x}=\tilde{x}^*} \xi^*(y,\eta) \;.$$

Differentiating this identity with respect to η we see that

$$(\partial x/\partial \tilde{x})^T \Big|_{\tilde{x}=\tilde{x}^*} \cdot \xi^*_\eta(y,\eta) \;=\; \tilde{\xi}^*_\eta(y,\eta) \;-\; C(y,\eta) \cdot \tilde{x}^*_\eta(y,\eta) \;, \tag{1.12}$$

where $C = \{C_{ij}\}$ is the symmetric matrix-function with elements

$$C_{ij}(y,\eta) \;=\; \sum_k \left(\frac{\partial^2 x_k}{\partial \tilde{x}_i \partial \tilde{x}_j} \right)\Bigg|_{\tilde{x}=\tilde{x}^*} \xi^*_k(y,\eta) \;.$$

Here $\partial^2 x_k / \partial \tilde{x}_i \partial \tilde{x}_j$ are the second Taylor coefficients of $x_k(\tilde{x})$ at the point $\tilde{x} = \tilde{x}^*$ which for fixed (y,η) can be chosen arbitrarily. Thus given coordinates $\tilde{x}$, an arbitrary real symmetric matrix C_0 and a fixed point (y,η) we can find coordinates x such that in (1.12) $C(y,\eta) = C_0$.

1.2. Existence of Global Phase Functions

To demonstrate the existence of phase functions satisfying the conditions (1.1)–(1.3) we shall prove the following lemma, which gives a natural example of the function φ.

Let us introduce on M a Riemannian metric g. For sufficiently close points $x \in M$, $y \in M$ let $\gamma_{y,x}(s)$ be the "shortest" geodesic connecting y and x, i.e., the geodesic defined in the normal system of coordinates with origin at y. We shall choose the parameter s such that $\gamma_{y,x}(0) = y$, $\gamma_{y,x}(1) = x$. We denote $\nu(y,x) = \dot\gamma_{y,x}(0) \in T_yM$, and smoothly extend ν onto $M \times M$. Let $b(x;y,\eta)$ be a smooth function positively homogeneous in η of degree 1 such that

$$b(x;y,\eta) \;=\; O(|x-x^*|^2|\eta|) \,, \qquad x \to x^* \,, \quad |\eta| \to \infty \,,$$

$$\operatorname{Im} B_{xx}(y,\eta) \;>\; 0 \,, \qquad B_{xx} \;=\; \partial_{xx} b\big|_{x=x^*} \,,$$

and

$$\operatorname{Im} b(x;y,\eta) \;>\; 0$$

for $x \neq x^*$.

Lemma 1.4. *The phase function*

$$\varphi(x;y,\eta) \;=\; \nu(x^*,x)\cdot\xi^* \;+\; b(x;y,\eta)$$

satisfies the conditions (1.1)–(1.3).

Proof: The conditions (1.1) and (1.2) are obviously fulfilled. Thus it remains to verify (1.3). Let us choose a local coordinate system. Then, in view of (1.10),

$$\operatorname{Re}\Phi_{x\eta} \;=\; \xi^*_\eta \;-\; \operatorname{Re}\Phi_{xx}\cdot x^*_\eta \,, \qquad \operatorname{Im}\Phi_{x\eta} \;=\; -\operatorname{Im}\Phi_{xx}\cdot x^*_\eta \,,$$

where

$$\operatorname{Im}\Phi_{xx} \;=\; \operatorname{Im} B_{xx}(y,\eta) \;>\; 0 \,.$$

By (1.10′) $\operatorname{Re}\Phi^T_{x\eta}\cdot x^*_\eta - (x^*_\eta)^T\cdot\operatorname{Re}\Phi_{x\eta} \;=\; 0$. This formula and formula $\operatorname{Im}\Phi_{x\eta} = -\operatorname{Im}\Phi_{xx}\cdot x^*_\eta$ imply

$$\begin{aligned}(1.13)\quad &(\operatorname{Re}\Phi^T_{x\eta} \;-\; i\operatorname{Im}\Phi^T_{x\eta})\cdot\operatorname{Im}\Phi^{-1}_{xx}\cdot(\operatorname{Re}\Phi_{x\eta} \;+\; i\operatorname{Im}\Phi_{x\eta})\\ &\;=\; \operatorname{Re}\Phi^T_{x\eta}\cdot\operatorname{Im}\Phi^{-1}_{xx}\cdot\operatorname{Re}\Phi_{x\eta} + \operatorname{Im}\Phi^T_{x\eta}\cdot\operatorname{Im}\Phi^{-1}_{xx}\cdot\operatorname{Im}\Phi_{x\eta} \,.\end{aligned}$$

The real matrix on the right-hand side of (1.13) is non-negative, and we obtain for any vector $\vec c$ from its kernel

$$x^*_\eta\,\vec c \;=\; 0 \,, \qquad \xi^*_\eta\,\vec c \;=\; 0 \,.$$

But since the transformation G is non-degenerate then $\vec c = 0$. This completes the proof.

Example 1.1. Given a Riemannian metric g on M we can take

$$\varphi(x;y,\eta) \;=\; \nu(x^*,x)\cdot\xi^* \;+\; \frac{i}{2}\,|\nu(x^*,x)|^2\,|\eta|$$

for x sufficiently close to x^* (for x far from x^* we can take an arbitrary smooth extension with $\operatorname{Im}\varphi > 0$). In particular, when $M = \mathbf{R}^n$ with Euclidian metric the conditions (1.1)–(1.3) are fulfilled for the phase function

$$\varphi(x;y,\eta) \;=\; (x-x^*)\cdot\xi^* \;+\; \frac{i}{2}\,|x-x^*|^2\,|\eta|\ .$$

We shall see below (Section 2.4) that in general there is no real phase function satisfying the conditions (1.1)–(1.3) globally. One can, however, always find a phase function which is real in a given small neighborhood.

Proposition 1.5. *Let (y_0,η_0) be a fixed point from $T^*M\backslash 0$, and $x_0 = x^*(y_0,\eta_0)$. Then in a neighborhood of the point x_0 there exists a coordinate system x such that*

$$\det \xi^*_\eta(y_0,\eta_0) \;\neq\; 0\ . \tag{1.14}$$

Proof: Let $\tilde x$ be arbitrary coordinates in a neighborhood of the point x_0. In view of (1.11)

$$\tilde x^*_\eta(y_0,\eta_0)\ :\ \ker \tilde\xi^*_\eta(y_0,\eta_0)\ \longrightarrow\ \ker\left(\tilde\xi^*_\eta(y_0,\eta_0)\right)^T\ ,$$

and since the transformation G is non-degenerate, the rank of this map is maximal. Let C^0 be the orthogonal projection on $\ker\left(\tilde\xi^*_\eta(y_0,\eta_0)\right)^T$. Then $C^0\cdot\tilde\xi^*_\eta(y_0,\eta_0)=0$ and therefore

$$\begin{aligned}\left(\tilde\xi^*_\eta(y_0,\eta_0)\ -\ C^0\cdot\tilde x^*_\eta(y_0,\eta_0)\right)^T\cdot\left(\tilde\xi^*_\eta(y_0,\eta_0)\ -\ C^0\cdot\tilde x^*_\eta(y_0,\eta_0)\right)&\\ =\ \left(\tilde\xi^*_\eta(y_0,\eta_0)\right)^T\cdot\tilde\xi^*_\eta(y_0,\eta_0)\ +\ \left(\tilde x^*_\eta(y_0,\eta_0)\right)^T\cdot C^0\cdot\tilde x^*_\eta(y_0,\eta_0)\ .&\end{aligned}$$

The matrix on the right-hand side of this equality is strictly positive, consequently the matrix

$$\tilde\xi^*_\eta(y_0,\eta_0)\ -\ C^0\cdot\tilde x^*_\eta(y_0,\eta_0)$$

is non-degenerate. Choosing now coordinates x such that

$$\sum_k \left.\left(\frac{\partial^2 x_k}{\partial\tilde x_i\partial\tilde x_j}\right)\right|_{\tilde x=x_0}\xi^*_k(y_0,\eta_0)\ =\ C^0_{ij}(y_0,\eta_0)$$

(see (1.12)) we obtain (1.14). The proof is complete.

Obviously, if (1.14) is fulfilled then in a neighborhood of the point $(x_0;y_0,\eta_0)$ the real phase function

$$(x-x^*)\cdot\xi^* \tag{1.15}$$

satisfies the conditions (1.1)–(1.3), and thus it gives a local parametrization of the Lagrangian manifold Λ.

We shall need the following two lemmas. The first one implies that a phase function satisfying the conditions (1.1)–(1.3) locally can always be extended up to a phase function satisfying the conditions (1.1)–(1.3) globally. The second lemma states that the class $\mathcal{F}$ is connected.

LEMMA 1.6. *Let $\mathbf{V}$ be a closed conic subset of $T^*M\backslash 0$ and let $\mathbf{W}=\{(x;y,\eta): (y,\eta)\in\mathbf{V},\ x=x^*(y,\eta)\}\subset M\times(T^*M\backslash 0)$. Let φ be a phase function satisfying* (1.3) *on $\mathbf{V}$, and* (1.1), (1.2) *on some* (*open*) *conic neighborhood $\mathcal{V}\subset T^*M\backslash 0$ of the set $\mathbf{V}$. Then there exist a phase function $\varphi_0\in\mathcal{F}$ and a conic neighborhood $\mathcal{W}\subset M\times(T^*M\backslash 0)$ of the set $\mathbf{W}$ such that $\varphi_0=\varphi$ on $\mathcal{W}$.*

Proof: Let $\psi(x;y,\eta)$ be an arbitrary phase function from the class $\mathcal{F}$ with $\operatorname{Im}\Psi_{xx}>0$, $\Psi_{xx}=\partial_{xx}\psi|_{x=x^*}$, and with $\operatorname{Im}\psi>0$ for $x\neq x^*$; such a phase function exists due to Lemma 1.4. Let us choose two small neighborhoods $\mathcal{W}\subset\tilde{\mathcal{W}}\subset M\times(T^*M\backslash 0)$ of the set $\mathbf{W}$ and a real-valued function $\rho(x;y,\eta)\in C^\infty(M\times(T^*M\backslash 0))$ positively homogeneous in η of degree 0 such that $0\leqq\rho(x;y,\eta)\leqq 1$ on $M\times(T^*M\backslash 0)$, $\rho=0$ on $\mathcal{W}$, and $\operatorname{supp}(1-\rho)\subset\tilde{\mathcal{W}}$. Set

$$\varphi_0(x;y,\eta) \;=\; \big(1-\rho(x;y,\eta)\big)\,\varphi(x;y,\eta) \;+\; \rho(x;y,\eta)\,\psi(x;y,\eta)\ .$$

It is easy to check, repeating the arguments from the proof of Lemma 1.4, that the constructed phase function φ_0 satisfies the requirements of Lemma 1.6 if $\tilde{\mathcal{W}}$ is a sufficiently small neighborhood of the set $\mathbf{W}$. This completes the proof.

LEMMA 1.7. *Any phase function $\varphi_0\in\mathcal{F}$ can be continuously transformed in the class $\mathcal{F}$ into any other phase function $\varphi_1\in\mathcal{F}$.*

Proof: For $0\leqq s\leqq 1$ set

$$\varphi_s(x;y,\eta) \;=\; (1-s)\,\varphi_0(x;y,\eta) \;+\; s\,\varphi_1(x;y,\eta) \;+\; s\,(1-s)\,b(x;y,\eta)\ , \tag{1.16}$$

where $b(x;y,\eta)$ is as in Lemma 1.4. By (1.8) and Lemma 1.4 φ_s satisfy (1.1)–(1.3) for all $s\in[0,1]$. The proof is complete.

1.3. Global Oscillatory Integrals

Let us introduce a "function" $d_\varphi\in C^\infty(M\times T^*M\backslash 0)$ homogeneous in η of degree zero such that

$$|d_\varphi| = \sqrt{|\det\partial_{x\eta}\varphi|} \tag{1.17}$$

for x close to x^*. Under change of coordinates $\sqrt{|\det \partial_{x\eta}\varphi|}$ behaves as a $\frac{1}{2}$-density with respect to x and as a $(-\frac{1}{2})$-density with respect to y, and we assume that $|d_\varphi|$ has the same property.

Lemma 1.2 immediately implies the following

THEOREM 1.8. *Let $u(x,y)$ be a Lagrangian distribution associated with the Lagrangian manifold Λ. Then for any phase function $\varphi \in \mathcal{F}$ there exists an amplitude $p(x;y,\eta)$ such that*

$$u(x,y) \;=\; (2\pi)^{-n} \int e^{i\varphi(x;y,\eta)}\, p(x;y,\eta)\, |d_\varphi(x;y,\eta)|\, d\eta \tag{1.18}$$

modulo a smooth half-density.

Remark 1.9. Usually Lagrangian distributions are supposed to be half-densities. When we consider p in (1.18) as a function on $M \times (T^*M\backslash 0)$ the properties of $|d_\varphi|$ yield that the integral $\int e^{i\varphi} p\; |d_\varphi|\, d\eta$ behaves precisely as a half-density in x and in y.

Proof of Theorem 1.8: By the condition (1.3) the phase function φ is non-degenerate. Therefore according to Theorem 25.4.7 from [8], a (local) oscillatory integral with an arbitrary phase function locally parametrizing Λ modulo a smooth function is equal to a (local) oscillatory integral with phase function φ. Since a Lagrangian distribution is a locally finite sum of such oscillatory integrals, the same is true for the whole Lagrangian distribution u with some global amplitude p.

DEFINITION 1.1. The Lagrangian distribution u and the oscillatory integral in (1.18) are said to be of order m if the amplitude $p \in S^m$.

LEMMA 1.10. *The oscillatory integral* (1.18) *of order m can be written in the form*

$$\begin{aligned} (2\pi)^{-n} \int e^{i\varphi}\, p(x;y,\eta)\, |d_\varphi(x;y,\eta)|\, d\eta &= (2\pi)^{-n} \int e^{i\varphi}\, p(x^*;y,\eta)\, |d_\varphi(x;y,\eta)|\, d\eta \\ &\quad + (2\pi)^{-n} \int e^{i\varphi}\, \tilde p(x;y,\eta)\, |d_\varphi(x;y,\eta)|\, d\eta \end{aligned} \tag{1.19}$$

with $\tilde p \in S^{m-1}$. If in a local coordinate system for x close to x^* we have

$$p(x;y,\eta) \;-\; p(x^*;y,\eta) \;=\; (x-x^*)\cdot r(x;y,\eta)\;, \qquad r \in S^m\;, \tag{1.20}$$

then in these coordinates

$$\tilde p(x;y,\eta) \;=\; L(x;y,\eta,D_\eta)\, r(x;y,\eta)\ ,$$

where L is a first-order differential operator such that

$$L(x;y,\lambda\eta,\lambda^{-1}D_\eta) \;=\; \lambda^{-1}L(x;y,\eta,D_\eta)\ , \qquad \lambda>0\ .$$

Proof: Without loss of generality we may assume that the amplitude p has a small support with respect to x. If p is equal to zero in a neighborhood of the set $\{x=x^*\}$, then $\operatorname{Im}\varphi(x;y,\eta)>0$ on the support of p, and (1.18) is a smooth half-density. Therefore we may assume also that x is sufficiently close to x^* and that (1.20) holds.

Let us consider the oscillatory integral with amplitude $(x-x^*)\cdot r$. We can replace $(x-x^*)e^{i\varphi}$ by $B^{-1}\nabla_\eta(e^{i\varphi})$, where $B=B(x;y,\eta)$ is a homogeneous non-degenerate matrix of degree zero, and $B(x^*;y,\eta)=-i\,\Phi_{x\eta}(y,\eta)$. Now, integrating by parts with respect to η we obtain an oscillatory integral with the same phase function and amplitude

$$\tilde p \;=\; |d_\varphi|^{-1}\mathrm{div}_\eta\big(|d_\varphi|\,(B^{-1})^T r\big) \;\in\; S^{m-1}\ .$$

The proof is complete.

Remark 1.11. Lemma 1.10 implies that if $p(x^*;y,\eta)=0$ then we can decrease the order of the amplitude by 1. If the amplitude $p(x;y,\eta)$ has a zero of the order $2N-1$ or $2N$ at $x=x^*$, however, then generally speaking, the order of the amplitude can be decreased by N. This happens because in the process of integrating by parts we differentiate the remaining factors $(x-x^*)$ with respect to η. (In the special case $x^*\equiv y$ these derivatives are equal to zero, and in this case the order of the amplitude can be decreased exactly by the order of the zero of p at $x=x^*$.)

Iterating formula (1.19) we obtain the following

Corollary 1.12. *For any amplitude $p(x;y,\eta)\in S^m$ there exists an amplitude $q(y,\eta)\in S^m$ independent of x such that modulo a smooth function*

$$(2\pi)^{-n}\int e^{i\varphi}p(x;y,\eta)\,|d_\varphi(x;y,\eta)|\,d\eta \;=\; (2\pi)^{-n}\int e^{i\varphi}q(y,\eta)\,|d_\varphi(x;y,\eta)|\,d\eta\ .$$

By Theorem 1.8 and Corollary 1.12 any Lagrangian distribution $u(x,y)$ can be written modulo C^∞ in the form

$$u(x,y) \;=\; (2\pi)^{-n}\int e^{i\varphi(x;y,\eta)}\,q(y,\eta)\,|d_\varphi(x;y,\eta)|\,d\eta\ . \tag{1.21}$$

1.4. Principal Symbol

Obviously, there are many amplitudes $p(x; y, \eta)$ determining the same Lagrangian distribution (1.18). In (1.21), however, the amplitude q (independent of x !) is defined almost uniquely by the Lagrangian distribution u and the phase function φ. We deduce this fact from the next theorem, but first we formulate

Definition 1.2. Let $C = C_1 + iC_2$ be an $n \times n$ complex symmetric matrix with $C_1 \geqq 0$ (C_1 and C_2 are real symmetric matrices). We denote by Π_C the orthogonal projection on $\ker C$ and introduce

$$\det{}_+ C = \det(C + \Pi_C) .$$

We choose the branch of the argument of $\det_+ C$ such that it is continuous with respect to C on the set of matrices C with a fixed kernel and is equal to zero when $C_2 = 0$.

Note that under the conditions of Definition 1.2 $\ker C = \ker C_1 \cap \ker C_2$. Indeed,

$$\vec{c} \in \ker C \quad \Rightarrow \quad \mathrm{Re}\,(C\vec{c}, \vec{c}) = (C_1 \vec{c}, \vec{c}) = 0 \quad \Rightarrow \quad \vec{c} \in \ker C_1 \quad \Rightarrow \quad \vec{c} \in \ker C_2 \ .$$

Therefore there exists a real orthogonal matrix J such that

$$J \cdot C \cdot J^T = \begin{pmatrix} \tilde{C} & 0 \\ 0 & 0 \end{pmatrix}$$

with some non-degenerate $\tilde{C}$. By Definition 1.2 $\det_+ C = \det \tilde{C}$ and the branch of the argument is chosen as explained in [6], Section 3.4. When $C_1 = 0$, we obtain

$$\arg \det{}_+ C = \frac{\pi}{2} \operatorname{sgn} C_2 , \tag{1.22}$$

where $\operatorname{sgn} C_2$ is the signature of C_2; see [6], Section 3.4.

We will also use the following simple

Lemma 1.13. *Let J be a non-degenerate real matrix. Then*

$$\arg \det{}_+ C = \arg \det{}_+ (JCJ^T) \ .$$

Proof: Let $k = \operatorname{rank} \Pi_C$. Then $\det_+ C$ coincides with the coefficient attached to ε^k in the polynomial $\det(C + \varepsilon I)$, i.e.,

$$\det{}_+ C = \varepsilon^{-k} \det(C + \varepsilon I)\big|_{\varepsilon=0} \ .$$

For any non-degenerate real matrix J we have

$$\begin{aligned}\det_+(JCJ^T) \;&=\; \varepsilon^{-k}\det(JCJ^T+\varepsilon I)\big|_{\varepsilon=0}\\ =\; \det{}^2 J\,\det(C+\Pi_C(J^TJ)^{-1}\Pi_C) \;&=\; \det{}^2 J\det_+(\Pi_C(J^TJ)^{-1}\Pi_C)\det_+ C\ ,\end{aligned}$$

and therefore $\arg\det_+ C = \arg\det_+(JCJ^T)+2\pi k$ for some integer k. Since $\arg\det_+ C$ continuously depends on C on the class of matrices with a fixed kernel and $\arg\det_+ C = \arg\det_+(JCJ^T)=0$ for a real non-negative matrix C, we obtain $k=0$. The proof is complete.

THEOREM 1.14. *Let q_m be the leading term of the amplitude q in* (1.21)*. Then the* (*non-smooth*) *function*

$$e^{-\frac{i}{2}(\arg\det_+(\Phi_{\eta\eta}/i))}q_m \tag{1.23}$$

is uniquely determined by the Lagrangian distribution u and is independent of the phase function $\varphi\in\mathcal{F}$ and of the choice of coordinates y.

Remark 1.15. Since (1.23) is independent of the choice of coordinates y and of the choice of the phase function, this function is an invariant. It can be called the *principal symbol* of the the Lagrangian distribution u. Usually the principal symbol is defined as a section of the Keller-Maslov bundle; see comment after Proposition 2.8, as well as [5] and [14]. Our aproach allows us to interpret the principal symbol as a non-smooth function defined on T^*M, and not as a section of the Keller-Maslov bundle.

Theorem 1.14 immediately implies

COROLLARY 1.16. *All the homogeneous terms of the amplitude q in* (1.21) *are uniquely determined by the Lagrangian distribution u and the phase function $\varphi\in\mathcal{F}$.*

Proof of Corollary 1.16: If u is a smooth half-density then by Theorem 1.14 all the homogeneous terms of the amplitude q are equal to zero. Thus, if u is represented by two different oscillatory integrals with the same phase function then all the homogeneous terms of the amplitudes must be equal.

The proof of Theorem 1.14 is based on two auxiliary lemmas. Let us fix a point (y_0,η_0) and denote $x_0=x^*(y_0,\eta_0)$, $\xi_0=\xi^*(y_0,\eta_0)$. Choose a local coordinate system x in a neighborhood of x_0 such that (1.14) is fulfilled, and introduce the matrix-function

$$\Psi \;=\; \Psi(y,\eta) \;=\; \begin{pmatrix}\Phi_{xx} & \Phi_{x\eta}\\ \Phi_{\eta x} & \Phi_{\eta\eta}\end{pmatrix}.$$

Taking into account (1.10) we obtain

$$\begin{pmatrix} \Phi_{xx} & \Phi_{x\eta} \\ \Phi_{\eta x} & \Phi_{\eta\eta} \end{pmatrix} \cdot \begin{pmatrix} I & x^*_\eta \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Phi_{xx} & \xi^*_\eta \\ \Phi_{\eta x} & 0 \end{pmatrix} .$$

Therefore, in view of (1.3) and (1.14), Ψ is non-degenerate and

$$|\det \Psi| = |\det \xi^*_\eta|\, |\det \Phi_{x\eta}| \;=\; |\det \xi^*_\eta|\, |d_\varphi|^2\Big|_{x=x^*} . \tag{1.24}$$

Lemma 1.17. *At the point* (y_0, η_0)

$$\arg\det{}_+ (\Psi/i) \;=\; \arg\det{}_+ (\Phi_{\eta\eta}/i) \;-\; \pi \operatorname{sgn}\big((x^*_\eta)^T \cdot \xi^*_\eta\big)/2 .$$

Proof: In view of (1.11) we have for $\varepsilon > 0$

$$(x^*_\eta - \varepsilon\xi^*_\eta)^T \cdot (x^*_\eta + \varepsilon\xi^*_\eta) \;=\; (x^*_\eta)^T \cdot x^*_\eta \;-\; \varepsilon^2 (\xi^*_\eta)^T \cdot \xi^*_\eta . \tag{1.25}$$

By (1.14) the matrix $(\xi^*_\eta)^T \cdot \xi^*_\eta$ is strictly positive at the point (y_0, η_0). Multiplying (1.25) from both sides by $\big((\xi^*_\eta)^T \cdot \xi^*_\eta\big)^{-1/2}$ we see that for sufficiently small $\varepsilon > 0$ the matrix (1.25), and thus the matrix $x^*_\eta + \varepsilon\xi^*_\eta$, is non-degenerate.

The equalities (1.10) imply

$$\begin{pmatrix} (x^*_\eta + \varepsilon\xi^*_\eta)^T & I \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} \Phi_{xx}/i & \Phi_{x\eta}/i \\ \Phi_{\eta x}/i & \Phi_{\eta\eta}/i \end{pmatrix} \cdot \begin{pmatrix} x^*_\eta + \varepsilon\xi^*_\eta & 0 \\ I & I \end{pmatrix} \tag{1.26}$$

$$= \begin{pmatrix} (x^*_\eta)^T \cdot \xi^*_\eta / i & 0 \\ 0 & \Phi_{\eta\eta}/i \end{pmatrix} \;-\; i\varepsilon \begin{pmatrix} 2(\xi^*_\eta)^T \cdot \xi^*_\eta & (\xi^*_\eta)^T \cdot \Phi_{x\eta} \\ \Phi_{\eta x} \cdot \xi^*_\eta & 0 \end{pmatrix}$$

$$-\; i\varepsilon^2 \begin{pmatrix} (\xi^*_\eta)^T \cdot \Phi_{xx} \cdot \xi^*_\eta & 0 \\ 0 & 0 \end{pmatrix} .$$

Since Ψ is non-degenerate, the deteminant $\det_+$ of the matrix on the left-hand side of (1.26) is equal to

$$\det{}^2 (x^*_\eta + \varepsilon\xi^*_\eta)\, \det{}_+ (\Psi/i) \;=\; c_k \varepsilon^{2k} \det{}_+ (\Psi/i) \;+\; O(\varepsilon^{2k+1}) ,$$

where $k = \dim \ker x^*_\eta$ and $c_k \neq 0$ is independent of φ.

Let $\tilde{C}$ be the restriction of the matrix

$$\begin{pmatrix} 2(\xi^*_\eta)^T \cdot \xi^*_\eta & (\xi^*_\eta)^T \cdot \Phi_{x\eta} \\ \Phi_{\eta x} \cdot \xi^*_\eta & 0 \end{pmatrix}$$

to the kernel of

$$\begin{pmatrix} (x^*_\eta)^T \cdot \xi^*_\eta & 0 \\ 0 & \Phi_{\eta\eta} \end{pmatrix} .$$

By (1.10′), (1.11)

$$\ker \begin{pmatrix} (x^*_\eta)^T \cdot \xi^*_\eta & 0 \\ 0 & \Phi_{\eta\eta} \end{pmatrix} = \ker \begin{pmatrix} x^*_\eta & 0 \\ 0 & x^*_\eta \end{pmatrix}$$

and $\Phi_{x\eta}\big|_{\ker x^*_\eta} = \xi^*_\eta\big|_{\ker x^*_\eta}$. Therefore the matrix $\tilde{C}$ is independent of φ. As the coefficient attached to ε^{2k} in the expansion for the determinant of (1.26) is non-zero, we have $\operatorname{rank} \tilde{C} = 2k$. Therefore, the determinant $\det_+$ of the matrix in the right-hand side of (1.26) modulo $O(\varepsilon^{2k+1})$ is equal to

$$\det{}_+ \left[\begin{pmatrix} (x^*_\eta)^T \cdot \xi^*_\eta / i & 0 \\ 0 & \Phi_{\eta\eta}/i \end{pmatrix} - i\varepsilon \tilde{C} \right]$$
$$= \varepsilon^{2k} \det{}_+ \big((x^*_\eta)^T \cdot \xi^*_\eta / i\big) \det{}_+ \tilde{C} \det{}_+ (\Phi_{\eta\eta}/i)$$

(in the proof of this lemma the sign "=" means also the equality between the respective branches of the arguments).

Thus at the point (y_0, η_0)

$$\det{}_+ (\Psi/i) = c \det{}_+ (\Phi_{\eta\eta}/i) ,$$

where $c \neq 0$ does not depend on φ. To compute the constant c we take a special phase function which coincides with (1.15) in a neighborhood of $(x_0; y_0, \eta_0)$ (by Lemma 1.6 such a phase function exists). For this phase function

$$\Phi_{xx} = 0 , \quad \Phi_{x\eta} = \xi^*_\eta , \quad \Phi_{\eta\eta} = -(x^*_\eta)^T \cdot \xi^*_\eta .$$

The upper left block Φ_{xx} of the corresponding matrix Ψ is zero, and so $\operatorname{sgn} \Psi = 0$. By (1.22) we have

$$\det{}_+ (\Psi/i) = \det{}^2 \xi^*_\eta$$

with zero branch of the argument. This implies

$$c = \det{}^2 \xi^*_\eta \det{}_+^{-1} \big(i (x^*_\eta)^T \xi^*_\eta\big)\big|_{(y_0,\eta_0)} .$$

Therefore for an arbitrary phase function φ

$$\det{}_+ (\Psi/i)\big|_{(y_0,\eta_0)} = \det{}^2 \xi^*_\eta \det{}_+^{-1} \big(i (x^*_\eta)^T \xi^*_\eta\big) \det{}_+ (\Phi_{\eta\eta}/i)\big|_{(y_0,\eta_0)}$$

and, consequently, at the point (y_0, η_0)

$$\arg \det{}_+ (\Psi/i) = \arg \det{}_+ (\Phi_{\eta\eta}/i) - \arg \det{}_+ \big(i (x^*_\eta)^T \cdot \xi^*_\eta\big) .$$

From (1.22) it follows that

(1.27) $$\arg \det{}_+ \big(i (x^*_\eta)^T \cdot \xi^*_\eta\big) = \frac{\pi}{2} \operatorname{sgn} \big((x^*_\eta)^T \cdot \xi^*_\eta\big) .$$

This completes the proof.

LEMMA 1.18. *Let u be a Lagrangian distribution* (1.21) *of order m, and q_m be the leading homogeneous term of the amplitude q. Let* (1.14) *be fulfilled. Then for any smooth function $\rho(x)$ which is equal to* 1 *at the point x_0 and which has sufficiently small support, we have*

$$\int e^{-i\lambda x\cdot\xi_0}\rho(x)u(x,y_0)\,dx \tag{1.28}$$

$$= \lambda^m e^{-i\lambda x_0\cdot\xi_0} \left(e^{i\pi\,\mathrm{sgn}\left((x^*_\eta)^T\cdot\xi^*_\eta\right)/4} e^{-i\left(\arg\det_+(\Phi_{\eta\eta}/i)\right)/2} |\det\xi^*_\eta|^{-1/2} q_m \right)\Big|_{(y_0,\eta_0)}$$

$$+\ O(\lambda^{m-1})\,, \qquad \lambda\to+\infty\,.$$

Proof: Let us replace in the left-hand side of (1.28) the distribution u by the oscillatory integral (1.21) and change the variables $\eta=\lambda\theta$. Then we obtain the integral

$$(2\pi)^{-n}\lambda^n \int e^{i\lambda(\varphi(x;y_0,\theta)-x\cdot\xi_0)}\rho(x)q(y_0,\lambda\theta)|d_\varphi(x;y_0,\theta)|\,dx\,d\theta\,.$$

Now we apply the stationary phase method. Recall that the equation

$$\varphi_\eta(x;y,\eta)\ =\ 0$$

has the unique solution $x=x^*$, and by (1.14) the equation

$$\varphi_x(x^*;y_0,\theta)\ -\ \xi_0\ =\ \xi^*(y_0,\theta)\ -\ \xi_0\ =\ 0$$

also has the unique solution $\theta=\eta_0$. Thus, the function $\varphi(x;y_0,\theta)-x\cdot\xi_0$ has a unique stationary point $x=x_0$, $\theta=\eta_0$. Obviously, its Hessian at this point coincides with Ψ. Using the stationary phase formula we obtain

$$\int e^{-i\lambda x\cdot\xi_0}\rho(x)u(x,y_0)\,dx$$

$$=\ \lambda^{n+m} e^{-i\lambda x_0\cdot\xi_0}\left(\det{}_+(\lambda\Psi/i)\right)^{-1/2}|d_\varphi|\,q_m\Big|_{(x_0;y_0,\eta_0)}\ +\ O(\lambda^{n+m-1})\,.$$

By (1.24) we have at the point $(x_0;y_0,\eta_0)$

$$\left(\det{}_+(\lambda\Psi/i)\right)^{-1/2}|d_\varphi|\ =\ \lambda^{-n}\left(\det{}_+(\Psi/i)\right)^{-1/2}|\det\xi^*_\eta|^{-1/2}\,|\det{}_+(\Psi/i)|^{1/2}$$

$$=\ \lambda^{-n}|\det\xi^*_\eta|^{-1/2}\,e^{-i\left(\arg\det_+(\Psi/i)\right)/2}\,.$$

This equality and Lemma 1.17 imply (1.28). The proof is complete.

Proof of Theorem 1.14: The coefficient attached to λ^m in the asymptotic formula (1.28) for the Fourier transform of the distribution $\rho(x)u(x, y_0)$ depends on the choice of the coordinate system but not on the phase function φ. Therefore the function (1.23) is also independent of φ. Lemma 1.13 and (1.7) imply that $\arg\det_+(\Phi_{\eta\eta}/i)$ (and thus (1.23)) is independent of the choice of coordinates y. Since this function cannot depend on the coordinates x, it is uniquely determined by the Lagrangian distribution u. The theorem is proved.

2. Cohomology Classes and Existence of Real Phase Functions

2.1. The Maslov Index

In this subsection we recall the definitions of some geometrical objects connected with a Lagrangian manifold. Almost all these objects are well known. Their definitions, however, are based on some auxiliary technical results which will be proved in the next subsection.

Let us choose a covering of Λ by small open neighborhoods U_α and real phase functions φ_α parametrizing Λ in these neighborhoods. We shall always assume for definiteness that the U_α are contractible.

For $(y, \eta) \in U_\alpha \cap U_\beta$ we denote

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(y,\eta) = \frac{1}{2}\,\operatorname{sgn}\partial_{\eta\eta}\varphi_\alpha\big|_{x=x^*} - \frac{1}{2}\,\operatorname{sgn}\partial_{\eta\eta}\varphi_\beta\big|_{x=x^*}\,. \tag{2.1}$$

It is known that $\sigma_{\alpha\beta}$ are integers independent of $(y, \eta) \in U_\alpha \cap U_\beta$ (see [5] or Lemma 2.5 below). The cocycle $\{U_\alpha \cap U_\beta, \sigma_{\alpha\beta}\}$ generates a cohomology class in the Čech cohomology group $H^1(\Lambda, \mathbf{Z})$. This class is said to be the *Maslov cohomology class* of the Lagrangian manifold Λ. The value of this class on a closed curve γ (i.e., the sum of $\sigma_{\alpha\beta}$ along γ) with sign minus is called the *Maslov index* of this curve and denoted by $\mathbf{ind}\,\gamma$. (Such an approach based on the local parametrization of Λ by real phase functions φ_α was suggested by L. Hörmander in [5].)

The factor class modulo 4 of the Maslov cohomology class from $H^1(\Lambda, \mathbf{Z}_4)$ is called the *reduced Maslov class* of Λ. The value

$$\mathbf{ind}_4\,\gamma = \mathbf{ind}\,\gamma \pmod 4$$

is said to be the *reduced Maslov index* of γ. The reduced Maslov class is naturally associated with a complex linear bundle over Λ which is called the *Keller-Maslov bundle*. By definition every real phase function φ_α gives a local trivialization of this bundle over U_α, and the transition function for two different phases φ_α and φ_β is equal to $\exp(i\pi\sigma_{\alpha\beta}/2)$.

All the given definitions are independent of the choice of U_α and the phase functions φ_α (see Theorem 2.6). Note that the reduced Maslov cohomology class (and therefore the Maslov cohomology class) might be non-trivial. The Keller-Maslov bundle, however, is always trivial, i.e., there exists its global section which is never equal to zero; see [5]. Such sections in general cannot be obtained by use of real phase functions. We shall see that they naturally appear when one deals with the global complex phase functions introduced in Section 1.

2.2. Auxiliary Functions $\mathcal{R}$ and Θ

Let us introduce the integer-valued function

$$\mathcal{R}(y,\eta) \;=\; \operatorname{rank} x^*_\eta(y,\eta)$$

on $T^*M\backslash 0$. Obviously, $\mathcal{R}(y,\eta) \leqq n-1$. The set of points (y,η) where $\mathcal{R}(y,\eta) < n-1$ is called the *singular set* of the Lagrangian manifold Λ. Its projection on the manifold M is said to be the *caustic set*.

We shall need the following lemmas.

Lemma 2.1. *Let the phase function* φ *satisfy the conditions* (1.1), (1.2) (*not necessarily* (1.3)). *For fixed local coordinates let* Π *and* $\tilde\Pi$ *be the orthogonal projections on* $\ker x^*_\eta$ *and* $\ker (x^*_\eta)^T$ *correspondingly,* $\Pi' = (I-\Pi)$, $\tilde\Pi' = (I-\tilde\Pi)$. *Then*

$$\widetilde{\det}\,(\tilde\Pi \cdot \xi^*_\eta \cdot \Pi) \neq \;0\;, \tag{2.2}$$

$$\det \Phi_{x\eta} \;=\; \widetilde{\det}\,(\tilde\Pi \cdot \xi^*_\eta \cdot \Pi)\;\widetilde{\det}\,(\tilde\Pi' \cdot \Phi_{x\eta} \cdot \Pi')\;, \tag{2.3}$$

where by $\widetilde{\det}$ *in the right-hand side we mean the determinants of the restrictions of these matrices to the corresponding subspaces.*

Proof: The identity $\ker x^*_\eta \cap \ker \xi^*_\eta = \{0\}$ and formula (1.11) imply that the restriction of $\tilde\Pi \cdot \xi^*_\eta \cdot \Pi$ is a non-degenerate $\mathcal{R}\times\mathcal{R}$-matrix. Therefore (2.2) is fulfilled. From (1.11) it follows also that

$$\tilde\Pi' \cdot \xi^*_\eta \cdot \Pi \;=\; 0\;.$$

This fact and (1.10) imply that in special bases associated with orthogonal decompositions $\mathbf{R}^n = \ker x^*_\eta \oplus \operatorname{Im}(x^*_\eta)^T$ and $\mathbf{R}^n = \ker (x^*_\eta)^T \oplus \operatorname{Im} x^*_\eta$ the matrix $\Phi_{x\eta}$ is triangular, and its diagonal blocks are $\tilde\Pi \cdot \xi^*_\eta \cdot \Pi$ and $\tilde\Pi' \cdot \Phi_{x\eta} \cdot \Pi'$. This implies (2.3) and completes the proof.

Lemma 2.2. *For any phase function* $\varphi \in \mathcal{F}$ *and any coordinate systems* x *and* y *we have*

$$\det{}_+\,(\Phi_{\eta\eta}/i) \;=\; i^{\mathcal{R}}\, f\, \det \Phi_{x\eta}\;, \tag{2.4}$$

where $f = f(y,\eta)$ *is a (non-smooth) real-valued function independent of* φ *(here we do not mean that the branches of the arguments necessarily coincide). If for coordinates* x *the inequality*

$$\det \xi^*_\eta(y,\eta) \neq 0 \tag{2.5}$$

holds (see (1.14)*), then*

$$f(y,\eta) \;=\; i^{-\mathcal{R}(y,\eta)}\,(\det \xi^*_\eta)^{-1}\det{}_+\left(i(x^*_\eta)^T\cdot\xi^*_\eta\right)\;. \tag{2.6}$$

Proof: By (1.10′)

$$\Phi_{\eta\eta} \;=\; -\bigl(\Pi'\cdot(x^*_\eta)^T\cdot\tilde\Pi'\bigr)\cdot\bigl(\tilde\Pi'\cdot\Phi_{x\eta}\cdot\Pi'\bigr)\;. \tag{2.7}$$

Thus

$$\det{}_+(\Phi_{\eta\eta}/i) \;=\; i^{\mathcal{R}}\,\widetilde{\det}\bigl(\Pi'\cdot(x^*_\eta)^T\cdot\tilde\Pi'\bigr)\,\widetilde{\det}\bigl(\tilde\Pi'\cdot\Phi_{x\eta}\cdot\Pi'\bigr)\;.$$

This equalitiy and (2.2), (2.3) imply (2.4) with a real function f independent of φ. When (2.5) is fulfilled we can take the phase function (1.15) and then obtain

$$\det{}_+(\Phi_{\eta\eta}/i) \;=\; \det{}_+\left(i(x^*_\eta)^T\cdot\xi^*_\eta\right)\,,\qquad \det\Phi_{x\eta} \;=\; \det\xi^*_\eta\;.$$

This yields (2.6). The lemma is proved.

Let us choose some coordinates and consider the complex function $\det^2\Phi_{x\eta}$. The argument of this function does not depend on the choice of local coordinates x and y (see Remark 1.1); this is why we use $\det^2\Phi_{x\eta}$ instead of $\det\Phi_{x\eta}$. We can now globally define on $T^*M\backslash 0$ the smooth multi-valued function

$$\vartheta_\Phi \;=\; \vartheta_\Phi(y,\eta) \;=\; \arg\det{}^2\Phi_{x\eta}(y,\eta)$$

(the branches of which differ by 2π).

Let us introduce the non-smooth multi-valued function

$$\Theta(y,\eta) \;=\; (2\pi)^{-1}\vartheta_\Phi \;-\; \pi^{-1}\arg\det{}_+(\Phi_{\eta\eta}/i) \;+\; \mathcal{R}/2\;. \tag{2.8}$$

Obviously, the function Θ is multi-valued only due to the fact that ϑ_Φ is multi-valued. On a simply connected open set we can always fix a particular smooth branch of ϑ_Φ, and this uniquely determines the values of Θ on this set. Since $\arg\det_+(\Phi_{\eta\eta}/i)$ is independent of the choice of local coordinates y (see the proof of Theorem 1.14), the function Θ is independent of the choice of local coordinates x and y.

Proposition 2.3. *The function* Θ *takes integer values, and it is independent of* φ*. The branches of* Θ *are continuous along any curve on which* $\operatorname{rank} x^*_\eta$ *is constant.*

Proof: The first statement of the proposition immediately follows from (2.4) and the fact that f is real and does not depend on φ.

By (1.10′) $\operatorname{rank}\Phi_{\eta\eta} = \operatorname{rank} x^*_\eta$. Therefore, $\arg\det_+(\Phi_{\eta\eta}/i)$ can have jumps only when $\operatorname{rank}\Phi_{\eta\eta}$ changes. This implies the second statement. The proof is complete.

Proposition 2.3 implies, in particular, that Θ is smooth outside the singular set of the Lagrangian manifold Λ.

Below we prove the important Lemma 2.4 which will allow us to compute the function Θ explicitily in some special cases (see Section 3).

Definition 2.1. For a real symmetric matrix C we denote by $r_+(C)$ and $r_-(C)$ the numbers of its positive and negative eigenvalues respectively.

Lemma 2.4. *Let $U \subset T^*M\backslash 0$ be a connected and simply connected open set, and let φ be a phase function satisfying* (1.1)–(1.3) *and such that $\Phi_{\eta\eta}$ is real on U. Then the difference*

$$\Theta \; - \; r_+(\Phi_{\eta\eta})$$

is constant on U. In other words, the jumps of the function Θ on U coincide with the jumps of $r_+(\Phi_{\eta\eta})$.

Proof: The equalities (1.22) and (2.8) give in U

$$\Theta \;=\; (\mathcal{R} + \operatorname{sgn}\Phi_{\eta\eta})/2 \;+\; k \;,$$

where k is an integer depending on the choice of the branch of ϑ_Φ . By (1.10′) we obtain

$$\operatorname{rank}\Phi_{\eta\eta} \;=\; \mathcal{R} \;.$$

These two equalities imply the lemma.

Lemma 2.4 implies that given local coordinates x satisfying (2.5), we have on a connected and simply connected open set

$$\Theta \;=\; r_-\big((x^*_\eta)^T \cdot \xi^*_\eta\big)$$

modulo some additive integer constant. Indeed, this fact immediately follows from Lemma 2.4 if we take the special phase function of the form (1.15).

2.3. Another Definition of the Maslov Index

Let us fix a complex global phase function φ and an open covering $\{U_\alpha\}$ with corresponding real phase functions φ_α (see Section 2.1). By Proposition 2.3 in every neighborhood U_α the difference

$$\begin{aligned}\big(\vartheta_\Phi \;-\; 2\arg\det_+(\Phi_{\eta\eta}/i)\big) \;-\; \big(\vartheta_{\Phi_\alpha} \;-\; 2\arg\det_+(\partial_{\eta\eta}\varphi_\alpha/i)|_{x=x^*}\big)\\ =\; \big(\vartheta_\Phi \;-\; 2\arg\det_+(\Phi_{\eta\eta}/i)\big) \;-\; \pi\operatorname{sgn}\partial_{\eta\eta}\varphi_\alpha\big|_{x=x^*}\end{aligned}$$

modulo 2π is equal to zero. It means that on U_α the function

$$\sigma_\alpha \;=\; \frac{1}{\pi}\arg\det{}_{+}\,(\Phi_{\eta\eta}/i) \;+\; \frac{1}{2}\;\operatorname{sgn}\partial_{\eta\eta}\varphi_\alpha\big|_{x=x^*} \tag{2.9}$$

coincides with a branch of the multi-valued function $(2\pi)^{-1}\vartheta_\Phi$. This fact immediately implies the following

LEMMA 2.5. *On the intersection $U_\alpha \cap U_\beta$ the difference $\sigma_\alpha - \sigma_\beta$ is an integer, and*

$$\sigma_\alpha \;-\; \sigma_\beta \;=\; \sigma_{\alpha\beta}\;,$$

where $\sigma_{\alpha\beta}$ is defined in (2.1).

Let us introduce the 1-form

$$\Omega_\Phi \;=\; (2\pi)^{-1}\,d\vartheta_\Phi\;.$$

In every neighborhood U_α we have $\Omega_\Phi = d\sigma_\alpha$. Therefore the value of the cocycle $\{U_\alpha \cap U_\beta, \sigma_{\alpha\beta}\}$ on a closed curve γ is equal to the integral of Ω_Φ over this curve. It means that the de Rham cohomology class generated by Ω_Φ is the image of the Maslov cohomology class provided by the standard isomorphism of the Čech cohomology group and the de Rham cohomology group; see, for example, [15]. This shows that the definition of the Maslov cohomology class does not depend on the choice of U_α and φ_α, and the corresponding de Rham cohomology class is independent of φ. The last statement also follows from Lemma 1.7. Indeed, in view of this lemma any two phase functions from $\mathcal{F}$ can be continuously transformed one into another in this class. But the considered cohomology class is integer-valued, and a continuous transformation can not change its values on closed curves.

Thus we have proved

THEOREM 2.6. *The definition of the Maslov cohomology class does not depend on the choice of U_α and φ_α. For any global phase function $\varphi \in \mathcal{F}$ its value on a closed curve γ is equal to*

$$\mathbf{ind}\,\gamma \;=\; -\int_\gamma \Omega_\Phi\;.$$

Since the reduced Maslov class is a factor of the Maslov cohomology class, it also does not depend on U_α and φ_α and

$$\mathbf{ind}_{\,4}\,\gamma \;=\; -\int_\gamma \Omega_\Phi \qquad (\mathrm{mod}\ 4)\;.$$

Theorem 2.6 allows us to interpret the Maslov index of a curve γ as the sum of jumps of the multi-valued function $-\Theta$ (introduced in (2.8)) along γ.

Theorem 2.7. *For any closed curve* γ

$$\mathbf{ind}\,\gamma \;=\; -\int_\gamma d\Theta\ ,$$

and, respectively,

$$\mathbf{ind}\,{}_4\gamma \;=\; -\int_\gamma d\Theta \qquad (\mathrm{mod}\ 4)\ .$$

Proof: By (2.8) we have

$$\Theta(y,\eta) \;-\; (2\pi)^{-1}\vartheta_\Phi \;=\; \mathcal{R}/2 \;-\; \pi^{-1}\arg\det{}_+\,(\Phi_{\eta\eta}/i)\ .$$

The right-hand side of this equality is a non-smooth, single-valued function. Therefore the integal of its differential along the closed γ is equal to zero. Consequently $\int_\gamma \Omega_\Phi = \int_\gamma d\Theta$. The proof is complete.

Using the function Θ we can define the Maslov index for an arbitrary (not necessarily closed) curve. Let γ be a curve on Λ with initial point (y_0,η_0) and end point (y_1,η_1). Since Θ is independent of φ and takes integer values (see Proposition 2.3), it implies that

$$-\int_\gamma d\Theta \;=\; -\;\left\{\mathcal{R}/2 \;-\; \pi^{-1}\arg\det{}_+\,(\Phi_{\eta\eta}/i)\right\}\Big|_{(y_0,\eta_0)}^{(y_1,\eta_1)} - \int_\gamma \Omega_\Phi \tag{2.10}$$

is an integer depending only on γ and the Lagrangian manifold Λ. Therefore it is natural to introduce the following

Definition 2.2. Let γ be a curve on Λ with initial point (y_0,η_0) and end point (y_1,η_1). The number (2.10) is called the Maslov index of γ, and its residue modulo 4 is called the reduced Maslov index.

2.4. Principal Symbol and Global Oscillatory Integrals Revisited

Let us consider now the multi-valued function $e^{i\vartheta_\Phi/4}$. Every real phase function φ_α determines the branch $e^{i\pi\sigma_\alpha/2}$ of this multi-valued function on U_α (here σ_α is defined by (2.9)). On the intersection $U_\alpha\cap U_\beta$ we have $e^{i\pi\sigma_\alpha/2} = e^{i\pi\sigma_{\alpha\beta}/2}\,e^{i\pi\sigma_\beta/2}$. This allows us to interpret $e^{i\vartheta_\Phi/4}$ as a global section of the Keller-Maslov bundle, the local trivialization of which on U_α is $e^{i\pi\sigma_\alpha/2}$ (here the procedure of local trivialization is simply the choice of a branch of the multi-valued function $e^{i\vartheta_\Phi/4}$). Obviously, this section is nowhere equal to zero, and it trivializes the Keller-Maslov bundle.

By (2.8) $\frac{1}{2}\arg\det_+(\Phi_{\eta\eta}/i) = \vartheta_\Phi/4 - \pi\Theta/2 + \pi\mathcal{R}/4$. Therefore Theorem 1.14 immediately implies

PROPOSITION 2.8. *The section $e^{-i\vartheta_\Phi/4}q_m$ of the Keller-Maslov bundle is uniquely determined by the Lagrangian distribution* (1.21).

The section $e^{-i\vartheta_\Phi/4}q_m$ is usually called the *principal symbol* of the Lagrangian distribution u. It determines u modulo a Lagrangian distribution of order $m-1$.

Assume now that the reduced Maslov cohomology class is trivial. In this case the variation of the multi-valued function ϑ_Φ along any closed curve is a number divisible by 8π. Then every branch of $e^{-i\vartheta_\Phi/4}$ is a smooth globally defined function on $T^*M\backslash 0$. Thus, in this case the sections of the Keller-Maslov bundle are canonically identified with complex functions on $T^*M\backslash 0$ (or on Λ).

Let us define in a small neighborhood of the set $\{x = x^*\}$ the multi-valued function

$$\vartheta_\varphi(x;y,\eta) \;=\; \arg\det{}^2\partial_{x\eta}\varphi(x;y,\eta)\;. \tag{2.11}$$

Fixing a smooth global branch of ϑ_Φ we obtain a smooth global branch of ϑ_φ. It allows us to define in this neighborhood the single-valued function $e^{i\vartheta_\varphi(x;y,\eta)/4}$. We arbitrarily extend it to all $(x;y,\eta)$ as a smooth function preserving the same notation, and put

$$d_\varphi(x;y,\eta) \;=\; e^{i\vartheta_\varphi(x;y,\eta)/4}\,|d_\varphi(x;y,\eta)|\;, \tag{2.12}$$

where $|d_\varphi|$ is defined by (1.17). It is clear that for x close to x^* the function d_φ is a global smooth branch of $(\det{}^2\partial_{x\eta}\varphi)^{1/4}$.

Now we obtain from (1.21) and Proposition 2.8 the following result.

PROPOSITION 2.9. *Let us assume that the reduced Maslov cohomology class of Λ is trivial. Then any Lagrangian distribution $u(x,y)$ of order m associated with Λ can be written as an oscillatory integral*

$$u(x,y) \;=\; (2\pi)^{-n}\int e^{i\varphi(x;y,\eta)}\,q(y,\eta)\,d_\varphi(x;y,\eta)\,d\eta \tag{2.13}$$

with an arbitrary phase function from $\mathcal{F}$. The leading homogeneous term q_m of the amplitude q in (2.13) *is independent of the choice of φ and is identified with the principal symbol of u.*

2.5. Existence of a Real Phase Function

Let Λ_0 be an open conic subset of Λ. The restriction of the Maslov cohomology class to Λ_0 is trivial (i.e., $\mathbf{ind}\,\gamma = 0$ for any closed curve γ lying in Λ_0) if and only if for some phase function $\varphi \in \mathcal{F}$ there exists a smooth branch of ϑ_Φ on Λ_0

(then, in view of Lemma 1.7, the latter is true for any $\varphi \in \mathcal{F}$). Consequently, if there exists a real phase function parametrizing Λ_0 then the restriction of the Maslov cohomology class to Λ_0 is trivial (in this case we can take $\vartheta_\Phi = 0$). Thus, the non-triviality of the Maslov class is an obstacle to existence of a real phase function parametrizing Λ_0. This obstacle, however, is not unique.

Theorem 2.10. *Let $\gamma \subset \Lambda$ be a closed simple (i.e., without self-intersections) curve. Then Λ can be parametrized by a real phase function satisfying the conditions* (1.1)–(1.3) *in a small neighborhood of γ if and only if the following two conditions are fulfilled:*

(1) $\mathbf{ind}\,\gamma = 0$;
(2) *there exists an integer p such that*

$$0 \leqq \Theta(y,\eta) + p \leqq \mathcal{R}(y,\eta) , \qquad \forall\,(y,\eta) \in \gamma . \tag{2.14}$$

The proof of this theorem is based on the following two auxiliary lemmas.

Lemma 2.11. *Let $(y_0,\eta_0) \in T^*M\backslash 0$ be an arbitrary point and k be an arbitrary integer such that*

$$k \in [-\mathcal{R}(y_0,\eta_0) , \mathcal{R}(y_0,\eta_0)]$$

and $k + \mathcal{R}(y_0,\eta_0)$ is even. Then there exists a neighborhood of the point

$$\big((x^*(y_0,\eta_0), \xi^*(y_0,\eta_0)) , (y_0, -\eta_0)\big) \in \Lambda$$

which is parametrized by a real phase function φ satisfying (1.1)–(1.3) *and such that* $\operatorname{sgn}\Phi_{\eta\eta}(y_0,\eta_0) = k$.

Proof: Let us fix local coordinates in the neighborhoods of the points y_0 and $x_0 = x^*(y_0,\eta_0)$ such that (2.5) is fulfilled, and denote by C the symmetric matrix $((x^*_\eta)^T \cdot \xi^*_\eta)(y_0,\eta_0)$. Choose a real phase function φ satisfying the conditions (1.1), (1.2) and such that at the point (y_0,η_0)

$$\Phi_{xx} = \xi^*_\eta \cdot f(C) \cdot (\xi^*_\eta)^T ,$$

where f is a real function. Then by (1.10) at (y_0,η_0)

$$\Phi_{\eta\eta} = -C + C^2 f(C) .$$

It is clear that we can find a function f which provides at (y_0,η_0) the equalities $\operatorname{rank}\Phi_{\eta\eta} = \mathcal{R}$ and $\operatorname{sgn}\Phi_{\eta\eta} = k$. By Lemma 2.1 and (2.7) the equality $\operatorname{rank}\Phi_{\eta\eta} = \mathcal{R}$ is equivalent to the non-degeneracy of the matrix $\Phi_{x\eta}$, and therefore the condition (1.3) is also fulfilled in a neighborhood of (y_0,η_0). The proof is complete.

Lemma 2.12. *Let φ_0 and φ_1 be real phase functions defined in a neighborhood of a fixed point $(x_0; y_0, \eta_0)$, $x_0 = x^*(y_0, \eta_0)$, and satisfying* (1.1)–(1.3). *If*

$$\operatorname{sgn} \partial_{\eta\eta}\varphi_0(x_0; y_0, \eta_0) \;=\; \operatorname{sgn} \partial_{\eta\eta}\varphi_1(x_0; y_0, \eta_0) \ ,$$

then there exists a smooth family of real phase functions φ_s, $0 \leqq s \leqq 1$, satisfying (1.1)–(1.3) *in a small neighborhood of $(x_0; y_0, \eta_0)$, such that $\varphi_s = \varphi_0$ for $s = 0$ and $\varphi_s = \varphi_1$ for $s = 1$.*

Proof: Choose local coordinates in the same way as in the proof of the previous lemma and denote

$$B_0(y,\eta) \;=\; \partial_{xx}\varphi_0|_{x=x^*(y,\eta)} \ , \qquad B_1(y,\eta) \;=\; \partial_{xx}\varphi_1|_{x=x^*(y,\eta)} \ .$$

It is sufficient to construct a smooth transformation $B_s(y,\eta)$ of these matrix-functions satisfying the condition

$$\det(\xi^*_\eta \;-\; B_s \cdot x^*_\eta) \;\neq\; 0 \ ,$$

and then to take a family of phase functions φ_s such that

$$\varphi_s \;=\; (x - x^*) \cdot \xi^* \;+\; B_s(x - x^*) \cdot (x - x^*) \;+\; O(|x - x^*|^3\, |\eta|) \ .$$

Since the matrices

$$\partial_{\eta\eta}\varphi_0(x_0; y_0, \eta_0) \;=\; (x^*_\eta)^T \cdot (\xi^*_\eta \;-\; B_0 \cdot x^*_\eta)\big|_{(y_0,\eta_0)}$$

and

$$\partial_{\eta\eta}\varphi_1(x_0; y_0, \eta_0) \;=\; (x^*_\eta)^T \cdot (\xi^*_\eta \;-\; B_1 \cdot x^*_\eta)\big|_{(y_0,\eta_0)}$$

have the same signatures and kernels they can be smoothly transformed one into another in the class of real symmetric matrices of the same form and with the same kernel. This generates the desired transformation B_s at (y_0, η_0); see also Lemma 1.2, Chapter 8, in [14]. Now we can take, for example,

$$B_s(y,\eta) \;=\; B_s(y_0,\eta_0) \;+\; s\,\big(B_1(y,\eta) \;-\; B_1(y_0,\eta_0)\big) \\ \;+\; (1-s)\,\big(B_0(y,\eta) \;-\; B_0(y_0,\eta_0)\big) \ .$$

By continuity of B_0 and B_1 the matrix

$$\xi^*_\eta(y,\eta) \;-\; B_s(y,\eta) \cdot x^*_\eta(y,\eta)$$

is non-degenerate when (y,η) is close to (y_0,η_0). This completes the proof.

Proof of Theorem 2.10: Firstly, it will be convenient for us to rewrite condition (2) from the statement of the theorem in the following equivalent form: there exists an even integer l such that

$$|2\Theta(y,\eta) - \mathcal{R}(y,\eta) + l| \leqq \mathcal{R}(y,\eta)\,, \qquad \forall\,(y,\eta)\in\gamma\,. \tag{2.14$'$}$$

Note that in view of Lemma 2.4 the expression $2\Theta(y,\eta) - \mathcal{R}(y,\eta)$ appearing in the left-hand side of (2.14$'$) equals $\operatorname{sgn}\Phi_{\eta\eta}$ modulo an integer additive constant.

Suppose there exists a real phase function φ satisfying the conditions of the theorem. We already know that this implies (1), and in this case we can consider Θ to be single-valued on γ. Let us prove (2). Suppose that (2) is false. Then for any even integer l the inequality (2.14$'$) fails at some point. Consequently, there exist two points $(y_1,\eta_1),(y_2,\eta_2)\in\gamma$ such that

$$\big|\,\big(2\Theta(y_2,\eta_2)-\mathcal{R}(y_2,\eta_2)\big)-\big(2\Theta(y_1,\eta_1)-\mathcal{R}(y_1,\eta_1)\big)\,\big| \;>\; \mathcal{R}(y_1,\eta_1)+\mathcal{R}(y_2,\eta_2)\,.$$

But the difference of signatures of two matrices cannot exceed the sum of their ranks. This contradiction proves the necessarity of condition (2).

Suppose that the conditions (1) and (2) hold. Let us prove the existence of a real phase function φ satisfying the conditions of the theorem. Let us choose a finite set of distinct points $(y_\alpha,\eta_\alpha)\in\gamma$, $\alpha=0,1,\dots,N$, and a set of their neighborhoods U_α such that $\gamma\subset\cup U_\alpha$ and that for each point (y_α,η_α) and any k the construction of Lemma 2.11 produces a real phase function φ_α parametrizing U_α and with

$$\operatorname{sgn}\partial_{\eta\eta}\varphi_\alpha|_{x=x^*(y_\alpha,\eta_\alpha)} \;=\; k\,.$$

Without loss of generality we assume that the covering $\{U_\alpha\}$ of γ is of multiplicity two.

As condition (1) is fulfilled, we can consider Θ to be single-valued on γ. Let us fix an even integer l for which (2.14$'$) holds, and let us parametrize each U_α by a real phase function φ_α satisfying (1.1)–(1.3) and such that

$$\operatorname{sgn}\partial_{\eta\eta}\varphi_\alpha|_{x=x^*(y_\alpha,\eta_\alpha)} \;=\; 2\Theta(y_\alpha,\eta_\alpha)\;-\;\mathcal{R}(y_\alpha,\eta_\alpha)\;+\;l\,.$$

By Lemma 2.4 on each U_α

$$\operatorname{sgn}\partial_{\eta\eta}\varphi_\alpha|_{x=x^*(y,\eta)} \;=\; 2\Theta(y,\eta)\;-\;\mathcal{R}(y,\eta)\;+\;l\,,$$

and therefore on each intersection $U_\alpha\cap U_\beta$

$$\operatorname{sgn}\partial_{\eta\eta}\varphi_\alpha|_{x=x^*(y,\eta)} \;=\; \operatorname{sgn}\partial_{\eta\eta}\varphi_\beta|_{x=x^*(y,\eta)}\,.$$

Let $U_\alpha\cap U_\beta\neq\emptyset$. Denote by $\varphi_s^{\alpha\beta}$ the family of phase functions from Lemma 2.12 defined for $(y,\eta)\in U_\alpha\cap U_\beta$ and with $\varphi_0^{\alpha\beta}=\varphi_\alpha$, $\varphi_1^{\alpha\beta}=\varphi_\beta$; here we reduce, if necessary, the neighborhoods U_α, $\alpha=0,1,\dots,N$, in order to be able to apply Lemma 2.12.

Let $\{\rho_\alpha\}$ be a partition of unity associated with the covering $\{U_\alpha\}$. We introduce

$$\varphi_{\alpha\beta}(x;y,\eta) \;=\; \varphi_s^{\alpha\beta}(x;y,\eta)\big|_{s=\rho_\beta(y,\eta)} \;,$$

Now we define the phase function φ such that $\varphi=\varphi_{\alpha\beta}$ if $(y,\eta)\in U_\alpha\cap U_\beta$, and $\varphi=\varphi_\alpha$ if $(y,\eta)\in U_\alpha$ and $(y,\eta)\notin U_\beta$ for all $\beta\neq\alpha$. It is a smooth real phase function parametrizing Λ in a neighborhood of γ and satisfying the conditions (1.1)–(1.3). The theorem is proved.

Exactly the same arguments lead us to

THEOREM 2.13. *Let $\gamma\subset\Lambda$ be a simple non-closed curve. Then Λ can be parametrized by a real phase function satisfying the conditions* (1.1)–(1.3) *in a small neighborhood of γ if and only if there exists an integer p such that* (2.14) *is fulfilled.*

Indeed, if the curve γ is not closed then we do not need the equality of signatures in the first and last neighborhoods from the covering $\{U_\alpha\}$, and therefore we can omit the first condition.

3. Lagrangian Distributions Associated with Hamiltonian Flows

3.1. Global Time-Dependent Phase Functions

Let $h(x,\xi)\in C^\infty(T^*M\backslash 0)$ be a positive function homogeneous in ξ of degree 1. Denote by $\big(x^t(y,\eta),\xi^t(y,\eta)\big)$ the Hamiltonian trajectory in $T^*M\backslash 0$ generated by the Hamiltonian h with initial data (y,η). In this and in the next section we deal with Lagrangian manifolds

$$\Lambda^t=\{\,(x,\xi),(y,-\eta)\,:\,x=x^t(y,\eta)\,,\,\xi=\xi^t(y,\eta)\,\}\subset(T^*M\backslash 0)\times(T^*M\backslash 0)$$

with fixed $t\in\mathbf{R}^1$ and

$$\Lambda_h=\{\,(t,\tau),(x,\xi),(y,-\eta)\,:\,\tau=-h(y,\eta)\,,\,x=x^t(y,\eta)\,,\,\xi=\xi^t(y,\eta)\,\}$$
$$\subset T^*\mathbf{R}^1\times(T^*M\backslash 0)\times(T^*M\backslash 0)\;.$$

By analogy with Section 1 we say that a complex function

$$\varphi(t;x;y,\eta)\in C^\infty(\mathbf{R}^1\times M\times T^*M\backslash 0)$$

is a *phase function* if it is homogeneous in η of degree 1 and $\operatorname{Im}\varphi\geqq 0$. As before we shall assume that $\operatorname{Im}\varphi(t;x;y,\eta)>0$ for x lying outside a small neighborhood of the point $x^t(y,\eta)$. By $\mathcal{F}_h$ we denote the class of phase functions satisfying the conditions

$$\varphi\big(t;x^t(y,\eta);y,\eta\big) \;=\; 0\;, \tag{3.1}$$

$$\varphi_x\big(t; x^t(y,\eta); y,\eta\big) \;=\; \xi^t(y,\eta)\ , \tag{3.2}$$

$$\det \partial_{x\eta}\varphi\big(t; x^t(y,\eta); y,\eta\big) \neq\; 0\ . \tag{3.3}$$

Then for fixed t all the results of Sections 1 and 2 remain valid with x^t and ξ^t instead of x^* and ξ^*. We preserve the notation from these sections, and considering t as a parameter we shall refer to the results obtained there. In particular, by Lemmas 1.4 and 1.2 phase functions satisfying (3.1)–(3.3) exist and they globally parametrize the Lagrangian manifolds Λ^t.

LEMMA 3.1. *Any phase function $\varphi \in \mathcal{F}_h$ gives a global parametrization of the expanded Lagrangian manifold Λ_h.*

Proof: In view of Lemma 1.2 it is sufficient to prove that

$$\varphi_t(t; x^t; y,\eta) \;=\; -h(y,\eta)\ .$$

Let us differentiate the identity (3.1) with respect to t. We obtain

$$\varphi_t(t; x^t; y,\eta) \;+\; \dot{x}^t\cdot\varphi_x(t; x^t; y,\eta) \;=\; 0\ .$$

Since $\dot{x}^t = h_\xi(x^t,\xi^t)$, by (3.2) and the Euler identity this implies

$$\varphi_t(t; x^t; y,\eta) \;=\; -h(x^t,\xi^t) \;=\; -h(y,\eta)\ .$$

The lemma is proved.

3.2. Existence of a Global Smooth Branch of $\arg(\det^2\partial_{x\eta}\varphi)$

Let us consider the complex "function" $\det^2\partial_{x\eta}\varphi(t;x;y,\eta)$ (it is a density in x and (-1)-density in y). When $t=0$ and $x=x^t$ this "function" is real and positive, i.e., its argument is equal to zero.

LEMMA 3.2. *For x close to x^t there exists a smooth branch of $\arg(\det^2\partial_{x\eta}\varphi)$ which is equal to zero when $t=0$.*

Proof: Let us take a closed curve γ lying in a small neighborhood of the set

$$C_\varphi \;=\; \{\,(t;x;y,\eta)\,:\,x = x^t(y,\eta)\,\}\ .$$

Since x is close to x^t the curve γ can be transformed into a closed curve on C_φ. After that we can continuously transform it along the trajectories x^t into a closed curve lying in the set $C_\varphi \cap \{t=0\}$, where $\arg(\det^2 \partial_{x\eta}\varphi)$ is equal to zero. Therefore the integral of the 1-form

$$\tilde{\Omega}_\varphi \;=\; \frac{1}{2\pi}\, d\big(\arg(\det^2 \partial_{x\eta}\varphi)\big)$$

over γ is also equal to zero. This proves the lemma.

3.3. Fixation of the Global Smooth Branch of $\arg(\det^2 \partial_{x\eta}\varphi)$

From now on we shall deal only with the branch ϑ_φ of $\arg(\det^2 \partial_{x\eta}\varphi)$ introduced in Lemma 3.2. *Respectively, we shall always choose the branches of the functions $\vartheta_\Phi, d_\varphi$ and Θ generated by this branch of ϑ_φ.*

3.4. Triviality of the Maslov Class

Since there exists a smooth global branch of ϑ_Φ (see Section 3.2), we obtain

Corollary 3.3. *The Maslov class of the Lagrangian manifold Λ_h is trivial.*

Let $d_\varphi(t;x;y,\eta) \in C^\infty(\mathbf{R}^1 \times M \times (T^*M\backslash 0))$ be a complex-valued "function" homogeneous in η of degree 0 defined by the formula

$$d_\varphi(t;x;y,\eta) \;=\; \exp\left(\frac{i}{4}\vartheta_\varphi(t;x;y,\eta)\right) |\det \partial_{x\eta}\varphi(t;x;y,\eta))|^{1/2}$$

for x close to x^t (cf. (2.11), (2.12)). As in Sections 1 and 2 (see (1.17)), we suppose that d_φ is a $(\frac{1}{2})$-density in x and a $(-\frac{1}{2})$-density in y.

By analogy with Proposition 2.9 we obtain the following result.

Theorem 3.4. *Let $u(t,x,y)$ be a Lagrangian distribution of order m associated with the Lagrangian manifold Λ_h. Then for any phase function $\varphi \in \mathcal{F}_h$ there exists an amplitude $q(t;y,\eta)$ of order m such that*

$$u(t,x,y) \;=\; (2\pi)^{-n} \int e^{i\varphi(t;x;y,\eta)}\, q(t;y,\eta)\, d_\varphi(t;x;y,\eta)\, d\eta \tag{3.4}$$

modulo a smooth half-density. The amplitude q is determined modulo $S^{-\infty}$ by the Lagrangian distribution u and the phase function φ, and its leading term q_m does not depend on φ.

3.5. The Maslov Index in the Time-Dependent Case

Now by our agreement (see Section 3.3) we have a single-valued function $\Theta(t;y,\eta)$ (see Section 2.2). This immediately implies the following

Proposition 3.5. *Let γ be a curve on Λ_h with initial point $(t_0; y_0, \eta_0)$ and end point $(t_1; y_1, \eta_1)$. Then*

$$\int_\gamma d\Theta \;=\; \Theta(t_1; y_1, \eta_1) \;-\; \Theta(t_0; y_0, \eta_0)\ .$$

Proposition 3.5 allows us to simplify Definition 2.2 for Lagrangian manifolds associated with Hamiltonian flows.

Definition 3.1. Let γ be a curve on Λ_h with initial point $(t_0; y_0, \eta_0)$ and end point $(t_1; y_1, \eta_1)$. The number $\Theta(t_0; y_0, \eta_0) - \Theta(t_1; y_1, \eta_1)$ is called the Maslov index of γ, and its residue modulo 4 is called the reduced Maslov index.

We shall see later (Theorems 3.8, 3.10, Corollaries 3.11, 3.12) that the integer $-\Theta(t; y, \eta)$ is an important invariant of the Lagrangian manifold Λ_h, which appears when one studies singularities of the Lagrangian distribution. This integer itself may now be interpreted as the Maslov index of the (non-closed) curve γ_0 with initial point $(0; y, \eta)$ and end point $(t; y, \eta)$. Indeed, since $\Theta(0; y, \eta) = 0$, we have

$$-\,\Theta(t; y, \eta) \;=\; -\int_{\gamma_0} d\Theta \;=\; \mathbf{ind}\,\gamma_0\ . \tag{3.5}$$

The next two propositions allow us to compute Θ explicitly in some cases.

Proposition 3.6. *Let* $\operatorname{rank} \partial_{\xi\xi} h(x, \xi) = n - 1$ *for all* $(x, \xi) \in T^*M \backslash 0$. *Then for all* t

$$\Theta(t - 0; y, \eta) \;=\; \Theta(t; y, \eta) \;=\; \Theta(t + 0; y, \eta)\ ,$$

if $\operatorname{rank} x^t_\eta = n - 1$, *and*

$$\begin{aligned} \Theta(t; y, \eta) \;-\; \Theta(t - 0; y, \eta) \;&=\; -r_+\big(\partial_{\xi\xi} h(x^t, \xi^t)\big)\ , \\ \Theta(t + 0; y, \eta) \;-\; \Theta(t; y, \eta) \;&=\; r_-\big(\partial_{\xi\xi} h(x^t, \xi^t)\big)\ , \end{aligned}$$

if $\operatorname{rank} x^t_\eta = 0$. *In particular, for sufficiently small* t

$$\Theta(t; y, \eta) \;=\; \begin{cases} r_+\big(\partial_{\eta\eta} h(y, \eta)\big)\ , & t < 0\ , \\ 0\ , & t = 0\ , \\ r_-\big(\partial_{\eta\eta} h(y, \eta)\big)\ , & t > 0\ . \end{cases}$$

Proposition 3.7. *Let* $r_+\big(\partial_{\xi\xi}h(x,\xi)\big) = n-1$ *for all* $(x,\xi)\in T^*M\backslash 0$. *Then for all* t

$$\begin{aligned}\Theta(t+0;y,\eta) - \Theta(t;y,\eta) &= 0\ ,\\ \Theta(t;y,\eta) - \Theta(t-0;y,\eta) &= 1 - \dim\ker x^t_\eta(y,\eta)\ .\end{aligned}$$

By Proposition 2.3 the function Θ can have jumps along a Hamiltonian trajectory only at points where $\operatorname{rank} x^t_\eta < n-1$. Under the conditions of Propositions 3.6 or 3.7 these jumps are determined by the matrix $\partial_{\xi\xi}h(x^t,\xi^t)$. In other cases, generally speaking, the jumps of Θ depend on higher order derivatives of h.

Proof of Propositions 3.6 and 3.7: Let us fix $(t_0;y,\eta)$ and choose local coordinates x and y such that $\det\xi^{t_0}_\eta(y,\eta)\neq 0$. According to Lemma 2.4 with $\varphi=(x-x^t)\cdot\xi^t$ the jumps of the function Θ coincide with the jumps of

$$r_-\big((\xi^t_\eta)^T\cdot x^t_\eta\big) = r_+(\partial_{\eta\eta}\varphi)|_{x=x^t}\ .$$

By the Taylor formula for any t close to t_0 we have

$$\begin{aligned}x^t_\eta &= x^{t_0}_\eta + (t-t_0)\big(h_\xi(x^t,\xi^t)\big)_\eta\Big|_{t=t_0} + O(|t-t_0|^2)\\ &= x^{t_0}_\eta + (t-t_0)\big(\partial_{\xi x}h(x^{t_0},\xi^{t_0})\cdot x^{t_0}_\eta + \partial_{\xi\xi}h(x^{t_0},\xi^{t_0})\cdot\xi^{t_0}_\eta\big) + O(|t-t_0|^2)\ ,\end{aligned}$$

$$\begin{aligned}\xi^t_\eta &= \xi^{t_0}_\eta - (t-t_0)\big(h_x(x^t,\xi^t)\big)_\eta\Big|_{t=t_0} + O(|t-t_0|^2)\\ &= \xi^{t_0}_\eta - (t-t_0)\big(\partial_{xx}h(x^{t_0},\xi^{t_0})\cdot x^{t_0}_\eta + \partial_{x\xi}h(x^{t_0},\xi^{t_0})\cdot\xi^{t_0}_\eta\big) + O(|t-t_0|^2)\ .\end{aligned}$$

Thus

$$\begin{aligned}(3.6)\quad (\xi^t_\eta)^T\cdot x^t_\eta &= (\xi^{t_0}_\eta)^T\cdot x^{t_0}_\eta\\ &+ (t-t_0)\big((\xi^{t_0}_\eta)^T\cdot\partial_{\xi\xi}h(x^{t_0},\xi^{t_0})\cdot\xi^{t_0}_\eta - (x^{t_0}_\eta)^T\cdot\partial_{xx}h(x^{t_0},\xi^{t_0})\cdot x^{t_0}_\eta\big)\\ &\qquad + O(|t-t_0|^2)\ .\end{aligned}$$

For all t the kernel of the matrix

$$(\xi^t_\eta)^T\cdot\partial_{\xi\xi}h(x^t,\xi^t)\cdot\xi^t_\eta$$

is the one-dimensional subspace $\{c\eta\,:\,c\in\mathbf{R}^1\}$, and this kernel is a subspace of the kernels of the matrices x^t_η and $O(|t-t_0|^2)$ in (3.6). Now (3.6) implies that if $\operatorname{rank}x^{t_0}_\eta=n-1$ then also $\operatorname{rank}x^t_\eta=n-1$ for t close to t_0, and Θ has no jump at t_0. When $\operatorname{rank}x^t_\eta=0$ it follows from (3.6) that

$$(\xi^t_\eta)^T\cdot x^t_\eta = (t-t_0)\,(\xi^{t_0}_\eta)^T\cdot\partial_{\xi\xi}h(x^{t_0},\xi^{t_0})\cdot\xi^{t_0}_\eta + O(|t-t_0|^2)$$

which implies Proposition 3.6.

Let now $r_+(\partial_{\xi\xi}h) = n-1$. Then there exists a real matrix C with kernel $\{c\eta \,:\, c \in \mathbf{R}^1\}$ such that

$$C^T \cdot (\xi^{t_0}_\eta)^T \cdot \partial_{\xi\xi} h(x^{t_0}, \xi^{t_0}) \cdot \xi^{t_0}_\eta \cdot C \;=\; I - \Pi_\eta \;,$$

where Π_η is the orthogonal projection on η in the chosen coordinates. It allows us to rewrite (3.6) as follows

$$C^T \cdot (\xi^t_\eta)^T \cdot x^t_\eta \cdot C \;=\; C^T \cdot \Big((\xi^{t_0}_\eta)^T \cdot x^{t_0}_\eta \;-\; (t-t_0)\,(x^{t_0}_\eta)^T \cdot \partial_{xx} h(x^{t_0}, \xi^{t_0}) \cdot x^{t_0}_\eta \Big) \cdot C$$
$$+\; (t-t_0)\,(I - \Pi_\eta) \;+\; O(|t-t_0|^2) \;.$$

The eigenvalues of the symmetric matrix

$$C^T \cdot \Big((\xi^{t_0}_\eta)^T \cdot x^{t_0}_\eta \;-\; (t-t_0)\,(x^{t_0}_\eta)^T \cdot \partial_{xx} h(x^{t_0}, \xi^{t_0}) \cdot x^{t_0}_\eta \Big) \cdot C$$

are either identically zero or uniformly separated from zero for t close to t_0. Therefore the jump of

$$r_-\big(C^T \cdot (\xi^t_\eta)^T \cdot x^t_\eta \cdot C\big) \;=\; r_-\big((\xi^t_\eta)^T \cdot x^t_\eta\big)$$

at the point $(t_0; y, \eta)$ equals the jump of the number of negative eigenvalues of the matrix-function $(t-t_0)\,(I - \Pi_\eta)$ restricted to $\ker(x^{t_0}_\eta \cdot C)$. This proves Proposition 3.7.

3.6. Asymptotics of Fourier Transforms

To clarify the role of the function Θ we prove two theorems on the asymptotic behavior of Fourier transforms.

THEOREM 3.8. *Let u be a Lagrangian distribution* (3.4) *of order m, and q_m be the leading homogeneous term of the amplitude q. Let $(t; y, \theta)$ be a fixed point from $\mathbf{R}^1 \times (T^*M\backslash 0)$, and let in local coordinates x in a neighborhood of $x^t(y,\theta)$*

$$\det \xi^t_\eta \;\neq\; 0 \;. \tag{3.7}$$

Then for any smooth function $\rho(x)$ with sufficiently small support which is equal to 1 at the point $x^t(y,\theta)$ we have

$$\int e^{-i\lambda x \cdot \xi^t(y,\theta)} \rho(x) u(t,x,y)\, dx \tag{3.8}$$
$$=\; \lambda^m \; \Big(e^{-i\lambda x^t \cdot \xi^t}\, e^{-i\pi\, r_-\left((x^t_\eta)^T \cdot \xi^t_\eta\right)/2}\, |\det \xi^t_\eta|^{-1/2}\, e^{i\pi\Theta/2}\, q_m \Big)\Big|_{\eta=\theta}$$
$$+\; O(\lambda^{m-1}) \;, \qquad \lambda \to +\infty \;.$$

Proof: From Lemma 1.18 and (2.8) we obtain for the Lagrangian distribution (3.4)

$$\int e^{-i\lambda x\cdot\xi^t(y,\theta)}\rho(x)u(t,x,y)\,dx \\ = \lambda^m\ \left(e^{-i\lambda x^t\cdot\xi^t}\, e^{i\pi\kappa/2}|\det\xi^t_\eta|^{-1/2}\, q_m\ \right)\Big|_{(t;y,\theta)} + O(\lambda^{m-1})\ ,$$

where

$$\begin{aligned}\kappa\ &=\ \operatorname{sgn}\big((x^t_\eta)^T\cdot\xi^t_\eta\big)/2\ +\ (2\pi)^{-1}\vartheta_\Phi\ -\ \pi^{-1}\arg\det{}_+(\Phi_{\eta\eta}/i)\\ &=\ r_+\big((x^t_\eta)^T\cdot\xi^t_\eta\big)/2\ -\ r_-\big((x^t_\eta)^T\cdot\xi^t_\eta\big)/2\ +\ \Theta\ -\ \mathcal{R}/2\\ &=\ \Theta\ -\ r_-\big((x^t_\eta)^T\cdot\xi^t_\eta\big)\ .\end{aligned}$$

The proof is complete.

Let us now fix points x_0 and y_0. Let $\rho\in C_0^\infty(\mathbf{R}^1)$ be a function with small support. We consider the asymptotics of the Fourier transform

$$\int e^{i\lambda t}\rho(t)\,u(t,x_0,y_0)\ , \tag{3.9}$$

where u is the Lagrangian distribution given by (3.4) and $|\lambda|\to\infty$. We assume that the amplitude q in (3.4) has small (conic) support, and that the following conditions are fulfilled.

Condition A. In $\operatorname{supp}\rho$ there is a unique $t=t_0$ such that $x^{t_0}(y_0,\eta)=x_0$ for some $\eta\in\operatorname{supp}q(t_0;y_0,\,\cdot\,)$.

Condition B. The set

$$W_0\ =\ \{\,\eta\in T^*_{y_0}M\backslash 0\ :\ x^{t_0}(y_0,\eta)=x_0\,\}$$

is a smooth connected $(n-\mathcal{R}_0)$-dimensional manifold in $T^*_{y_0}M\backslash 0$, where

$$\mathcal{R}_0\ :=\ \mathcal{R}(t_0;y_0,\eta)\ =\ \operatorname{rank}x^{t_0}_\eta(y_0,\eta)$$

is constant on W_0.

Condition B implies that the function $\Theta(t_0;y_0,\eta)$ is constant on W_0. We denote by $\mathcal{R}_0$ and Θ_0 the values of $\mathcal{R}(t_0;y_0,\eta)$ and $\Theta(t_0;y_0,\eta)$ for $\eta\in W_0$.

Proposition 3.9. *The tangent space $T_\eta W_0$ at the point η coincides with* $\ker x^{t_0}_\eta(y_0,\eta)$.

Proof: Obviously,

$$x_\eta^{t_0}\cdot\vec c \;=\; \sum_k c_k\partial_{\eta_k}(x^t-x_0) \;=\; 0$$

for any vector $\vec c=(c_1,\dots,c_n)$ from the tangent space $T_\eta(W_0)$. Since $\dim W_0 = \dim\ker x_\eta^{t_0}(y_0,\eta)$ this implies the proposition. The proof is complete.

The Fourier transform (3.9) behaves as a half-density with respect to (x_0,y_0). Therefore it is sufficient to study its asymptotic behavior only for some fixed coordinates x and y in the neighborhoods of the points x_0 and y_0 (the asymptotic formula for other coordinate systems $\tilde x$ and $\tilde y$ is obtained by multiplication by $|\det(\partial x/\partial\tilde x)\,\det(\partial y/\partial\tilde y)|^{1/2}$). We choose an arbitrary coordinate system y and coordinates x satisfying (3.7).

Let

$$\theta=(\theta',\theta'')\ ,\qquad \theta'=(\theta_1,\dots,\theta_l)\ ,\qquad \theta''=(\theta_{l+1},\dots,\theta_n)\ ,\qquad l=\mathcal{R}_0\ ,$$

be (non-linear) coordinates in a neighborhood of $\operatorname{supp} q_m(t_0;y_0,\,\cdot\,)$ such that

$$W_0 \;=\; \{\theta'=0\}\ ,\qquad \eta(\lambda\theta) \;=\; \lambda\eta(\theta) \text{ for } \lambda>0\ ,$$

and θ'' are coordinates on W_0. We introduce on W_0 the positive density

$$d\mu_0 \;=\; |\det\nolimits_+(i(x_\theta^{t_0})^T\cdot\xi_\theta^{t_0})|^{-1/2}\,|\det\xi_\theta^{t_0}|^{1/2}\,|\det(\partial\eta/\partial\theta)|^{1/2}\,d\theta''\ ,$$

which is obviously homogeneous with respect to θ'' of degree $n-\mathcal{R}_0/2$. By Condition B

$$|\det\nolimits_+(i(x_\theta^{t_0})^T\cdot\xi_\theta^{t_0})| \;=\; |\det(x_{\theta'}^{t_0})^T\cdot\xi_{\theta'}^{t_0}|\ ,\qquad \mathcal{R}_0>0\ .$$

Let

$$\hat W_0 \;=\; \{\,\eta\in W_0\,:\,h(y_0,\eta)=1\,\} \;=\; \{\,\theta''\,:\,h\big(y_0,\eta(0,\theta'')\big)=1\,\}\ ,$$

and $d\hat\mu_0$ be the smooth density on $\hat W_0$ defined in coordinates θ by the equality $d\mu_0=h^{n-\mathcal{R}_0/2-1}dh\,d\hat\mu_0$.

Theorem 3.10. *Under Conditions A and B in coordinates satisfying* (3.7) *the following asymptotic formulas hold*

$$\int e^{i\lambda t}\rho(t)u(t,x_0,y_0)\,dt \;=\; O(|\lambda|^{-\infty})\ ,\qquad \lambda\to-\infty\ , \tag{3.10}$$

$$
\begin{aligned}
(3.11)\quad & \int e^{i\lambda t}\rho(t)u(t,x_0,y_0)\,dt \\
&= (2\pi)^{(\mathcal{R}_0/2-n+1)}\, i^{(\Theta_0-\mathcal{R}_0/2)}\,\lambda^{(m+n-\mathcal{R}_0/2-1)}\, e^{i\lambda t_0}\,\rho(t_0)\int_{\hat W_0} q_m\, d\hat\mu_0 \\
&\qquad + O(\lambda^{m+n-\mathcal{R}_0/2-2}), \qquad \lambda\to+\infty .
\end{aligned}
$$

Proof: Let us substitute in (3.9) the oscillatory integral (3.4) with the standard phase function

$$
(3.12)\qquad \varphi = (x-x^t)\cdot\xi^t
$$

and change variables $\eta\to|\lambda|\theta$. Then we obtain

$$
\begin{aligned}
(3.13)\quad (2\pi)^{-n}\,|\lambda|^n \int & e^{i\,|\lambda|\,(t\,\mathrm{sign}\,\lambda+\varphi(t;x_0;y_0,\eta(\theta)))}\,\rho(t) \\
&\times q\big(t;y_0,|\lambda|\eta(\theta)\big)\, d_\varphi\big(t;x_0;y_0,\eta(\theta)\big)\,|\det(\partial\eta/\partial\theta)|\,d\theta\,dt .
\end{aligned}
$$

Since the gradient $\varphi_\eta(t_0;x_0;y_0,\eta)$ does not vanish outside the set W_0, we can assume without loss of generality that $\mathrm{supp}\, q(t_0;y_0,\,\cdot\,)$ lies in a small neighborhood of W_0.

We apply the stationary phase method with respect to the variables θ'. The stationary points are determined by the equation

$$
(3.14)\qquad \partial_{\theta'}\varphi\big(t;x_0;y_0,\eta(\theta)\big) \;=\; \big(x_0-x^t\big(y_0,\eta(\theta)\big)\big)\cdot\partial_{\theta'}\xi^t\big(y_0,\eta(\theta)\big) \;=\; 0 ,
$$

which has the solution $\theta'=0$ for $t=t_0$. By Condition B the matrix

$$
\partial_{\theta'\theta'}\varphi\big(t_0;x_0;y_0,\eta(0,\theta'')\big) \;=\; -\,(x^{t_0}_{\theta'})^T\cdot\xi^{t_0}_{\theta'}\big|_{\,y=y_0,\theta'=0}
$$

is non-degenerate on W_0, and thus for t close to t_0 there exists the unique solution of (3.14) $\theta'_*=\theta'_*(t,\theta'')$, such that $\theta'_*(t_0,\theta'')\equiv 0$ and $(\theta'_*(t,\theta''),\theta'')$ is close to W_0. Using the stationary phase formula we see that (3.13) is equal to

$$
\begin{aligned}
(3.15)\quad (2\pi)^{\mathcal{R}_0/2-n}\,|\lambda|^{n-\mathcal{R}_0/2} \iint & \Big\{ e^{i\,|\lambda|\,(t\,\mathrm{sign}\,\lambda+\varphi(t;x_0;y_0,\eta(\theta)))}\,\rho(t) \\
&\times q_m\big(t;y_0,|\lambda|\eta(\theta)\big)\,|\det(\partial\eta/\partial\theta)| \\
\times \big(\mathrm{det}_+\,\big(\partial_{\theta'\theta'}\varphi\big(t;x_0;y_0,\eta(\theta)\big)/i\big)\big)^{-1/2}\, & d_\varphi\big(t;x_0;y_0,\eta(\theta)\big)\Big\}\Big|_{\theta'=\theta'_*(t,\theta'')}\, d\theta''\,dt
\end{aligned}
$$

modulo lower-order terms. We define the following time-dependent density

$$
\begin{aligned}
d\mu \;=\; \Big\{ \big|\mathrm{det}_+\,\big(\partial_{\theta'\theta'}\varphi\big(t;x_0;y_0,\eta(\theta)\big)/i\big)\big|^{-1/2} & \\
\times |d_\varphi\big(t;x_0;y_0,\eta(\theta)\big)|\,|\det(\partial\eta/\partial\theta)| & \Big\}\Big|_{\,\theta'=\theta'_*}\, d\theta''
\end{aligned}
$$

homogeneous in θ'' of degree $n - \mathcal{R}_0/2$. It is clear that for the chosen phase function

$$d\mu|_{\,t=t_0} \;=\; d\mu_0 \;.$$

By (2.8) for the function

$$\mathcal{A} = \arg\,\big(\det{}_+\,\big(\partial_{\theta'\theta'}\varphi\big(t;x_0;y_0,\eta(\theta)\big)/i\big)\big)^{-1/2}\,d_\varphi\big(t;x_0;y_0,\eta(\theta)\big)\,\Big|_{\theta'=\theta'_*}$$

we obtain

$$\mathcal{A}|_{\,t=t_0} \;=\; \frac{\pi}{2}\,(\Theta_0 - \mathcal{R}_0/2)\;.$$

Now we can rewrite (3.15) as follows

$$(3.16)\quad (2\pi)^{(\mathcal{R}_0/2-n)}\,|\lambda|^{(n-\mathcal{R}_0/2)}\iint\Big\{e^{i\mathcal{A}}\,e^{i\,|\lambda\,|(t\,\mathrm{sign}\,\lambda+\varphi(t;x_0;y_0,\eta(\theta)))}\,\rho(t) \times q_m\big(t;y_0,|\lambda|\eta(\theta)\big)\Big\}\Big|_{\,\theta'=\theta'_*}\,d\mu\,dt\;.$$

Let us now introduce polar coordinates $(r,\hat\theta'')$ such that

$$\theta'' \;=\; r\,\hat\theta''\;,\qquad h\big(y_0,\eta(0,\hat\theta'')\big) \;=\; 1\;;$$

here $\hat\theta''$ is a point on the "sphere" $h\big(y_0,\eta(0,\hat\theta'')\big)=1$. We define the density $d\hat\mu$ on this "sphere" by the equality $d\mu = r^{n-\mathcal{R}_0/2-1}dr\,d\hat\mu$. Obviously, $d\hat\mu|_{t=t_0} = d\hat\mu_0$. Now (3.16) can be written as

$$(3.17)\quad (2\pi)^{(\mathcal{R}_0/2-n)}\,|\lambda|^{(n-\mathcal{R}_0/2)}\iint\Big\{e^{i\,\mathcal{A}}\,e^{i\,|\lambda\,|(t\,\mathrm{sign}\,\lambda+r\,\varphi(t;x_0;y_0,\eta(\theta)))}\,\rho(t) \times r^m\,|\lambda|^m\,q_m\big(t;y_0,\eta(\theta)\big)\Big\}\Big|_{\,\theta=(\theta'_*(t,\hat\theta''),\hat\theta'')}\,r^{n-\mathcal{R}_0/2-1}\,dr\,dt\,d\hat\mu\;.$$

We apply to (3.17) the stationary phase method with respect to the variables r and t. The stationary points are determined by the equations

$$(3.18)\qquad \varphi\big(t;x_0;y_0,\eta(\theta'_*(t,\hat\theta''),\hat\theta'')\big) \;=\; 0\;,$$

$$(3.19)\qquad \mathrm{sign}\,\lambda \;+\; r\,\varphi_t\big(t;x_0;y_0,\eta(\theta'_*(t,\hat\theta''),\hat\theta'')\big) \;=\; 0\;.$$

Since

$$\varphi\big(t_0;x_0;y_0,\eta(0,\hat\theta'')\big) \;=\; 0\;,$$

and

$$(3.20)\qquad \frac{d}{dt}\varphi\big(t;x_0;y_0,\eta(\theta'_*(t,\hat\theta''),\hat\theta'')\big)\Big|_{\,t=t_0} \;=\; \varphi_t\big(t_0;x_0;y_0,\eta(0,\hat\theta'')\big) \;=\; -\,h\big(y_0,\eta(0,\hat\theta'')\big) \;=\; -1\;,$$

t_0 is the unique solution of the equation (3.18) with respect to t. In view of (3.20) for $\lambda < 0$ the equation (3.19) has no solution. It means that for negative λ there is no stationary point. This implies (3.10).

For $\lambda > 0$ there is the unique stationary point

$$t = t_0 , \qquad r = 1 .$$

The Hessian at this point is

$$H_{r,t} = \begin{pmatrix} \varphi_{tt} & -1 \\ -1 & 0 \end{pmatrix},$$

and on account of (1.22)

$$\bigl(\det_+ (H_{r,t}/i)\bigr)^{-1/2} = 1 .$$

Now from (3.17) by the stationary phase formula we obtain (3.11). The proof is complete.

Corollary 3.11. *Let Condition* A *be fulfilled and* $x^{t_0}(y_0,\eta) = x_0$ *for all* $\eta \in T^*_{y_0}M\backslash 0$. *Then for any coordinates* x *and* y

$$\int e^{i\lambda t}\rho(t)u(t,x_0,y_0)\,dt$$
$$= (2\pi)^{1-n}\, i^{\Theta_0}\,\lambda^{m+n-1}\,e^{i\lambda t_0}\,\rho(t_0) \int\limits_{\{h(y_0,\eta)=1\}} q_m\,|\det \xi^{t_0}_\eta|^{1/2}\,d\hat\eta$$
$$+ \; O(\lambda^{m+n-2}) , \qquad \lambda \to +\infty ,$$

where $d\hat\eta$ *is defined by the equality* $d\eta = h^{n-1}\,dh\,d\hat\eta$.

Proof: We can take in (3.11) $\theta = \theta'' = \eta$ which immediately implies the corollary.

Corollary 3.12. *Let the conditions of Theorem* 3.10 *be fulfilled with* $\mathcal{R}_0 = n-1$. *Then there exists the unique point* $\eta_0 \in T^*_{y_0}M\backslash 0$ *such that*

$$x^{t_0}(y_0,\eta_0) = x_0 , \qquad h(y_0,\eta_0) = 1 ,$$

and

$$(3.21) \qquad \int e^{i\lambda t}\rho(t)u(t,x_0,y_0)\,dt$$
$$= (2\pi)^{(1-n)/2}\, i^{\Theta_0+(1-n)/2}\,\lambda^{m+(n-1)/2}\,e^{i\lambda t_0}\,\rho(t_0)$$
$$\times\,|\det\bigl(x^{t_0}_\eta\cdot(x^{t_0}_\eta)^T + |\varsigma_0|^2\,\hat\Pi_0\bigr)|^{-1/4}\,|\eta_0|\,q_m(t_0;y_0,\eta_0) + O(\lambda^{m+(n-3)/2})$$

as $\lambda \to +\infty$. *Here* $\varsigma_0 = |\eta_0|^{-1}\big((\xi_\eta^{t_0})^T\big)^{-1}\eta_0$, $x_\eta^{t_0} = x_\eta^{t_0}(y_0,\eta_0)$, $\xi_\eta^{t_0} = \xi_\eta^{t_0}(y_0,\eta_0)$, *and* $\hat\Pi_0$ *is the orthogonal projection on the one-dimensional linear subspace generated by the vector* ς_0.

Proof: The set W_0 is a one-dimensional conic submanifold, i.e., it is a one-dimensional ray generated by the vector η_0. Without loss of generality we choose orthonormal coordinates $\eta = (\eta_1, \eta')$ such that η_1 goes along the vector η_0. Then we can take $\theta' = \eta'$, $\theta'' = \eta_1$, and $d\hat\mu_0$ is the δ-measure at the point η_0 with coefficient

$$|\det{}_+\,(i(x_\eta^{t_0})^T\cdot\xi_\eta^{t_0})|^{-1/2}\,|\det\xi_\eta^{t_0}|^{1/2}\,|\eta_0|\;. \tag{3.22}$$

Since the kernel of $x_\eta^{t_0}$ at the point (y_0,η_0) is parallel to W_0, taking into account (1.11) we obtain at this point

$$\begin{aligned}|\det{}_+\,(i(x_\eta^{t_0})^T\cdot\xi_\eta^{t_0})|^2 \;&=\; |\det\big((\xi_\eta^{t_0})^T\cdot x_\eta^{t_0}\cdot(x_\eta^{t_0})^T\cdot\xi_\eta^{t_0} \;+\; \Pi_{\eta_0}\big)|\\ &=\; |\det\xi_\eta^{t_0}|^2\,|\det\big(x_\eta^{t_0}\cdot(x_\eta^{t_0})^T \;+\; ((\xi_\eta^{t_0})^T)^{-1}\cdot\Pi_{\eta_0}\cdot(\xi_\eta^{t_0})^{-1}\big)|\;.\end{aligned}$$

Here Π_{η_0} is the orthogonal projection on the one-dimensional linear subspace generated by the vector η_0. Obviously, for any vector $\vec c$ we have

$$((\xi_\eta^{t_0})^T)^{-1}\cdot\Pi_{\eta_0}\cdot(\xi_\eta^{t_0})^{-1}\cdot\vec c \;=\; (\vec c\cdot\varsigma_0)\,\varsigma_0\;.$$

Therefore (3.22) is equal to

$$|\det\big(x_\eta^{t_0}\cdot(x_\eta^{t_0})^T \;+\; |\varsigma_0|^2\,\hat\Pi_0\big)|^{-1/4}\,|\eta_0|\;,$$

and (3.11) implies (3.21). The proof is complete.

4. Applications to Hyperbolic Equations

4.1. Fundamental Solution of a Hyperbolic Operator

Let $A = A(x, D_x)$ be a first-order elliptic pseudodifferential operator in the space of half-densities on M (as usual, $D_x = -i\partial_x$). Assume that the principal symbol a_1 of A is real and positive. We consider the solution $u(t,x,y)$ of the following initial value problem

$$D_t u \;+\; A(x,D_x)u \;=\; 0\;, \tag{4.1}$$

$$u(0,x,y) \;=\; \delta(x-y)\;, \tag{4.2}$$

which is often called the fundamental solution. It is well known that u is a Lagrangian distribution of order zero associated with the Lagrangian manifold Λ_h introduced in Section 3, where $h = a_1$ (see, for example, [14]). Therefore

Theorem 3.4 allows us to represent the fundamental solution u modulo C^∞ by only one oscillatory integral (3.4) with an arbitrary phase function from the class $\mathcal{F}_h$ and an amplitude q of order zero. Now we give an independent proof of this result, and as a by-product we deduce the transport equations for the homogeneous terms q_{-j} of the amplitude q. For simplicity we assume that in any coordinate system all the homogeneous terms of the full symbol of A and its derivatives can be analytically extended with respect to ξ onto $\mathbf{C}^n\backslash 0$. This condition is fulfilled, for example, when A is a root of an elliptic differential operator. (In general one ought to use the almost analytic extentions of these homogeneous terms, see [14]).

Let us substitute the oscillatory integral (3.4) into (4.1). Then we obtain an oscillatory integral with the same phase function and amplitude

$$p(t;x;y,\eta) \;=\; e^{-i\varphi}\, D_t(q\, e^{i\varphi}\, d_\varphi) \;+\; q\, e^{-i\varphi} A(x,D_x)(e^{i\varphi}\, d_\varphi)\;. \tag{4.3}$$

By the theorem on the action of a pseudodifferential operator on an exponential function (see, for example, [14]) we have

$$q\, e^{-i\varphi} A(x,D_x)(e^{i\varphi}\, d_\varphi) \;=\; q\big(a_1(x,\varphi_x) \;+\; F(t;x;y,\eta)\big)\, d_\varphi\;, \tag{4.4}$$

where F is an amplitude of order zero. Thus we can rewrite (4.3) in the form

$$p(t;x;y,\eta) \;=\; d_\varphi\, \big(\varphi_t \;+\; a_1(x,\varphi_x)\big)\, q \;+\; d_\varphi\, D_t q \;+\; d_\varphi\, (d_\varphi^{-1}\, D_t d_\varphi \;+\; F)\, q\;. \tag{4.5}$$

By Corollary 1.12 (with d_φ instead of $|d_\varphi|$) the oscillatory integral with amplitude (4.5) can be transformed into an oscillatory integral of the form (3.4) with amplitude $\tilde{q}$ independent of x. In view of Lemma 3.1

$$\varphi_t \;+\; a_1(x,\varphi_x) \;=\; 0$$

for $x = x^t$. Therefore by Lemma 1.10 the amplitude $\tilde{q}$ is of order zero. Iterating the formula (1.19) we see that the homogeneous terms $\tilde{q}_{-j}$ have the form

$$\begin{aligned} \tilde{q}_0 \;&=\; D_t q_0 \;+\; F_0\, q_0\;, \\ \tilde{q}_{-j} \;&=\; D_t q_{-j} \;+\; F_0\, q_{-j} \;+\; \sum_{i<j} L_{j,i}\, q_{-i}\;. \end{aligned}$$

Here $F_0 = F_0(t;y,\eta)$ is a homogeneous function of degree zero, and $L_{j,i} = L_{j,i}(t;y,\eta,D_\eta)$ are differential operators such that

$$L_{j,i}(t;y,\lambda\eta,\lambda^{-1}D_\eta) \;=\; \lambda^{i-j}\, L_{j,i}(t;y,\eta,D_\eta)\;, \qquad \forall\, \lambda > 0\;.$$

Thus if the q_{-j} are the solutions of the recurrent system

$$D_t q_0 \;+\; F_0\, q_0 \;=\; 0\;, \tag{4.6}$$

$$D_t q_{-j} \;+\; F_0\, q_{-j} \;+\; \sum_{i<j} L_{j,i}\, q_{-i} \;=\; 0\;, \tag{4.7}$$

then after substituting the oscillatory integral (3.4) into the equation (4.1) we obtain a smooth half-density.

Now let us satisfy the initial condition (4.2). In local coordinates

$$\delta(x-y) \;=\; (2\pi)^{-n}\int e^{i(x-y)\cdot\eta}\,d\eta\;.$$

By Lemma A.1 from the Appendix this implies that

$$\delta(x-y) \;=\; (2\pi)^{-n}\int e^{i\varphi(0;x;y,\eta)}\,w(y,\eta)\,d_\varphi(0;x;y,\eta)\,d\eta \qquad (\operatorname{mod} C^\infty)$$

with an amplitude w of order zero (of course, this fact also follows from Theorem 3.4). The homogeneous terms w_{-j} of the amplitude w are computed in accordance with (A.4), (A.5), and by (A.4) $w_0\equiv 1$. Thus if

$$q_0(0;y,\eta) \;=\; 1\;, \tag{4.8}$$

$$q_{-j}(0;y,\eta) \;=\; w_{-j}(y,\eta)\;, \tag{4.9}$$

then the oscillatory integral (3.4) satisfies the initial condition (4.2) modulo C^∞. Solving the ordinary differential equations (4.6), (4.7) with initial conditions (4.8), (4.9) we find all the homogeneous terms q_{-j}. In virtue of the known a priori estimates, the oscillatory integral (3.4) with amplitude $q\sim\sum q_{-j}$ differs from the fundamental solution u by a smooth half-density.

Let us now compute the leading term q_0. By (4.6) and (4.8)

$$q_0 \;=\; \exp\left(-\int_0^t F_0(s;y,\eta)\,ds\right).$$

Since q_0 is independent of φ the function F_0 also does not depend on φ. Let us fix a conic open subset of $\mathbf{R}^1\times M\times(T^*M\backslash 0)$ and calculate F_0 assuming that in this subset our global phase function φ is $(x-x^t)\cdot\xi^t$, and $\det\xi^t_\eta\neq 0$ (see Section 1.2). Obviously,

$$\varphi_{x\eta} \;=\; \xi^t_\eta \tag{4.10}$$

and

$$\begin{aligned}\varphi_t \;&=\; -\dot{x}^t\cdot\xi^t \;+\; (x-x^t)\cdot\dot{\xi}^t\\ &=\; -\,\xi^t\cdot\nabla_\xi a_1(x^t,\xi^t) \;-\; (x-x^t)\cdot\nabla_x a_1(x^t,\xi^t)\\ &=\; -a_1(x^t,\xi^t) \;-\; (x-x^t)\cdot\nabla_x a_1(x^t,\xi^t)\;.\end{aligned} \tag{4.11}$$

Hence in (4.5)

$$\begin{aligned}\varphi_t \;+\; a_1(x,\varphi_x) \;&=\; a_1(x,\xi^t) \;-\; a_1(x^t,\xi^t) \;-\; (x-x^t)\cdot\nabla_x a_1(x^t,\xi^t)\\ &=\; \partial_{xx}a_1(x^t,\xi^t)(x-x^t)\cdot(x-x^t)/2 \;+\; O(|x-x^t|^3|\eta|)\;.\end{aligned}$$

We have in our neighborhood

$$(x - x^t)e^{i\varphi} \;=\; -i(\xi^t_\eta)^{-1}\nabla_\eta e^{i\varphi}\ .$$

Therefore integrating by parts we obtain

$$\int e^{i\varphi}\,\big(\partial_{xx}a_1(x^t,\xi^t)(x-x^t)\cdot(x-x^t)/2 \,+\, O(|x-x^t|^3|\eta|)\big)\,d_\varphi\,d\eta$$
$$=\;(2i)^{-1}\int e^{i\varphi}\,\mathrm{Tr}\,\big((\xi^t_\eta)^{-1}\cdot\partial_{xx}a_1(x^t,\xi^t)\cdot x^t_\eta\big)\,d_\varphi\,d\eta$$

up to an oscillatory integral of order 0 with an amplitude containing the factor $(x-x^t)$. This residual oscillatory integral can be reduced by the same procedure to an oscillatory integral of order -1. It follows that

$$F_0 \;=\; (2i)^{-1}\,\mathrm{Tr}\,\big((\xi^t_\eta)^{-1}\cdot\partial_{xx}a_1(x^t,\xi^t)\cdot x^t_\eta\big) \;+\; d_\varphi^{-1}D_t d_\varphi \;+\; F$$

modulo an amplitude of order -1. Locally, since φ is real, $\vartheta_\varphi \equiv 2\pi\kappa$ for some integer κ which is determined by the global phase function φ (see Section 3.3). Therefore

$$d_\varphi \;=\; \exp(i\kappa\pi/2)\,|\det\xi^t_\eta|^{1/2}\ ,$$

and by Liouville formula

$$d_\varphi^{-1}D_t d_\varphi \;=\; (2i)^{-1}\,\mathrm{Tr}\,\big(\dot\xi^t_\eta\cdot(\xi^t_\eta)^{-1}\big) \;=\; -(2i)^{-1}\,\mathrm{Tr}\Big(\big(\nabla_x a_1(x^t,\xi^t)\big)_\eta\cdot(\xi^t_\eta)^{-1}\Big)$$
$$=\; -(2i)^{-1}\,\mathrm{Tr}\big((\xi^t_\eta)^{-1}\cdot\partial_{xx}a_1(x^t,\xi^t)\cdot x^t_\eta \;+\; \partial_{x\xi}a_1(x^t,\xi^t)\big)\ .$$

Since the phase function φ is locally linear with respect to x, we have

$$F \;=\; a_0(x,\varphi_x) \;=\; a_0(x^t,\xi^t) \;+\; O(|x-x^t|)$$

modulo an amplitude of order -1; here a_0 is the second symbol of the operator A in the chosen coordinates. The oscillatory integral with amplitude $O(|x-x^t|)$ is reduced to an oscillatory integral of order -1, and so it is not contained in F_0.

Combining the above formulas we obtain

$$F_0 \;=\; a_0(x^t,\xi^t) \;-\; (2i)^{-1}\,\mathrm{Tr}\big(\partial_{x\xi}a_1(x^t,\xi^t)\big) \;=\; a_{\mathrm{sub}}(x^t,\xi^t)\ ,$$

where a_{sub} is said to be the *subprincipal symbol* of the pseudodifferential operator A; see, for example, [7] and [14]. Thus we have proved the following result.

THEOREM 4.1. *The solution u of the Cauchy problem* (4.1), (4.2) *is represented by an oscillatory integral* (3.4) *with any phase function $\varphi\in\mathcal{F}_h$ and with amplitude q of order zero the leading homogeneous term of which is*

$$q_0 \;=\; \exp\left(-i\int_0^t a_{\mathrm{sub}}(x^s,\xi^s)\,ds\right). \tag{4.12}$$

Theorem 4.1 allows us to apply the results of the previous section to the fundamental solution u. In particular, Theorems 3.8, 3.10, and Corollaries 3.11, 3.12 are valid for u with $m = 0$ and q_0 defined by (4.12).

4.2. The Riemannian Case

Now let M be a Riemannian manifold with metric g, and let the principal symbol $a_1(x,\xi)$ of the operator A be equal to

$$|\xi|_x = \left(\sum_{i,j} g^{ij}(x)\xi_i\xi_j \right)^{1/2} . \tag{4.13}$$

Then $x^t(y,\eta)$ is the geodesic with the initial conditions

$$x^t \big|_{t=0} = y , \qquad \dot{x}^t \big|_{t=0} = \partial_\eta a_1(y,\eta) .$$

The vector fields $x^t_{\eta_k}(y,\eta)$, $k = 1,\dots,n$, are the Jacobi vector fields with the initial conditions

$$x^t_{\eta_k}(y,\eta) \big|_{t=0} = 0 , \qquad \dot{x}^t_{\eta_k}(y,\eta) \big|_{t=0} = \partial_{\eta\eta_k} a_1(y,\eta) ,$$

where the dot means the covariant derivative. Therefore by Proposition 3.7 the value of the function $-\Theta$ (introduced in (2.8)) at the point $(t;y,\eta)$ is the Morse index of the geodesic $x^s(y,\eta)$, $0 \leqq s \leqq t$, i.e., the number of conjugate points counted with their multiplicities. Now Definition 3.1 and (3.5) imply that in this case the Morse and the Maslov indices coincide; this has already been noted in [2].

In this case we can simplify the formulation of Corollary 3.12.

THEOREM 4.2. *Let M be a Riemannian manifold with metric g, and let A be a pseudodifferential operator on M with principal symbol* (4.13). *Let u be the solution of the Cauchy problem* (4.1), (4.2), *and ρ be a function from $C_0^\infty(\mathbf{R}^1)$. Let $x_0, y_0 \in M$ be fixed, and suppose that in* $(\operatorname{supp}\rho) \times (T^*_{y_0}\backslash 0)$ *there is a unique point (t_0,η_0) such that $x^{t_0}(y_0,\eta_0) = x_0$ and $a_1(y_0,\eta_0) = 1$. Let* $\operatorname{rank} x^{t_0}_\eta(y_0,\eta_0) = n-1$. *Let x and y be local coordinates such that*

$$a_1(x_0,\xi) = |\xi| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2} , \tag{4.14}$$

$$a_1(y_0,\eta) = |\eta| = (\eta_1^2 + \cdots + \eta_n^2)^{1/2} . \tag{4.15}$$

Then the following asymptotic formula holds as $\lambda \to +\infty$

$$\begin{aligned} \int e^{i\lambda t}\rho(t)u(t,x_0,y_0)\,dt &= (2\pi)^{(1-n)/2}\, i^{\Theta_0+(1-n)/2}\, \lambda^{(n-1)/2}\, e^{i\lambda t_0}\, \rho(t_0) \\ &\quad \times |\det{}_+ (i x^{t_0}_\eta \cdot (x^{t_0}_\eta)^T)|^{-1/4}\, q_0(t_0;y_0,\eta_0) \;+\; O(\lambda^{(n-3)/2}) . \end{aligned} \tag{4.16}$$

Here $x_\eta^{t_0} = x_\eta^{t_0}(y_0,\eta_0)$ *and* q_0 *is defined by* (4.12).

Remark 4.3. In Theorem 4.2 we can take x and y to be normal geodesic coordinates with origins x_0 and y_0 respectively.

Proof of Theorem 4.2: It is sufficient to prove Theorem 4.2 for *some* coordinate system x satisfying the condition (4.14) because both sides of (4.16) are invariant under changes of local coordinates x preserving (4.14). So further on we can assume, without loss of generality that x are local coordinates satisfying the additional condition

$$\partial_x a_1(x,\xi)|_{\,x=x_0} \;=\; 0\,. \tag{4.17}$$

For example, normal geodesic coordinates x with origin x_0 satisfy (4.17).

At this stage we cannot yet apply Corollary 3.12 because we do not know whether the condition $\det \xi_\eta^{t_0}(y_0,\eta_0)\neq 0$ is satisfied. So let us choose coordinates $\tilde{x}$ in a neighborhood of x_0 such that at the point $(t_0;x_0;y_0,\eta_0)$ we have

$$\partial\tilde{x}/\partial x \;=\; I\;, \qquad \sum_k \left(\frac{\partial^2 \tilde{x}_k}{\partial x_i \partial x_j}\right) \tilde{\xi}_k^{t_0} \;=\; c\,\delta_j^i\;,$$

where $c \gg 1$ is a constant and δ_j^i are the Kronecker symbols (see also the proof of Proposition 1.5). Then

$$\tilde{x}_\eta^{t_0}(y_0,\eta_0) \;=\; x_\eta^{t_0}(y_0,\eta_0)$$

$$\tilde{\xi}_\eta^{t_0}(y_0,\eta_0) \;=\; \xi_\eta^{t_0}(y_0,\eta_0) \;-\; c\,x_\eta^{t_0}(y_0,\eta_0)\,.$$

By analogy with (1.25) multiplying the latter matrix by

$$(\xi_\eta^{t_0})^T(y_0,\eta_0) \;+\; c\,(x_\eta^{t_0})^T(y_0,\eta_0)$$

we obtain the matrix

$$(\xi_\eta^{t_0})^T(y_0,\eta_0)\cdot\xi_\eta^{t_0}(y_0,\eta_0) \;-\; c^2\,(x_\eta^{t_0})^T(y_0,\eta_0)\cdot x_\eta^{t_0}(y_0,\eta_0)\,. \tag{4.18}$$

In the chosen coordinates x and y at the point (y_0,η_0) we have $|\xi^{t_0}| = |\eta_0| = 1$ and, by (4.17),

$$(\xi_\eta^{t_0})^T\,\xi^{t_0} \;=\; (\xi_\eta^{t_0})^T\,a_{1\xi}(x_0,\xi^{t_0}) \;=\; \frac{d}{d\eta}\,a_1(x^{t_0},\xi^{t_0}) \;=\; a_{1\eta}(y,\eta) \;=\; \eta_0\,. \tag{4.19}$$

From the Euler identities $\xi_\eta^{t_0}\cdot\eta = \xi^{t_0}$, $x_\eta^{t_0}\cdot\eta = 0$ and (4.19) it follows that η_0 is the eigenvector of the matrix (4.18) corresponding to the eigenvalue 1. On the subspace orthogonal to η_0 the matrix $(x_\eta^{t_0})^T(y_0,\eta_0)\cdot x_\eta^{t_0}(y_0,\eta_0)$ is non-degenerate and positive. Arguments similar to those in the proof of Lemma 1.17 show that the matrix (4.18) is non-degenerate for sufficiently large c, and consequently $\tilde{\xi}^{t_0}_\eta(y_0,\eta_0)$ is non-degenerate for sufficiently large c.

Now comparing the required formula (4.16) with (3.21) (written in coordinates $\tilde{x}$) we see that it is sufficient to prove that

$$\tilde{\varsigma}_0 = \xi^{t_0}(y_0,\eta_0) \tag{4.20}$$

and that

$$(x^{t_0}_\eta)^T(y_0,\eta_0)\cdot\xi^{t_0}(y_0,\eta_0) = 0\ . \tag{4.21}$$

But (4.21) follows from the preservation of the 1-form $\xi\,dx$ under a homogeneous canonical transformation, whereas (4.20) follows from (4.19) and (4.21).

Remark 4.4. The right-hand side in (4.16) can be simplified in the following obvious way. Denote $x_0 = (x_{01}, x_0')$, $y_0 = (y_{01}, y_0')$, $x = (x_1, x')$, $y = (y_1, y')$, $\xi = (\xi_1, \xi')$, $\eta = (\eta_1, \eta')$, where x_0', y_0', x', y', ξ', η' are $(n-1)$-component. Suppose that our local coordinate systems are oriented in such a way that the hypersurfaces $x_1 = x_{01}$ and $y_1 = y_{01}$ are orthogonal to the covectors $\xi^{t_0}(y_0,\eta_0)$ and η_0 respectively (that is, the last $n-1$ components of these covectors are zero in the chosen coordinate systems). For the sake of brevity let us denote by x_η and $x'_{\eta'}$ the (square) matrices of derivatives of $x(\eta)$ and $x'(\eta)$ with respect to η and η', where $x(\eta) = (x_1(\eta), x'(\eta)) = x^{t_0}(y_0,\eta)$. Then at $\eta=\eta_0$ we have

$$|{\det}_+\,(ix_\eta\cdot x_\eta^T)| \;=\; |\det(x'_{\eta'}\cdot(x'_{\eta'})^T)| \;=\; |\det x'_{\eta'}|^2\ .$$

Appendix

By the stationary phase method used in the proof of Lemma 1.18 we can find all the asymptotic terms in (3.8). For the sake of simplicity we assume below that $\rho\equiv 1$ in a neighborhood of the point $x^t(y,\theta)$.

According to Theorem 7.7.5 from [6] we have

$$\begin{aligned} &\int e^{-i\lambda x\cdot\xi^t(y,\theta)}\,\rho(x)\,u(t,x,y)\,dx \\ &\qquad = \lambda^m\,e^{-i\lambda\,x^t\cdot\xi^t}\,e^{-i\pi\,r_-\left((x^t_\theta)^T\cdot\xi^t_\theta\right)/2}\,e^{i\pi\Theta/2} \\ &\times|\det\xi^t_\theta|^{-1/2}\sum_{j+l<N}\lambda^{-(j+l)}\,\left(L_j\,q_{m-l}\right)\big|_{\eta=\theta} + O(\lambda^{m-N})\,, \qquad \lambda\to+\infty\,, \end{aligned} \tag{A.1}$$

where $x^t = x^t(y,\theta)$, $\xi^t = \xi^t(y,\theta)$, $\Theta = \Theta(t;y,\theta)$, $q_{m-l} = q_{m-l}(t;y,\eta)$, and the $L_j = L_j(t;y,\eta,\theta,D_\eta)$ are differential operators of order $2j$ which are defined as follows. Let

$$\begin{aligned}\psi(t;x;y,\eta,\theta) \;&=\; \varphi(t;x;y,\eta) \;-\; \big(x - x^t(y,\theta)\big)\cdot \xi^t(y,\theta)\\ &-\; \Phi_{xx}(t;y,\theta)\big(x - x^t(y,\theta)\big)\cdot\big(x - x^t(y,\theta)\big)/2\\ -\; \mathrm{Re}\;\big(\Phi_{x\eta}(t;y,\theta)(\eta-\theta)\cdot\big(x - x^t(y,\theta)\big)\big) \;&-\; \Phi_{\eta\eta}(t;y,\theta)(\eta-\theta)\cdot(\eta-\theta)/2 \;.\end{aligned}$$

Then

$$L_j\, f \;=\; i^{-j}\sum_{\substack{\nu-\mu=j\\ 2\nu\geqq 3\mu}} (2^\nu \mu!\nu!)^{-1}\;\big(\Psi^{-1}(D_{(x,\eta)})\cdot D_{(x,\eta)}\big)^\nu (\psi^\mu\, f)\big|_{x=x^t(y,\theta)}\;,$$

where

$$\Psi \;=\; \Psi(t;y,\theta) \;=\; \begin{pmatrix}\Phi_{xx} & \Phi_{x\eta}\\ \Phi_{\eta x} & \Phi_{\eta\eta}\end{pmatrix}$$

and $D_{(x,\eta)} = (D_{x_1},\dots,D_{x_n}, D_{\eta_1},\dots,D_{\eta_n})$.

Let us introduce the differential operators

$$\mathcal{L}_j = \mathcal{L}_j(t;y,\eta,D_\eta) = L_j(t;y,\eta,\eta,D_\eta)\;.$$

Then formula (A.1) can be rewritten in the equivalent form

$$\text{(A.2)}\quad \begin{aligned}&\int e^{-i\lambda x\cdot\xi^t(y,\eta)}\,\rho(x)\,u(t,x,y)\,dx\\ &\quad=\;\lambda^m\, e^{-i\lambda\, x^t\cdot\xi^t}\, e^{-i\pi\, r_-\left((x^t_\eta)^T\cdot\xi^t_\eta\right)/2}\, e^{i\pi\Theta/2}\\ &\quad\times|\det\xi^t_\eta|^{-1/2}\sum_{j+l<N}\lambda^{-(j+l)}\,\mathcal{L}_j\, q_{m-l} \;+\; O(\lambda^{m-N})\;,\qquad \lambda\to+\infty\;,\end{aligned}$$

where $x^t = x^t(y,\eta)$, $\xi^t = \xi^t(y,\eta)$, $\Theta = \Theta(t;y,\eta)$, $q_{m-l} = q_{m-l}(t;y,\eta)$. The operators $\mathcal{L}_j$ are homogeneous in η of degree $-j$ (that is if f is homogeneous of degree p then $\mathcal{L}_j f$ is homogeneous of degree $p-j$); this fact follows from (A.2) because the left-hand side of (A.2) is invariant under changes $\lambda\to c\lambda$, $\eta\to\eta/c$, for all $c>0$.

Formula (A.2) and the homogeneity of the operators $\mathcal{L}_j$ imply

$$\text{(A.3)}\quad \begin{aligned}&\int e^{-ix\cdot\xi^t(y,\eta)}\,\rho(x)\,u(t,x,y)\,dx\\ &\quad=\; e^{-i\,x^t\cdot\xi^t}\, e^{-i\pi\, r_-\left((x^t_\eta)^T\cdot\xi^t_\eta\right)/2}\, e^{i\pi\Theta/2}\\ &\quad\times|\det\xi^t_\eta|^{-1/2}\sum_{j+l<N}\mathcal{L}_j\, q_{m-l} \;+\; O(|\eta|^{m-N})\;,\qquad |\eta|\to\infty\;.\end{aligned}$$

Note that by the stationary phase method the asymptotic formula (A.3) is uniform with respect to $(t; y, \eta/|\eta|)$, and we can differentiate it with respect to these parameters.

LEMMA A.1. *Let φ and $\tilde{\varphi}$ be phase functions satisfying the conditions* (3.1)–(3.3). *Then for any $q \in S^m$ there exists an amplitude $\tilde{q} \in S_m$ such that the oscillatory integral* (3.4) *coincides modulo C^∞ with*

$$(2\pi)^{-n} \int e^{i\tilde{\varphi}(t;x;y,\eta)}\, \tilde{q}(t; y, \eta)\, d_{\tilde{\varphi}}(t; x; y, \eta)\, d\eta \ .$$

The homogeneous terms $\tilde{q}_{m-j}$ are uniquely determined by the equations

$$\tilde{q}_m \;=\; q_m \ , \tag{A.4}$$

$$\tilde{q}_{m-l} \;=\; q_{m-l} \;+ \sum_{\substack{j+k=l\\ j\geqq 1}} (\mathcal{L}_j q_{m-k} \;-\; \tilde{\mathcal{L}}_j \tilde{q}_{m-k}) \ , \qquad l = 1, 2, \dots \ . \tag{A.5}$$

Proof: Obviously, if we have (A.4) and (A.5) then the Fourier transforms of the oscillatory integrals with respect to x have the same asymptotics in coordinates for which (3.7) is fulfilled. This means that the difference of these oscillatory integrals is a smooth half-density with respect to x. But since we can differentiate (A.3), it is also smooth with respect to t and y.

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