CM322 COMPLEX ANALYSIS CLASS TEST 1, 2010

You have 45 minutes for this test. You may not use calculators or formula sheets. All questions will be marked. You may use your own paper but must put the final answers on this sheet.

PRINT YOUR NAME HERE:

PRINT YOUR STUDENT NUMBER HERE:

1. In each of the following cases determine whether the set Ω is connected, open and/or bounded:
(a) $\Omega = \{z \,:\, \operatorname{Re} z > 0,\ \operatorname{Im} z > \sin(\frac{1}{\operatorname{Re} z})\}$, **(b)** $\Omega = \{z \,:\, 0 < |z| < 1\}$.

The set **(a)** *is the domain bounded by the imaginary axis* $\operatorname{Im} z > 0$ *and by the graph of the function* $\sin(\frac{1}{x})$. *The set is obviously unbounded, is open because the boundary is not included, and connected because every two points can be joined by a path consisting of two vertical and one horizontal line segments.*

The set **(b)** *is obtained from the open unit disc by removing the origin. It is bounded, open (since the boundary is not included) and connected.*

2. State precisely what it means to say that a function $f(z)$ is
(a) differentiable, **(b)** entire.
If $f(z)$ is entire and $f'(z) = 0$ at any point $z \in \mathbb{C}$, prove that f is identically equal to a constant. (Any results from real analysis may be used without justification.)

We say that f *is differentiable at* $z_0 \in \mathbb{C}$ *if* $\dfrac{f(z) - f(z_0)}{z - z_0} \to c$ *as* $z \to z_0$ *where* $c := f'(z_0)$ *is a complex number called the derivative of* f *at* z_0. *The function is entire if it is differentiable at every point of the complex plane.*

If it is true and, in addition, $f'(z) = 0$ *for all* $z \in \mathbb{C}$ *then, putting* $z = z_0 + \varepsilon$ *and* $z = z_0 + i\varepsilon$ *with* $\varepsilon \in \mathbb{R}$, *we see that* $\dfrac{\partial}{\partial x} f(x + iy) = 0$ *and* $\dfrac{\partial}{\partial y} f(x + iy) = 0$ *for all* $x, y \in \mathbb{R}$ *(alternatively, this follows from the Cauchy–Riemann equations). This implies that the function* f *is constant along any horizontal or vertical line segment in* $\mathbb{C}$. *Since every two point in* $\mathbb{C}$ *can be joined by a path consisting of two such line segments, this implies that* f *is constant on the whole complex plane.*

3. Prove rigorously that the function $f(z) = \operatorname{Re} z$ is nowhere differentiable.
(In this question you **may not** use the Cauchy–Riemann equations.)

Let $z = z_0 + i\varepsilon$ *where* $\varepsilon \in \mathbb{R}$. *Then* $\dfrac{\operatorname{Re} z - \operatorname{Re} z_0}{z - z_0} = 0$. *On the other hand, if* $z = z_0 + \varepsilon$ *with* $\varepsilon \in \mathbb{R}$ *then* $\dfrac{\operatorname{Re} z - \operatorname{Re} z_0}{z - z_0} = 1$. *This shows that* $\dfrac{\operatorname{Re} z - \operatorname{Re} z_0}{z - z_0}$ *does not converge to a (unique) limit as* $z \to z_0$. *Therefore* $f'(z_0)$ *is not defined.*

4. Define the radius of convergence $\hat{R}$ of a power series $\sum_{n=0}^{\infty} a_n z^n$. Can $\hat{R}$ be equal to 0 or ∞? Is there a power series which converges for all $z \in \mathcal{D}(0,2)$ and diverges for all $z \in \mathbb{C}$ with $|z| = 2$? Justify your answers by giving examples or/and quoting appropriate results from the course.

The radius of convergence is the least upper bound of the set of nonnegative numbers $r \in \mathbb{R}_+$ for which the set $\{a_0, a_1 r, a_2 r^2, a_3 r^3 \ldots\}$ is bounded.

The radius of convergence of the series $\sum \frac{z^n}{n!}$ is ∞ because $n!$ grows faster than r^n for any $r \geqslant 0$. For the same reason, the radius of convergence of the series $\sum n!\, z^n$ is 0. The series $\sum \left(\frac{z}{2}\right)^n$ diverges if $|z| = 2$ because in this case $\left(\frac{z}{2}\right)^n \not\to 0$. However, it converges for all $z \in \mathcal{D}(0,2)$ by the geometric progression formula.

5. Define the exponential function and prove that $\exp(z+w) = \exp z \exp w$.
(You may use any results on series and differentiation rules without justification.)

By definition, $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Since the radius of convergence of this series is ∞, the exponential function is differentiable on the whole complex plain.

Let $w \in \mathbb{C}$ be a fixed number and $f(z) = \exp(z+w)\exp(-z)$. Differentiating f and applying the formulae for the derivative of the product and the chain rule, we see that $f'(z) = 0$ for all $z \in \mathbb{C}$. Thus f is identically equal to a constant. To find this constant, we can take $z = 0$ which yields $f(z) = f(w)$. Now, putting $a = z + w$ and $b = -z$, we obtain $\exp a \exp b = \exp(a+b)$ (where a and b can be arbitrary complex numbers).

6. Let $T(z) = \frac{az+b}{cz+d}$ be a Möbius transformation with $c \neq 0$. Prove that T can be represented as the composition of four basic transformations: translation, dilatation, rotation and inversion.

The multiplication by a complex number z_0 is the composition of a dilation and a rotation because $z_0 = r\, e^{i\theta}$ with $r, \theta \in \mathbb{R}$ and $r > 0$. Denote $k = (bc - ad)/c$. Then $az + b = c^{-1}\,[a(cz+d) + ck]$ and

$$T(z) = \frac{a}{c} + \frac{k}{cz+d}\,.$$

This is the composition of

(1) the combination of dilation and rotation $z \mapsto c\,z$,

(2) the translation $cz \mapsto cz + d$,

(3) the inversion $cz + d \mapsto (cz+d)^{-1}$,

(4) the combination of of dilation and rotation $(cz+d)^{-1} \mapsto k\,(cz+d)^{-1}$,

(5) $k\,(cz+d)^{-1} \mapsto \frac{a}{c} + k\,(cz+d)^{-1}$.