

S1. (i) The function $1/z^3$ has a primitive (antiderivative) $-1/(2z^2)$ in $\mathbb{C} \setminus \{0\}$ and so its integral around any closed contour not passing through 0 must vanish. Since the winding number of the contour γ is one, by Cauchy's integral formula, $\int_{\gamma} \frac{1}{z} dz = 2\pi i$.

(ii) The function $1/z$ has antiderivative $\text{Log } z$ in the slit plane $\mathbb{C} \setminus \{z + |z| = 0\}$, and the path γ does not meet the negative half-line (where the logarithm has a jump). Therefore

$$\int_{\gamma} \frac{dz}{z} = \text{Log } \gamma(1) - \text{Log } \gamma(0) = \text{Log}(2+i) - \text{Log } 1 = \text{Log}(2+i) = \frac{1}{2} \ln 5 + i \tan^{-1}\left(\frac{1}{2}\right).$$

S2. An upper bound for the modulus of $I(R) = \int_{C_R} z^{-2} \log(z) dz$ is obtained from the basic estimate using the maximum value of the modulus of the integrand and length of circle:

$$|f(z)| \leq R^{-2} \sqrt{(\log R)^2 + \pi^2} < \frac{\pi + \ln R}{R^2}.$$

This bound implies that the integral on $\gamma = C_R$ tends to 0 as $R \rightarrow \infty$, since $R^{-1} \ln(R) \rightarrow 0$.

S3. To parametrise the ellipse, write $\gamma(t) = a \cos t + ib \sin t$ with $0 \leq t \leq 2\pi$. Then the integral $n(\gamma, 0)$ (which is equal to 1) becomes

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)} dt = \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{-a \sin t + ib \cos t}{a \cos t + ib \sin t} \right) dt,$$

which reduces to two (real) integrals for the real/imaginary parts, the latter of which gives the value 0.i, while the former reduces to the form

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt.$$

But the winding number of this ellipse around 0 is equal to 1. The result follows.

S4. First of all, $f(1/2) = f(1)/2$, $f(1/4) = f(1)/4, \dots$ and so $f(0) = \lim f(1/2^n) = f(1)/2^n = 0$, by continuity of f .

Thus we have $f(z) = zg(z)$ with some entire function g . Since $2zg(2z) = f(2z) = 2f(z) = 2zg(z)$, we have $g(2z) = g(z)$. This implies that g is a bounded function on the whole plane (it's modulus does not exceed $\max_{|z| \leq 1} |g(z)|$). By Liouville's Theorem, g is constant.