1. Prove Liouville's Theorem (see course notes on Week 9).

Letting $r \to \infty$ in Cauchy's estimate with $n = 1$, we see that $f'(z_0) = 0$ for all z_0. Therefore f is constant (see notes on Week 4).

2. Prove the fundamental theorem of algebra (see course notes on Week 9).

Let $P(z)$ be a nonconstant polynomial on $\mathbb{C}$. If it does not vanish at any point $z \in \mathbb{C}$ then $\frac{1}{P(z)}$ is an entire function (as the quotient of analytic functions is analytic). This function is bounded because $|P(z)| \to \infty$ as $|z| \to \infty$, which contradicts to Liouville's Theorem.

3. Evaluate $\int_\gamma \frac{\sin z}{z}\,\mathrm{d}z$ over a closed contour γ.

The function $\sin z$ is given by a series $\sum_{n=1}^{\infty} a_n\, z^n$ which is absolutely convergent for all $z \in \mathbb{C}$. By the theorem from Week 3, this series is uniformly convergent on γ (because γ lies in a disc $\mathcal{D}(0, \hat{R})$ with a sufficiently large $\hat{R}$). Since the functions $a_n z_n$ have primitives, their integrals over γ are equal to zero. Therefore, integrating the series term by term, we obtain $\int_\gamma \frac{\sin z}{z}\,\mathrm{d}z = 0$.

4. Evaluate $\int_\gamma \frac{1}{z}\,\mathrm{d}z$ over the closed anticlockwise oriented contour whose trace coincides with the boundary of the square with corners at $1+i$, $-1+i$, $-1-i$ and $1-i$.

The integral is equal to $2\pi i$. The proof repeats, word by word, the proof of the corresponding proof for the integral over a circle around $z_0 = 0$ (see the lemma in the course notes on Week 8).