S1. The condition $f'(z) = f(z)$ implies that

$$(\exp(-z)\, f(z))' = -\exp(-z)\, f(z) + \exp(-z)\, f(z) = 0$$

for all z. Since an open disc is a connected set, the function $\exp(-z)\, f(z) = \frac{f(z)}{\exp z}$ is constant.

S2. $\cos z = \frac{1}{2}(\exp z + \exp(-z))$ and $\sin z = \frac{1}{2i}(\exp z - \exp(-z))$. By the binomial formula,

$$\sin^2 z + \cos^2 z = \frac{1}{4}(\exp z + \exp(-z))^2 - \frac{1}{4}(\exp z - \exp(-z))^2 = \exp z\, \exp(-z) = 1\,.$$

S3. Differentiating $(\operatorname{Re} f)^2 - (\operatorname{Im} f)^2$, we obtain $u\, u_x = v\, v_x$ and $u\, u_y = v\, v_y$ where $u(x,y) = \operatorname{Re} f$, $v(x,y) = \operatorname{Im} f$ and $z = x + iy$. If $u \neq 0$ then the Cauchy–Riemann equations imply

$$u\, u_x = v\, v_x = -v\, u_y = -u^{-1} v^2\, v_y = -u^{-1} v^2\, u_x\,,$$

$$u\, u_y = v\, v_y = v\, u_x = u^{-1} v^2\, v_x = -u^{-1} v^2\, u_y$$

and, consequently, $(u^2 + v^2)\, u_x = (u^2 + v^2)\, u_y = 0$. If $u \neq 0$ then, in a similar way, we obtain $(u^2 + v^2)\, v_x = (u^2 + v^2)\, v_y = 0$. In both cases these identities together with $u\, u_x = v\, v_x$ and $u\, u_y = v\, v_y$ imply that $u_x = u_y = v_x = v_y = 0$. Thus, $f' = 0$ and f is a constant function because a disc is a connected set.

S4. (i) If B is closed then its complement is open. Therefore there exists an open disc D centred at a which does not overlap with B. Clearly, the distance from $\{a\}$ to B is not smaller than the radius of this disc.

(ii) Let A be the closed lower half-plane, and let B be the set of points $z = x + iy$ such that $x > 0$ and $y \geq x^{-1}$. Then B coincides with the closed domain bounded by the hyperbola $y = x^{-1}$ and $\operatorname{dist}(A, B) = 0$ because the hyperbola converges to the real line as $x \to +\infty$.