S1. (a) For any $z \in \mathbb{R}$, with $z \neq -d/c$, the formula shows that $T(z)$ is real. As $z \to -d/c$, writing $z = (-d/c) + \epsilon$ with $\epsilon \to 0$, we find

$$\frac{az+b}{cz+d} = \frac{a(\frac{-d}{c}+\epsilon)+b}{\epsilon} = a + \frac{-ad+cb}{c\epsilon}$$

which tends to $+\infty$ or $-\infty$ (depending on the sign of ϵ).

We have $T(\infty) = a/c$. Indeed, if $\theta = \arg z$ then $z = e^{i\theta}|z|$ and

$$T(z) \;=\; \frac{a\,e^{i\theta}|z|+b}{c\,e^{i\theta}|z|+d} \;=\; \frac{a\,e^{i\theta}+|z|^{-1}b}{c\,e^{i\theta}+|z|^{-1}d} \underset{|z|\to\infty}{\to} \frac{a}{c}.$$

Now to calculate the $T-$image of a typical point $z \in \mathcal{H}$, we find

$$\operatorname{Im} T(z) = \frac{1}{2i}(T(z) - \overline{T(z)}) = \frac{1}{2i}\frac{(ad-bc)(z-\overline{z})}{|cz+d|^2},$$

which is in $\mathcal{H}$ too, since $ad - bc > 0$. Similarly, for the lower half plane, we find $T(\mathcal{L}) \subset \mathcal{L}$. So each (real coefficient) Möbius transformation T preserves the decomposition of the Riemann sphere into $\mathcal{H} \cup \widehat{\mathbb{R}} \cup \mathcal{L}$.

Notes: (1) An easier way to see this, perhaps is to show the image of one point (e.g. i) lies in $\mathcal{H}$, which is easily verified. Then any other point of $\mathcal{H}$ is connected to i by a path which does not cross the real line, and this property holds for the image under T, which finishes the proof.

(2) A similar result holds for any circle C in place of $\widehat{\mathbb{R}}$, but the set of transformations which preserve C do not usually have such a simple description.

S2. Let $z = 1+it$ and let $w = 1/z = u+iv$. Then $u+iv = 1/(1+it) = (1-it)/(1+t^2)$ so that $u = 1/(1+t^2)$ and $v = -t/(1+t^2)$. Hence $v/u = -t$ and so $u = 1/(1+(v/u)^2))$ giving $1 = u/(u^2+v^2)$. Rearranging this, we see that u and v satisfy the equation $u^2+v^2-u = 0$ or $(u-\frac{1}{2})^2 + v^2 = \frac{1}{4}$. That is, $u+iv$ lies on the circle (in the w-plane) with centre at $\frac{1}{2}+i0$ and with radius $\frac{1}{2}$.

Conversely, if $u+iv$ lies on this circle and is not zero, then by working backwards we see that if $t = -v/u$ then $z = 1/w$ is given by $z = 1+it$.

It follows that the line $\{\, z : \operatorname{Re} z = 1 \,\}$ is mapped onto $S \setminus \{\, 0 \,\}$, where S is the circle $\{\, w : |w - \frac{1}{2}| = \frac{1}{2} \,\}$.

S3. Here is just one of many ways to carry this out.

First, move $\mathcal{D}(i,1)$ to $\mathcal{D}(0,1)$ by the translation $T_1(z) = z - i$.

Then consider the map $T_2(z) = \frac{iz+i}{1-z}$. By direct calculation, $\operatorname{Im} T_2(z) > 0$ whenever $|z| < 1$. Therefore T_2 maps $\mathcal{D}(0,1)$ into the u.h.p. $\mathcal{U}$. The inverse $T_2^{-1}(z)$ is the map $z \mapsto \frac{z-i}{z+i}$. A direct calculation shows that $|T_2^{-1}(z)| < 1$ whenever $\operatorname{Im} z > 0$. Thus, $T_2(z)$ maps $\mathcal{D}(0,1)$ onto $\mathcal{U}$.

Then rotate about 0 via $T_3(z) = iz$ to turn $\mathcal{U}$ into the left half plane $\mathcal{L}$. Altogether we have

$$T(z) = T_3 \circ T_2 \circ T_1(z) = i\frac{i(z-i)+i}{-z+i+1} = \frac{-z+i-1}{-z+i+1}.$$

Image of 0 is i, so the required mapping is

$$V(z) = T(z) - i = \frac{(1+i)z}{z-1-i}.$$

S4. This is clear if you draw a diagram, but here is a possible way to put it in formulae. Let $z_0 \in G$ be an arbitrary point of G. Then $1 < \operatorname{Im} z_0 < 4$. This implies that $1 + \delta < \operatorname{Im} z_0 < 4 - \delta$ for some positive δ. If $|z - z_0| < \delta$ then $|\operatorname{Im}(z - z_0)| < \delta$ and, consequently, $1 < \operatorname{Im} z < 4$. Thus, the set G contains the open disc $\mathcal{D}(z_0, \delta)$.

S5. The intersection $\bigcap_{n\in\mathbb{N}} G_n$ is just one point 0. Indeed, this point belongs to every set G_n, and any other point z of the complex plain is not contained in G_n with $n > |z|^{-1}$. The one point set is not open as it does not contain any open discs.

In the example above the intersection is a closed set. But it is not always the case for an infinite intersection of open sets. Assume, for example, that G_n are open strips $\{z \in \mathbb{C} : -1 < \operatorname{Im} z < \frac{1}{n}\}$. Then the intersection of G_n is the half open strip $\{z \in \mathbb{C} : -1 < \operatorname{Im} z \leq 0\}$.