NOTES ON HARMONIC FUNCTIONS

Definition. A function $u(x,y)$ of two real variable x and y is said to be *harmonic* on a open domain $\Omega \subset \mathbb{C}$ if it satisfies the *Laplace equation*

$$\Delta u \ := \ u_{xx} + u_{yy} = 0 \quad \text{on } \Omega,$$

where u_{xx} and y_{yy} are the second derivatives with respect to x and y. As usual, here and further on we identify $\mathbb{C}$ with $\mathbb{R}^2$ and assume that $z = x + iy$.

In polar coordinates (r,θ) the Laplace equation takes the form

$$\Delta u \ = \ r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} \,.$$

This shows that the function $\ln|z|$ is harmonic away from the origin $z = 0$, and the function $\arg z$ is harmonic on every domain Ω where it is continuous. Recall that $\ln|z| = \operatorname{Re}(\ln z)$ and $\arg z = \operatorname{Im}(\ln z)$. Thus the real and imaginary parts of the logarithm are harmonic functions on every domain Ω on which there exists an analytic branch of $\ln z$. The same statement remains true for all analytic functions.

Indeed, let $f(z)$ be an analytic function on Ω. Then the functions $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ satisfy the Cauchy–Riemann equations, that is, $u_x = v_y$ and $u_y = -v_x$. Differentiating these equations with respect to x and y, we see that $\Delta u = \Delta v = 0$. In other words, the real and imaginary parts of every analytic function are harmonic functions.

The other way round, it turns out that every real-valued harmonic function on a simply connected domain is the real part of an analytic function. More precisely, the following theorem holds.

Theorem 1. Let u be a real-valued harmonic function on an open simply connected set Ω. Then there exists a real-valued harmonic function v (called a *conjugate harmonic function*) such that $f = u + iv$ is analytic on Ω.

In order to prove Theorem 1, we shall need

Lemma. If f satisfies the Cauchy–Riemann equations on an open set Ω and the partial derivatives f_x and f_y are continuous then f is analytic on Ω.

Proof. We have to show that the limit $\lim\limits_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists for all $z \in \Omega$ (here $h \in \mathbb{C}$, so that the limit does not depend on the direction). Let $h = s + it$. Then $h \to 0$ if and only if $s \to 0$ and $t \to 0$.

Putting $f(x,y) := f(x+iy)$, we can rewrite $\frac{f(z+h)-f(z)}{h}$ as follows

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{f(x+s,y+t)-f(x,y)}{s+it} \\ &= \frac{f(x+s,y+t)-f(x,y+t)}{s}\,\frac{s}{s+it} + \frac{f(x,y+t)-f(x,y)}{t}\,\frac{t}{s+it}\,.\end{aligned}$$

By the mean value theorem, $\dfrac{f(x+s,y+t)-f(x,y+t)}{s} = f_x(x+s^*,y+t)$ and $\dfrac{f(x,y+t)-f(x,y)}{t} = f_y(x,y+t^*)$, where s^* and t^* are some real numbers lying in the intervals $[-s,s]$ and $[-t,t]$ respectively. Thus we have

$$\frac{f(z+h)-f(z)}{h} \;=\; \frac{s}{s+it}\,f_x(x+s^*,y+t)+\frac{t}{s+it}\,f_y(x,y+t^*)\,.$$

Note that $\left|\frac{s}{s+it}\right| \leqslant 1$ and $\left|\frac{t}{s+it}\right| \leqslant 1$. Since the partial derivatives of f are continuous, $f_x(x+s^*,y+t)-f_x(x,y)\to 0$ and $f_y(x,y+t^*)\to 0$ as $s,t\to 0$. Therefore

$$\frac{f(z+h)-f(z)}{h} \;=\; \frac{s}{s+it}\,f_x(x,y)+\frac{t}{s+it}\,f_y(x,y) \;+\; R(h,x,y)\,,$$

where R is some function such that $R(h,x,y)\to 0$ as $h\to 0$. Applying the Cauchy–Riemann equations, we see that $\dfrac{s}{s+it}\,f_x+\dfrac{t}{s+it}\,f_y \;=\; f_x\,$. Thus the limit $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ exists and is equal to $f_x(x,y)$.

Proof of Theorem 1. It is sufficient to construct the function v on every connected component of the set Ω. Therefore, without loss of generality, we shall be assuming that Ω is connected.

If u is a harmonic real-valued function on Ω then the function $g=u_x-iu_y$ satisfies the Cauchy–Riemann equations and, therefore, is analytic.

Let us fix an arbitrary point $a\in\Omega$ and consider the function $f_0(z):=\int_{\gamma_z} g(w)\,\mathrm{d}w$ where $\gamma_z:[0,1]\mapsto\Omega$ is a path joining a and z. Note that $f_0(z)$ does not depend on the choice of γ_z because the integral $\int_\gamma g$ over any closed contour γ in Ω is equal to zero (see Week 10).

If $z\in\Omega$ and $z_0\in\Omega$ are sufficiently close to each other, then the line segment $\gamma_{z_0,z}(t)=z_0+t(z-z_0)$ joining z_0 and z lies in Ω. By the above, the integral of f over the closed contour formed by γ_{z_0}, $\gamma_{z_0,z}$ and the reverse path to γ_z is zero. This implies that $f_0(z)-f_0(z_0)=\int_{\gamma_{z_0,z}} g(w)\,\mathrm{d}w$. Using this formula, in the same way as in the proof of Morera's theorem (see Week 10), one can show that f_0 is analytic.

Let $\gamma_z(t)=x(t)+iy(t)$. By the definition of the integral,

$$\begin{aligned} f_0(z) \;=\; \int_{\gamma_z} g(w)\,\mathrm{d}w \;&=\; \int_0^1 g(\gamma(t))\,\dot\gamma(t)\,\mathrm{d}t \\ &=\; \int_0^1 \big(u_x(x(t),y(t))-iu_y(x(t),y(t))\big)\,\big(x'(t)+iy'(t)\big)\,\mathrm{d}t \end{aligned}$$

and, consequently,

$$\begin{aligned} \operatorname{Re} f_0(z) \;&=\; \int_0^1 \big(u_x(x(t),y(t))\,x'(t)+u_y(x(t),y(t))\,y'(t)\big)\,\mathrm{d}t \\ =\; \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}u(x(t),y(t))\,\mathrm{d}t \;&=\; u(x(1),y(1))-u(x(0),y(0)) \;=\; u(z)-u(z_0)\,. \end{aligned}$$

The function $v(z) := \operatorname{Im} f_0(z)$ is harmonic (as the imaginary part of an analytic function), and the function $f(z) = u(z) + iv(z) = f_0(z) + u(z_0)$ is analytic. Thus v is a conjugate to u harmonic function.

Remark. If Ω is connected then any other conjugate to u harmonic function coincides with $v + C$ where C is some constant. Indeed, if the functions $u + iv$ and $u + i\tilde{v}$ are analytic then $v - \tilde{v}$ is a real-valued analytic function and, consequently, is constant (see Week 4).

Remark. Since analytic functions are infinitely differentiable, Theorem 1 implies that all harmonic functions are also infinitely differentiable.

Theorem 2 (the mean value property). Let u be a harmonic real-valued function on an open set Ω. If Ω contains a closed disc of radius r centred at z_0 then

$$u(z_0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,\mathrm{d}t\,.$$

Proof. By Theorem 1, u is the real part of an analytic function f. Consider the closed path $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, going around z_0. By Cauchy's formula,

$$\begin{aligned} f(z_0) \;&=\; \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z \;=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z_0}\,\dot{\gamma}(t)\,\mathrm{d}t \\ &=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}}\,ire^{it}\,\mathrm{d}t \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t\,. \end{aligned}$$

Taking the real parts, we obtain the required formula.

Theorem 3 (the maximum principle). Let u be a harmonic real-valued function on an open connected set Ω. If u attains its maximal value at some point $z_0 \in \Omega$ then u is constant.

Proof. Assume that $u(z_0) = M$ for some $z_0 \in \Omega$ and $u(z) \leqslant M$ for all $z \in \Omega$. Since Ω is open, it contains an open disc $\mathcal{D}(z_0, R)$. By Theorem 2, for all $r < R$ we have

$$M \;=\; u(z_0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,\mathrm{d}t \;\leqslant\; \frac{1}{2\pi}\int_0^{2\pi} M\,\mathrm{d}t \;=\; M\,.$$

If $u(z_0 + re^{it})$ is strictly smaller than M at some point $t_0 \in [0, 2\pi]$ then, by continuity, the same is true for all t sufficiently close to t_0. In this case $\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,\mathrm{d}t$ is strictly smaller than $\frac{1}{2\pi}\int_0^{2\pi} M\,\mathrm{d}t$, which contradicts to the above formula. Thus $u(z_0 + re^{it}) = M$ for all $r < R$ and $t \in [0, 2\pi]$ or, in other words, $u(z) = M$ for all $z \in \mathcal{D}(z_0, R)$.

The rest of proof repeats that of the maximum modulus principle for analytic functions (see Week 9).

Remark. Applying Theorem 3 to the function $-u$, we see that a non-constant harmonic function on an open connected set cannot attain its minimal value.