1. (i) Evaluate

$$\int_\gamma \frac{z^2+1}{z^3}\ dz$$

where γ is the simple closed polygonal contour, oriented counter clockwise, which traces the boundary of the triangle with corners at $-2-i$, i and $2-i$.

(ii) Sketch the path $\gamma(t) = 1 + t + i\sin\left(\frac{5}{2}\,\pi t\right)$, $0 \le t \le 1$, and evaluate

$$\int_\gamma \frac{1}{z}\ dz.$$

Hint: it is usually better not to try to compute the integral directly via parametrisation of γ; look instead for a primitive or apply a suitable theorem.

2. Let $\gamma_R : [0, 2\pi] \to \mathbb{C}$ be the circle $\gamma_R(\theta) = Re^{i\theta}$. Find an upper bound for the modulus of the integral

$$I(R) = \int_{\gamma_R} z^{-2}\,\mathrm{Log}(z)dz,$$

and show that $I(R)$ tends to 0 as $R \to \infty$.

3. Define a parametrised path γ which traces the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ once anticlockwise. Deduce that

$$\int_0^{2\pi} \left(\frac{1}{a^2\cos^2 t + b^2\sin^2 t}\right) dt = \frac{2\pi}{ab}.$$

4. Let f be an entire function satisfying $f(2z) = 2f(z)$ for all $z \in \mathbb{C}$. Show that f is a polynomial of degree at most 1 with $f(0) = 0$.