

1. Let f be an analytic functions in an open disc such that $f'(z) = f(z)$. Prove that $f(z) = c \exp z$ with some complex constant c .

2. Prove that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$
(you are supposed to give a direct proof and may not use known result on functions of a real variable).

3. Let f be an analytic functions in an open disc such that $f(z) \neq 0$ for all z and $(\operatorname{Re} f)^2 - (\operatorname{Im} f)^2$ is a constant function. Prove that $f(z)$ is constant.

4. The distance between non-empty sets A and B in \mathbb{C} is defined to be

$$\operatorname{dist}(A, B) = \inf\{|z - w| : z \in A, w \in B\}.$$

If B is closed and $a \notin B$, show that $\operatorname{dist}(\{a\}, B) > 0$. (Here $\{a\}$ is the set consisting of one element a).

Give an example of disjoint sets A and B with $\operatorname{dist}(A, B) = 0$. Is it possible for both A and B to be closed sets?