1. Let a and b be real numbers such that $b > a$, and let f and g be continuous functions on the interval $[a, b]$. State whether the following statements are correct. In each case write down few words of explanation or give a counterexample.

   1a. If $f(x) > g(x)$ for all x then $\int_a^b f(x)\,\mathrm{d}x > \int_a^b g(x)\,\mathrm{d}x$.

   1b. If $|f(x)| > 2\,|g(x)|$ for all x then $\int_a^b f(x)\,\mathrm{d}x \geq 2\int_a^b g(x)\,\mathrm{d}x$.

   1c. If $c \in (a, b)$ then $\int_a^b |f(x)|\,\mathrm{d}x \geq |\int_a^c f(x)\,\mathrm{d}x|$.

   Solution 1a. True. The axioms (1) and (2) imply that

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x = \int_a^b (f(x) - g(x))\,\mathrm{d}x \geq 0\,.$$

Moreover, $f - g$ is a continuous strictly positive function. If m is its minimum value on the closed interval $[a, b]$ then, by the axioms (2) and (3)

$$\int_a^b (f(x) - g(x))\,\mathrm{d}x \geq \int_a^b m\,\mathrm{d}x = m\,(b-a) > 0.$$

Solution 1b. False; $f(x) = -3$ and $g(x) = 1$ is a counterexample.

Solution 1c. True, because $|\int_a^c f(x)\,\mathrm{d}x| \leq \int_a^c |f(x)|\,\mathrm{d}x$ by the property (6), and $\int_a^c |f(x)|\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x$ by the axioms (2) and (4).

2. Give a complete and careful proof that $\lim_{n\to\infty} \int_1^n e^{-x^2}\,\mathrm{d}x$ exists and is finite. You may use any theorems about sequences without proof, but you must not use the integral comparison test.

   Solution 2. Denote $a_n = \int_1^n e^{-x^2}\,\mathrm{d}x$. Since $e^{-x^2} \leq e^{-x}$ for all $x \in [1, \infty)$, by (1a)

$$a_n = \int_1^n e^{-x^2}\,\mathrm{d}x < \int_1^n e^{-x}\,\mathrm{d}x = e - e^{-n} < e$$

for all $n = 1, 2, \ldots$ Also, by the axioms (2) and (4), $a_n < a_{n+1}$ for all n. Now the monotone convergence theorem implies that the sequence $\{a_n\}$ converges to its l.u.b. $\sup\{a_n\}$ and

$$\lim_{n\to\infty} \int_1^n e^{-x^2}\,\mathrm{d}x = \lim_{n\to\infty} a_n = \sup\{a_n\} \leq e.$$

3. Using the fundamental theorem of calculus and the chain rule, evaluate the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_0^{x^2} \frac{1}{1+t^3}\,\mathrm{d}t\right).$$

Solution 3. Denote $F(y) = \int_0^y \frac{1}{1+t^3}\,\mathrm{d}t$. Then

$$F'(y) = \frac{1}{1+y^3}, \qquad \int_0^{x^2} \frac{1}{1+t^3}\,\mathrm{d}t = F(x^2)$$

and, by the chain rule,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_0^{x^2} \frac{1}{1+t^3}\,\mathrm{d}t\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x^2) = \frac{\mathrm{d}}{\mathrm{d}x}\left(2x\,F'(x^2)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x}{1+x^6}\right)$$

$$= \frac{2(1+x^6) - 12x^6}{(1+x^6)^2} = \frac{2-10x^6}{(1+x^6)^2}.$$