

1. Suppose that f is differentiable on the interval (a, b) . Prove that if $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly monotonically increasing in the sense that $f(u) > f(v)$ whenever $u > v$. Write down an example that proves the converse statement is false: there exists a differentiable function f such that $f(u) > f(v)$ whenever $u > v$ but $f'(x) > 0$ for all x is false.

The first part follows directly by the mean value theorem. For the second part consider, $f(x) = x^3$ which has a zero derivative at $x = 0$.

2. By applying the mean value theorem to $f(x)e^{-cx}$, prove that if f is differentiable on \mathbf{R} and $f'(x) = cf(x)$ for all $x \in \mathbf{R}$ then $f(x) = e^{cx}f(0)$ for all $x \in \mathbf{R}$.

The derivative of $f(x)e^{-cx}$ is zero everywhere, so this function is constant by the mean value theorem.

3. Use an appropriate theorem to find the range of values of $x \in \mathbf{R}$ for which each of the following power series converges.

In each of the above parts the series $\sum a_n x^n$ converges for $|x| < \hat{R}$ where $\hat{R} = \lim_{n \rightarrow \infty} |a_n/a_{n+1}|$. The limit is obtained by identifying the dominant term in the numerator and denominator.

$$(3a) \quad \sum_{n=1}^{\infty} \frac{n^9 - 1}{n^8 + 1} x^n$$

$$\hat{R} = \lim_{n \rightarrow \infty} \frac{n^9 - 1}{n^8 + 1} \frac{(n+1)^8 + 1}{(n+1)^9 - 1} = 1.$$

$$(3b) \quad \sum_{n=0}^{\infty} \frac{2^n + 3^n}{4^n + 5^n} x^n$$

$$\hat{R} = \lim_{n \rightarrow \infty} \frac{2^n + 3^n}{4^n + 5^n} \frac{4^{n+1} + 5^{n+1}}{2^{n+1} + 3^{n+1}} = \frac{5}{3}.$$

$$(3c) \quad \sum_{n=0}^{\infty} \frac{(2n)! x^n}{(n!)^2}$$

$$\hat{R} = \lim_{n \rightarrow \infty} \frac{(2n)!}{(n!)^2} \frac{((n+1)!)^2}{(2(n+1))!} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}.$$