Calculate the derivatives of each of the following two functions. Find all of the local and global maximum and minimum values, if there are any, of the functions on the intervals given. Sketch the graphs of the functions in these regions **and explain your results!**

1. $f(x) = \frac{1}{1+\log(x)^2}$ on $(0, \mathrm{e}]$;

   $f(x)$ is positive and vanishes as $x \to +\infty$ and also as $x \to 0+0$. It is not defined at 0 and so has a g.l.b. but not a min. Its only turning point is where $f'(x) = 0$, i.e. at $x = 1$, which is its maximum. It has a local min at $x = \mathrm{e}$, which is an end point of the interval.

2. $f(x) = xe^{-x^2/2}$ on $[0, \infty)$;

   It is positive everywhere on the interval except at $x = 0$, where it vanishes. Thus it has a local and global minimum at $x = 0$. It converges to zero as $x \to +\infty$. It has a maximum where its derivative vanishes, at $x = 1$.

3. Evaluate the following derivative (you must state the conditions under which the result holds)
$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)^2}{g(x)^3}.$$

   The conditions are that f and g should be differentiable at x and that $g(x) \neq 0$. I would expect you to refer the rules for differentiating products, quotients and powers, which have all been proved in the course. The answer is $2f(x)f'(x)g(x)^{-3} - f(x)^2 3g(x)^{-4}g'(x)$.

4. Evaluate the derivatives of the function $f(x) = x^x$ for all real numbers $x > 0$. You will need to use the standard rules for differentiating the exponential and log functions.

   Recall the definition $a^b = \mathrm{e}^{b\log(a)}$. Using the formula $f(x) = \mathrm{e}^{x\log(x)}$ together with the rule for differentiating a function of a function, yields
$$f'(x) = \mathrm{e}^{x\log(x)} \frac{\mathrm{d}}{\mathrm{d}x} \{x\log(x)\} = x^x(\log(x) + 1).$$

5. Calculate the derivative from the left and derivative from the right at $x = 0$ of the function
$$f(x) = \tan(2|x| + x).$$

   If $x \geq 0$ then $f(x) = \tan(3x)$ so the right derivative at 0 is 3. If $x \leq 0$ then $f(x) = \tan(-x)$ so the left derivative at 0 is -1.

6. The function $f(x)$ is defined for all $x \in \mathbf{R}$ by

$$f(x) = \begin{cases} x^2(\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the derivative of $f(x)$ at all non-zero x. Using only the ε, δ definition of differentiation prove that f is differentiable at $x = 0$ and find its derivative at that point. Sketch the graph of the derivative. Is the derivative continuous at $x = 0$?

$x = 0$ is a special point so one must use the definition of differentiation. The bound

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = |x \sin(1/x)| \leq |x|$$

implies that $f'(0) = 0$ by the sandwich theorem. For $x \neq 0$ the rules for differentiation yield

$$f'(x) = 2x \sin(1/x) + x^2 \cos(1/x)(-1/x^2) = 2x \sin(1/x) - \cos(1/x).$$

The derivative oscillates more and more rapidly between ± 1 as $x \to 0$ so the derivative is not continuous at 0.

7. Let $g(x) = \mathrm{e}^{-1/x^2}$ if $x \neq 0$ and $g(0) = 0$. Calculate the first two derivatives of $g(x)$ at $x = 0$. **Explain your answer!**

For $x \neq 0$ this involves using the formula for differentiating a function of a function. $x = 0$ is a special point. By definition,

$$g'(0) = \lim_{x \to 0} \frac{\mathrm{e}^{-1/x^2} - 0}{x - 0} = \lim_{x \to 0} x^{-1} \mathrm{e}^{-1/x^2} = 0.$$

You can prove this using the substitution $1/x^2 = s$ and letting $s \to +\infty$. The treatment of the second derivative is similar.