5CCM221a (CM221A) Real Analysis I
Summer examination, May 2011

Model solutions and marking scheme
(all theoretical questions are coursework)

Section A

Question 1. Let $\{a_n\}$, $n = 1, 2, ...$ be a bounded sequence of real numbers. State precisely, what it means to say that
(a) the sequence $\{a_n\}$ converges to a;
(b) a is an accumulation point of the sequence $\{a_n\}$.
In both cases write down a precise definition, using $\varepsilon > 0$.

(c) Define the upper limit $\limsup a_n$ of the bounded sequence $\{a_n\}$.

Determine whether the following statements are correct. Rigorous proofs are not required, but you should write some words of explanation in each case.

(d) If $\{a_n\}$ converges to a then a is an accumulation point of $\{a_n\}$.

(e) Every sequence has a finite accumulation point.

(f) $\limsup(b\, a_n) = b \limsup a_n$ for all real numbers b and all bounded real sequences $\{a_n\}$.

Solution 1.

(a) $\lim a_n = a$ if for each $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$ for all $n > n_\varepsilon$. **(2)**

(b) a is an accumulation point of $\{a_n\}$ if for every $\varepsilon > 0$ the interval $(a-\varepsilon, a+\varepsilon)$ contains infinitely many elements of $\{a_n\}$. **(2)**

(c) The upper limit is the largest accumulation point. **(2)**

(d) True. If $|a - a_n| < \varepsilon$ for all $n > n_\varepsilon$ then the interval $(a - \varepsilon, a + \varepsilon)$ contains infinitely many elements of $\{a_n\}$. **(2)**

(e) False, $a_n = n$ is a counterexample. **(2)**

(f) False, unless $b \geqslant 0$. If $b < 0$ then $\limsup(b\, a_n) = b \liminf a_n$ **(3)**

Question 2.

(a) Using $\varepsilon > 0$, write down the precise definition of absolute convergence of a series $\sum_{n=1}^{\infty} a_n$ of real numbers.

(b) Write down without proof the ratio test theorem for the series $\sum_{n=1}^{\infty} a_n$.

(c) Write down without proof the integral test theorem for the series $\sum_{n=1}^{\infty} a_n$.

Determine which of the following series converge. You should state which test for convergence you use and explain your answer briefly.

(d) $$\sum_{n=1}^{\infty} \frac{1}{2 + (\sin n)^2 + \cos n}$$

(e) $$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

(f) $$\sum_{n=1}^{\infty} n^{-3/2}$$

Solution 2.

(a) The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if for each $\varepsilon > 0$ there is $m_\varepsilon \in \mathbb{N}$ such that $|\sum_{n=1}^{m} |a_n| - a| < \varepsilon$, $\forall m > m_\varepsilon$, where a is some fixed number **(2)**

(b) Assume that $a_n \neq 0$ and $\lim_{n\to\infty} |a_{n+1}/a_n| \to c$ as $n \to \infty$. If $c < 1$ then the series is absolutely convergent, if $c > 1$ then the series diverges. **(2)**

(c) Let $a_n = f(n)$ where f is a nonincreasing nonnegative function. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,\mathrm{d}x$ is finite. **(3)**

(d) The series diverges because $(2 + (\sin n)^2 + \cos n)^{-1} \not\to 0$ as $n \to \infty$. **(2)**

(e) The series converges by the ratio test. **(2)**

(f) The series converges by the integral test. **(2)**

Question 3.

Let $f : [a, b] \mapsto \mathbb{R}$ be a function, and let $c \in [a, b]$. Using ε and δ, define
(a) the limit $\lim_{x\to c} f(x)$,
(b) the left limit $\lim_{x\to c-0} f(x)$.

(c) What does it mean to say that f is continuous on the half-open interval $(a, b]$?

In each of the following cases, determine whether the statement is true or false. You must write some words of explanation or give a counterexample.

(d) If f is continuous on the closed interval $[a, b]$, $f(a) = 1$ and $f(b) = 2$ then there exists a point $c \in (a, b)$ such that $f(c) = \sqrt{2}$.

(e) If the function f is continuous on the half-open interval $(a, b]$ then either the right limit $\lim_{x\to a+0} |f(x)|$ exists and finite, or $|f(x)| \to \infty$ as $x \to a + 0$.

(f) If the function f is continuous on the closed interval $[a, b]$ then it is bounded.

Solution 3.

(a) $\lim_{x\to c} f(x) = y$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - y| < \varepsilon$ whenever $0 < |x - c| < \delta$. **(2)**

(b) $\lim_{x\to c-0} f(x) = y$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - y| < \varepsilon$ whenever $c - \varepsilon < x < c$. **(2)**

(c) f is continuous on $(a, b]$ if $\lim_{x\to c} f(x) = f(c)$ for all $c \in (a, b)$ and $\lim_{x\to b-0} f(x) = f(b)$. **(2)**

(d) True. This is a particular case of the intermediate value theorem. **(2)**

(e) False, $f(x) = \sin\left((x - a)^{-1}\right)$ is a counterexample. **(2)**

(f) True, it is a theorem. **(2)**

Question 4.

(a) What does it mean to say that a function f defined on an interval (a, b) is differentiable at a point $c \in (a, b)$? What is the derivative $f'(c)$?

(b) State (but do not prove) Rolle's theorem.

In each of the following cases, determine whether the statement is true or false. You must write some words of explanation or give a counterexample.

(c) If a function f is differentiable at every point $x \in (0, 1)$ then it is bounded on $(0, 1)$.

(d) The function $\cos(x^{-1})$ is differentiable on the open interval $(0, 1)$.

(e) If f is a bounded function on (a, b) then there exists a point $x \in (a, b)$ at which f is differentiable.

Solution 4.

(a) f is differentiable at c if the $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. This limit is called the derivative of f at c and is denoted $f'(c)$. **(2)**

(b) If $f : [a, b] \to \mathbb{R}$ is continuous and is differentiable at every $x \in (a, b)$ and $f(a) = f(b) = 0$ then there exists a point $c \in (a, b)$ at which $f'(c) = 0$. **(3)**

(c) False, $f(x) = 1/x$ is a counterexample **(2)**

(d) True. At every point $x > 0$ the derivatives exists and can be calculated with the use of the chain rule. **(2)**

(e) False, $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational,} \end{cases}$ is a counterexample. **(Seen)** **(3)**

Section B

Question 5.

(a) State and prove the nth root test theorem for a number series $\sum_{n=0}^{\infty} b_n$.
You may use the comparison theorem for series without justification.

(b) Using the nth root test theorem, show that for every power series $\sum_{n=0}^{\infty} a_n\, x^n$ there exists $\hat{R}$ such that the series converges for all $x \in (-\hat{R}, \hat{R})$ and diverges for all x with $|x| > \hat{R}$.

Solution 5.

(a) Let $\limsup |b_n|^{1/n} = c$. If $c < 1$ then the series $\sum_{n=1}^{\infty} b_n$ is absolutely convergent. If $c > 1$ then the series diverges. (If $c = 1$ nothing can be said.)

Proof. Assume that $c < 1$. Then there exists a positive number b such that $c < b < 1$. From the definition of the upper limit it follows that $|b_n|^{1/n} \leqslant b$ for all sufficiently large n or, in other words, $|b_n| \leqslant b^n$ for all $n \geqslant m$, where m is some positive integer. Since the series $\sum_{n=1}^{\infty} b^n$ converges, by the Comparison Theorem the series $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

Assume now that $c > 1$. Then there exists a positive number b such that $1 < b < c$. From the definition of the upper limit it follows that $|b_n|^{1/n} \geqslant b$ for infinitely many values of n. In other words, there exist positive integers $n_1 < n_2 < n_3 \dots$ such that $|b_{n_k}| \geqslant b^{n_k}$. Since $n_k \to \infty$ and, therefore, $b^{n_k} \to \infty$ as $k \to \infty$, we see that b_n do not converge to 0 as $n \to \infty$. This implies that the series $\sum_{n=1}^{\infty} b_n$ diverges. **(18)**

(b) Define $\hat{R} = \left(\limsup |a_n|^{1/n}\right)^{-1}$. Then the series $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent for $|x| < \hat{R}$ and is divergent for $|x| > \hat{R}$.

Indeed, if $b_n := a_n\, x^n$ then $\limsup |b_n|^{1/n} = \limsup(|a_n|^{1/n}\, |x|) = C\, |x|$ where $C := \limsup |a_n|^{1/n}$. Clearly, $\limsup |b_n|^{1/n} < 1$ if and only if $|x| < C^{-1}$, and $\limsup |b_n|^{1/n} > 1$ if and only if $|x| > C^{-1}$. Therefore the statement follows from the nth root test. **(7)**

Question 6.

Let f be a continuous function on a closed bounded interval $[a, b]$.

(a) Prove that f is bounded.

(b) State (but do not prove) the maximum and minimum theorems for f.

(c) State and proof the intermediate value theorem for f.

Any results on sequences and the nested intervals theorem may be used without justification.

Solution 6.

(a) If f were unbounded then there would exist a sequence $x_n \in [a, b]$ such that $|f(x_n)| \geqslant n$ for all n. Since $\{x_n\}$ is bounded, by the Bolzano-Weierstrass Theorem it has a subsequence $\{x_{n_k}\}$ which converges to a limit $c \in [a, b]$ as $k \to \infty$.
Since f is continuous, we must have $f(x_{n_k}) \to f(c)$ as $k \to \infty$. However, $f(c)$ is a finite number and $|f(x_{n_k})| \geqslant n_k \to \infty$ as $k \to \infty$. The obtained contradiction proves the theorem. **(8)**

(b) The function f attains its maximum and minimum values. **(2)**

(c) If $f(a) \leqslant d \leqslant f(b)$, then there exists $c \in [a, b]$ such that $f(c) = d$.
Proof. We proceed by subdividing the interval into halves repeatedly, starting with $[a_1, b_1] = [a, b]$. The idea is to construct intervals $[a_n, b_n]$ such that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all n and $b_{n+1} - a_{n+1} = (b_n - a_n)/2$ and $f(a_n) \leqslant d \leqslant f(b_n)$ for all n.

This is done inductively. If it has been done for some n then we make the next subdivision as follows. If $f((a_n + b_n)/2) \geqslant d$ then put $[a_{n+1}, b_{n+1}] = [a_n, (a_n + b_n)/2]$. Otherwise $f((a_n + b_n)/2) < d$ and we put $[a_{n+1}, b_{n+1}] = [(a_n + b_n)/2, b_n]$.

Now, by the nested intervals theorem, there exists $c \in [a, b]$ such that $a_n \to c$ and $b_n \to c$. Also, we have $f(a_n) \leqslant d \leqslant f(b_n)$ for all n. Using the continuity of f at c, we conclude that $f(c) = \lim_{n\to\infty} f(a_n) \leqslant d \leqslant \lim_{n\to\infty} f(b_n) \leqslant f(c)$, so $f(c) = d$. **(15)**

Question 7. Let f be a function on a closed interval $[a, b]$.

(a) What does it mean to say that f has a local maximum at a point $c \in (a, b)$? What does it mean to say that f has a local minimum at a point $c \in (a, b)$?

(b) Let f have a local maximum or a local minimum at $c \in (a, b)$, and let f be differentiable at c. Prove that $f'(c) = 0$.

(c) Does the equality $f'(c) = 0$ imply that f has a local minimum or a local maximum at c? Justify your statement or give a counterexample.

(d) Find the maximum and minimum values of the function $f(x) = (x-1)^2 e^x$ on the interval $[-2, 2]$.

Solution 7.

(a) f has a local maximum at a point $c \in (a, b)$ if there exists $\varepsilon > 0$ such that $f(c) \geqslant f(x)$ for all $x \in (c-\varepsilon, c+\varepsilon)$. Similarly, f has a local minimum at $c \in (a, b)$ if there exists $\varepsilon > 0$ such that $f(c) \leqslant f(x)$ for all $x \in (c-\varepsilon, c+\varepsilon)$. **(4)**

(b) If f has a local maximum at c then there exists $\varepsilon > 0$ such that $\frac{f(x)-f(c)}{x-c} \geqslant 0$ for all $x \in (c-\varepsilon, c)$ and $\frac{f(x)-f(c)}{x-c} \leqslant 0$ for all $x \in (c, c+\varepsilon)$. Therefore, in the definition of the derivative, the left limit is nonnegative and the right limit is nonpositive. Since f is differentiable, both these limits exist and coincide. This implies that they are equal to zero. The corresponding result for a local minimum is obtained in a similar way (or by applying the local maximum result to the function $g(x) = -f(x)$). **(8)**

(c) No. A counterexample is $f(x) = x^3$. **(3)**

(d) We have $f'(x) = (x^2 - 2x + 1 + 2x - 2)\, e^x = (x^2 - 1)\, e^x$, so that $f'(x) = 0$ at $x = \pm 1$. By direct calculation, $f(-1) = 4\, e^{-1}$, $f(1) = 0$, $f(2) = e^2$ and $f(-2) = 9\, e^{-2}$. Comparing these values, we see that the maximum value is e^2 and the minimum value is 0. **(10)**