FINDING MAXIMAL AND MINIMAL VALUES

Definition. Let f be a function defined on an interval (a,b). We say that f has a local maximum at a point $c \in (a,b)$ if there exists $\varepsilon > 0$ such that $f(c) \geqslant f(x)$ for all $x \in (c-\varepsilon, c+\varepsilon)$. Similarly, f has a local minimum at $c \in (a,b)$ if there exists $\varepsilon > 0$ such that $f(c) \leqslant f(x)$ for all $x \in (c-\varepsilon, c+\varepsilon)$.

Theorem. Let f be a differentiable function on the interval (a,b). If f has a local maximum or a local minimum at $c \in (a,b)$ then $f'(c) = 0$.

Proof. If f has a local maximum at c then there exists $\varepsilon > 0$ such that $\frac{f(x)-f(c)}{x-c} \geqslant 0$ for all $x \in (c-\varepsilon, c)$ and $\frac{f(x)-f(c)}{x-c} \leqslant 0$ for all $x \in (c, c+\varepsilon)$. Therefore, in the definition of the derivative, the left limit is nonnegative and the right limit is nonpositive. Since f is differentiable, both these limits exist and coincide. This implies that they are equal to zero. The corresponding result for a local minimum is obtained in a similar way (or by applying the local maximum result to the function $g(x) = -f(x)$).

A function may have several local maxima and minima. For example, $f(x) = \cos x$ has local maxima at the points $x = 2\pi n$ and local minima at the points $\pi + 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$

It may well happen that $f'(c) = 0$ but c is not a local mimimum or local maximum of f. Such a point c is called a *saddle* point.

Example. Let $f(x) = x^3$ then $f'(x) = 3x^2 = 0$ at $x = 0$. However, the function f does not have a local minimum or a local maximum at this point.

Finding the maximum value of a differentiable function on an interval. The maximal value of a function f on an interval I either coincides with a local maxima or is attained at an end point of the interval. In order to find it, one has to do the following:
(a) to find all points $c_1, c_2, \ldots \in I$ at which $f'(c_k) = 0$;
(b) to evaluate $f(c_k)$;
(c) to evaluate f at the end points of the interval **if they are included in I;**
(d) to select the point at which f takes the maximal value. This may be either one of the points c_k, or an end point of the interval.

In a similar way one can find the minimum value. **Do not forget (c) and (d)!**

Warning: the equality $f'(c) = 0$ does not imply that $f(c)$ is the maximal (or minimal) value of f. It may well happen that c is a local maximum (or minimum), or that the value of f at an end point is greater (or smaller) than $f(c)$.

MEAN VALUE THEOREMS

Rolle's Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous and it is differentiable at every $x \in (a, b)$ and $f(a) = f(b) = 0$ then there exists a point c in (a, b) at which $f'(c) = 0$.

Proof. We know that a continuous function on a bounded closed interval attains its maximum and minimum values. If both these values are zero, the function is identically equal to zero and $f' = 0$ everywhere. If one of these values is not zero and is attained at the point c then $c \in (a, b)$ and, by the previous theorem $f'(c) = 0$.

Mean Value Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous and it is differentiable at every $x \in (a, b)$ then there exists a point c in (a, b) at which $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Consider the function

$$g(x) \;=\; f(x) - f(a) - \frac{f(b) - f(a)}{b - a}\,(x - a)\,.$$

Since f is continuous on $[a, b]$ and differentiable on (a, b), the same is true about g. Also, $g(a) = g(b) = 0$. Applying Rolle's Theorem to g, we obtain the required result.

Corollary. If f is differentiable on an interval (a, b) and $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on (a, b).

Proof. Let $a_1, b_1 \in (a, b)$ and $a_1 < b_1$. Applying Mean Value Theorem to the interval $[a_1, b_1]$, we obtain $\frac{f(b_1)-f(a_1)}{b_1-a_1} = 0$, that is, $f(b_1) = f(a_1)$. Since this is true for all $a_1, b_1 \in (a, b)$, the function f is constant.

Cauchy's Mean Value Theorem. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose further that g' is never zero on (a, b). Then there is some $c \in (a, b)$ such that

$$\frac{f(a) - f(b)}{g(a) - g(b)} \;=\; \frac{f'(c)}{g'(c)}\,.$$

Proof. Note first of all that $g(a) - g(b) \neq 0$. Indeed, if $g(a) = g(b)$ then, by Rolle's theorem, $g'(x) = 0$ at some point $x \in (a, b)$.

Let $\varphi(x) = (g(b) - g(a))\; f(x) - (f(b) - f(a))\; g(x) + f(b)g(a) - f(a)g(b)$. Then $\varphi(a) = \varphi(b) = 0$ and φ satisfies the conditions of Rolle's theorem. Thus there exists $c \in (a, b)$ such that

$$\varphi'(c) = (g(b) - g(a))\; f'(c) - (f(b) - f(a))\; g'(c) = 0\,.$$

This implies the required result.

TAYLOR'S THEOREM

Definition. We say that f is n times differentiable on (a, b) if each derivative of order up to n exists at every point of the interval (the derivative of order two is the derivative of the first derivative, the derivative of order three is obtained by differentiation the derivative of order three and so on). We say that it is n times continuously differentiable if the final derivative is continuous (the function and its first $(n-1)$ derivatives are automatically continuous). If the interval is $[a, b]$ then we require the one-sided derivatives of all orders up to n to exist at the end-points if the interval. The usual notation for the derivative of order n is $f^{(n)}$, so that $f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(n-1)}(x)$.

Theorem (Taylor's formula). If f is n times continuously differentiable on the interval $(a-\varepsilon, a+\varepsilon)$ then, for each $x \in (a-\varepsilon, a+\varepsilon)$, there exists a point c lying between a and x (that is, $c \in (a, x)$ if $x > a$ and $c \in (x, a)$ if $x < a$), such that

$$f(x) = f(a) + f'(a)\,(x-a) + f^{(2)}(a)\,\frac{(x-a)^2}{2!} + f^{(3)}(a)\,\frac{(x-a)^3}{3!} + \cdots + f^{(n-1)}(a)\,\frac{(x-a)^{n-1}}{(n-1)!} + R_n\,,$$

where $R_n = f^{(n)}(c)\,\dfrac{(x-a)^n}{n!}$.

Remark. Sometimes it is more convenient to write down Taylor's formula as

$$f(a+h) = f(a) + f'(a)\,h + f^{(2)}(a)\,\frac{h^2}{2!} + f^{(3)}(a)\,\frac{h^3}{3!} + \cdots + f^{(n-1)}(a)\,\frac{h^{n-1}}{(n-1)!} + R_n\,,$$

where $R_n = f^{(n)}(c)\,\dfrac{h^n}{n!}$ and c is a point lying between a and $a+h$.

Proof of Taylor's formula. Let $g(x) = (x-a)^m$ and

$$\tilde{f}(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(a)\,\frac{(x-a)^k}{k!}\,.$$

Note that the functions $\tilde{f}$ and g and their first $(n-1)$ derivatives are equal to zero at the point $x = a$. Therefore, applying Cauchy's Mean Value Theorem, we obtain

$$\frac{\tilde{f}(x)}{g(x)} = \frac{\tilde{f}(x) - \tilde{f}(a)}{g(x) - g(a)} = \frac{\tilde{f}'(c_1)}{g'(c_1)} = \frac{\tilde{f}'(c_1) - f'(a)}{g'(c_1) - g'(a)} = \frac{\tilde{f}^{(2)}(c_2)}{g^{(2)}(c_2)}$$
$$= \frac{\tilde{f}^{(2)}(c_2) - \tilde{f}^{(2)}(a)}{g^{(2)}(c_2) - g^{(2)}(a)} = \frac{\tilde{f}^{(3)}(c_3)}{g^{(3)}(c_3)} = \cdots = \frac{\tilde{f}^{(n-1)}(c_{n-1}) - \tilde{f}^{(n-1)}(a)}{g^{(n-1)}(c_{n-1}) - g^{(n-1)}(a)} = \frac{\tilde{f}^{(n)}(c_n)}{g^{(n)}(c_n)}\,,$$

where $c_1 \in (a, x)$, $c_2 \in (a, c_1)$, $c_3 \in (a, c_2)$, $\ldots$, $c_n \in (a, c_{n-1}) \subset (a, x)$ if $x > a$, and $c_1 \in (x, a)$, $c_2 \in (c_1, a)$, $c_3 \in (c_2, a)$, $\ldots$, $c_n \in (c_{n-1}, a) \subset (x, a)$ if $x < a$.
Since $g^{(n)}(x) = n!$ and $\tilde{f}^{(n)}(x) = f^{(n)}(x)$ for all x, we get $\tilde{f}(x) = \frac{g(x)}{n!}\,f^{(n)}(c_n)$. This is equivalent to Taylor's formula with $c = c_n$.

Remark. Note that the mean value theorem is a particular case of Taylor's theorem corresponding to $n = 1$.