DIFFERENTIATION

Let f be a function defined on an interval I. We say that $f'(x) = k$ if

$$\forall\varepsilon > 0\ \exists\,\delta_{x,\varepsilon} > 0 :\ 0 < |x-y| < \delta_{x,\varepsilon} \Rightarrow \left|\frac{f(x)-f(y)}{x-y} - k\right| < \varepsilon\,.$$

Equivalently

$$\forall\varepsilon > 0\ \exists\delta_{x,\varepsilon} > 0 :\ 0 < |\delta| < \delta_{x,\varepsilon} \Rightarrow \left|\frac{f(x+\delta)-f(x)}{\delta} - k\right| < \varepsilon.$$

and

$$\lim_{y\to x}\frac{f(x)-f(y)}{x-y} = k.$$

The number k is called the *derivative of f at the point x.* If $f'(x)$ exists for every point x from the interval, we can consider f' as a function of the variable x. This function is called *the derivative of f.* Another notation for the derivative is $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$.

It may well happen that the limit does not exist, in which case we say that f is not differentiable at x.
The right and left limits

$$\lim_{y\to x+0}\frac{f(x)-f(y)}{x-y} \quad \text{and} \quad \lim_{y\to x-0}\frac{f(x)-f(y)}{x-y}$$

(if they exist) are said to be the right and left derivatives of f at the point x. The function f is differentiable if both the right and left derivative exist and have the same value. If x is an end point of the interval I then one can speak only about one of these derivatives (the other limit does not make sense).

Theorem. If the derivative $f'(x)$ exists then f is continuous at x.
Proof. If f is differentiable at x then there exists a number $\delta_{x,1} > 0$ such that $\left|\frac{f(x)-f(y)}{x-y} - k\right| < 1$ whenever $0 < |x-y| < \delta_{x,1}$ (because we can take $\varepsilon = 1$ in the definition). Since $\left|\frac{f(x)-f(y)}{x-y} - k\right| \leqslant \left|\frac{f(x)-f(y)}{x-y}\right| + |k|$, this implies that $\left|\frac{f(x)-f(y)}{x-y}\right| < |k| + 1$. Multiplying both parts of this inequality by $|x-y|$, we see that $|f(x)-f(y)| < (|k|+1)|x-y|$ whenever $0 < |x-y| < \delta_{x,1}$.
Let us fix an arbitrary $\varepsilon > 0$ and choose $\delta > 0$ so small that $\delta < \delta_{x,1}$ and $\delta < ((|k|+1))^{-1}\varepsilon$. Then, by the above, we have $|f(x)-f(y)| < \varepsilon$ whenever $|x-y| < \delta$. This means that f is continuous at x.

The converse is not true; a continuous function may not be differentiable.

Example. The function $f(x) = |x|$ is continuous at the point $x = 0$ because $|f(0) - f(y)| = |y| \to 0$ as $y \to 0$. However,

$$\left| \frac{f(0+\delta) - f(0)}{\delta} - k \right| = \left| \frac{|\delta|}{\delta} - k \right| .$$

If $\delta < 0$ then the right hand side is equal to $|k+1|$. If $\delta > 0$ then the right hand side is equal to $|1-k|$. Clearly, for all sufficiently small ε at least one of these numbers is greater than ε. Therefore the derivative at $x = 0$ does not exist.

Theorem. Let f and g be differentiable functions. Then
(a) $\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = f'(x) + g'(x)$;
(b) $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$;
(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ provided that $g(x) \neq 0$.

Proofs.

(a) $\lim_{y\to x} \frac{f(x)+g(x)-(f(y)+g(y))}{x-y} = \lim_{y\to x}\left(\frac{f(x)-f(y)}{x-y} + \frac{f(x)-f(y)}{x-y}\right)$
$= \lim_{y\to x} \frac{f(x)-f(y)}{x-y} + \lim_{y\to x} \frac{g(x)-g(y)}{x-y} = f'(x) + g'(x)$
because the limit of the sum is equal to the sum of limits.

(b) $\lim_{y\to x} \frac{f(x)g(x)-f(y)g(y)}{x-y} = \lim_{y\to x}\left(\frac{(f(x)-f(y))g(x)}{x-y} + \frac{f(y)(g(x)-g(y)}{x-y}\right)$
$\lim_{y\to x} \frac{(f(x)-f(y))g(x)}{x-y} + \lim_{y\to x} \frac{f(y)(g(x)-g(y))}{x-y}$
$= \left(\lim_{y\to x} g(x)\right)\left(\lim_{y\to x} \frac{f(x)-f(y)}{x-y}\right) + \left(\lim_{y\to x} f(y)\right)\left(\lim_{y\to x} \frac{g(x)-g(y)}{x-y}\right)$
$= f'(x)g(x) + f(x)g'(x)$
because the limits of the products are equal to the products of limits and $\lim_{y\to x} f(y) = f(x)$ as f is continuous.

(c) $\lim_{y\to x} \frac{f(x)g^{-1}(x)-f(y)g^{-1}(y)}{x-y} = \lim_{y\to x} \frac{f(x)g(y)-f(y)g(x)}{(x-y)g(x)g(y)}$
$= \lim_{y\to x} \frac{(f(x)-f(y))g(x)-f(x)(g(x)-g(y))}{(x-y)g(x)g(y)}$
$\lim_{y\to x}\left(\frac{1}{g(x)g(y)}\right)\left(\lim_{y\to x} \frac{(f(x)-f(y))g(x)}{(x-y)} - \lim_{y\to x} \frac{f(x)(g(x)-g(y))}{(x-y)}\right)$
$= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$
because $\lim_{y\to x}\left(\frac{1}{g(x)g(y)}\right) = \frac{1}{\lim_{y\to x} g(x)g(y)} = \frac{1}{g^2(x)}$ as g is continuous.

Example. The derivative of a constant function is identically equal to zero because in this case $f(x) - f(y) = 0$ for all x and y. This fact and the part (b) imply that $\frac{\mathrm{d}}{\mathrm{d}x}(c\,g(x)) = c\,g'(x)$ for any constant c.

Example. If n is a positive integer then $\frac{\mathrm{d}}{\mathrm{d}x}x^n = \lim_{x\to y}\frac{x^n-y^n}{x-y}$. Substituting the expansion $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ (see Sheet 1) and passing to the limit, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = \lim_{y\to x}(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$$
$$= (x^{n-1} + x^{n-2}x + \cdots + xx^{n-2} + x^{n-1}) = nx^{n-1}\,.$$

Example. $\frac{\mathrm{d}}{\mathrm{d}x}e^x = \lim_{y\to x}\frac{e^x-e^y}{x-y} = e^x \lim_{y\to x}\frac{e^{y-x}-1}{y-x}$. The exponential function is given by the series $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$ (take it for granted). This implies that $\frac{e^{y-x}-1}{y-x} = 1 + \frac{y-x}{2!} + \frac{(y-x)^2}{3!} + \cdots = 1 + (y-x)F(y-x)$
where $F(y-x) = \frac{1}{2!} + \frac{y-x}{3!} + \frac{(y-x)^2}{4!} + \ldots$ The ratio test shows that this series absolutely convergent for all x and y. If $|x-y| < 1$ then
$|F(y-x)| \leqslant \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ because the modulus of each partial sum of the series defining F is estimated by the corresponding partial sum of the series on the right hand side. Thus F is estimated by some constant C whenever $|y-x| < 1$. This implies that $\left|\frac{e^{y-x}-1}{y-x} - 1\right| < C\,|x-y|$ for all y such that $|y-x| < 1$. Since $|x-y| \to 0$ as $y \to x$, we obtain $\lim_{y\to x}\left|\frac{e^{y-x}-1}{y-x} - 1\right| = 0$. Consequently, $\lim_{y\to x}\frac{e^{y-x}-1}{y-x} = 1$ and $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$.

Example. In a similar manner, using power expansions for $\sin x$ and $\cos x$ and the trigonometric formulae for $(\sin x - \sin y)$ and $(\cos x - \cos y)$, one can show that $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$.

Chain rule (differentiating a function of a function). If $f(x)$ is differentiable at $x = c$ and $g(y)$ is differentiable at $y = f(c)$ then $g(f(x))$ is differentiable at $x = c$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}g(f(x))|_{x=c} = g'(f(c))f'(c).$$

Proof. By definition,

$$\lim_{x\to c}\frac{g(f(x)) - g(f(c))}{x-c} = \lim_{x\to c}\left(\frac{g(f(x)) - g(f(c))}{f(x)-f(c)}\;\frac{f(x)-f(c)}{x-c}\right)$$
$$\left(\lim_{x\to c}\frac{g(f(x)) - g(f(c))}{f(x)-f(c)}\right)\left(\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right) = \left(\lim_{x\to c}\frac{g(f(x)) - g(f(c))}{f(x)-f(c)}\right)f'(c)\,.$$

Since f is continuous at c, we have $f(x) \to f(c)$ as $x \to c$. Therefore

$$\lim_{x\to c}\frac{g(f(x)) - g(f(c))}{f(x)-f(c)} = \lim_{f(x)\to f(c)}\frac{g(f(x)) - g(f(c))}{f(x)-f(c)} = g'(f(c))\,.$$

Example. Let $f(x) = \ln x$ where $x > 0$. By definition, $g(f(x)) = x$ for all x where $g(y) = e^y$. Since $(e^y)' = e^y$, differentiation this identity and applying the previous theorem, we obtain $\frac{\mathrm{d}}{\mathrm{d}x} e^{\ln x} = (\ln x)' e^{\ln x} = (\ln x)' x = 1$. Thus we have $(\ln x)' = x^{-1}$.