There will be no lectures on the week starting 7 November. Use the opportunity to revise the material covered in lectures. Some of you may need to go through solutions to the last year class test and exercise sheets to see what you got wrong and why.

DO NOT LEAVE THIS UNTIL THE SPRING VACATION!

CONTINUOUS FUNCTION II

Recall that intervals of the form (a, b), $(-\infty, b)$, $(a, +\infty)$ and $(-\infty, +\infty)$ are said to be open. The intervals $[a, b]$, $(-\infty, b]$, $[a, +\infty)$ and $(-\infty, +\infty)$ are said to be closed. Note that the interval $(-\infty, +\infty)$ is both open and closed.

The following four theorems are only true for functions that are continuous on **closed and bounded intervals** $[a, b]$.

Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous function on the closed bounded interval $[a, b]$ then f is bounded.

Proof. If f were unbounded then there would exist a sequence $x_n \in [a, b]$ such that $|f(x_n)| \geqslant n$ for all n. Since the sequence $\{x_n\}$ is bounded, by the Bolzano–Weierstrass theorem it has a subsequence $\{x_{n_k}\}$ which converges to a limit $c \in [a, b]$ as $k \to \infty$. Since f is continuous, we must have $f(x_{n_k}) \to f(c)$ as $k \to \infty$. However, $f(c)$ is a finite number and $|f(x_{n_k})| \geqslant n_k \to \infty$ as $k \to \infty$. The obtained contradiction proves the theorem.

Example. The function $f(x) = x$ is continuous on the closed interval $[1, \infty)$ but is not bounded. The function $f(x) = x^{-1}$ is continuous on the bounded half-open interval $(0, 1]$ but is not bounded.

Maximum Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous function on the closed bounded interval $[a, b]$ then it attains a maximum value.

Proof. Since f is bounded, its range has the least upper bound $M = \sup f$. Then, for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n) - M| < n^{-1}$ (otherwise M would be separated from the range of f and would not be its **least** upper bound). The sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to a limit $c \in [a, b]$ as $k \to \infty$. Since f is continuous, we have $f(c) = \lim_{k\to\infty} f(x_{n_k}) = M$.

Minimum Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous function on the closed bounded interval $[a, b]$ then it attains a minimum value.

Proof. If $m = \inf f$ then $-m = \sup(-f)$. By the maximum theorem, the continuous function $-f$ attains the maximum value $-m$. This means that $-f(x) = -m$ and, consequently, $f(x) = m$ for some $x \in [a, b]$.

Example. The function $f(x) = (1 - x)\sin(1/x)$ is continuous on the bounded half-open interval $(0, 1]$ but does not attain the minimum and maximum values ± 1. The function $f(x) = x^{-1}$ is continuous on the closed unbounded interval $[1, +\infty)$ but does not attain the minimum value 0.

Intermediate Value Theorem. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed bounded interval $[a, b]$, and let d be a number lying between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = d$.

Proof. Assume that $f(a) \leqslant f(b)$, so that $f(a) \leqslant d \leqslant f(b)$. Let S be the set of numbers $x \in [a, b]$ such that $f(x) \leqslant d$. The set S is bounded from above by b and is not empty because $a \in S$. By the completeness axiom, it has the least upper bound $c = \sup S$. In other words, c is the smallest number in the interval $[a, b]$ such that $f(x) > d$ for all $x > c$.

If $f(c) < d$ then $c < b$ and $f(c) < d - \varepsilon$ for some $\varepsilon > 0$. By continuity, there exists $\delta > 0$ such that $c + \delta < b$ and $f(x) < f(c) + \varepsilon < d$ for all $x \in [c, c + \delta]$. In this case c is not an upper bound for S (since $c + \delta \in S$).

If $f(c) > d$ then $c > a$ and $f(c) > d + \varepsilon$ for some $\varepsilon > 0$. By continuity, there exists $\delta > 0$ such that $c - \delta > a$ and $f(x) > f(c) - \varepsilon > d$ for all $x \in [c - \delta, c]$. In this case c is not the **least** upper bound for S (since $c - \delta$ is also an upper bound).

Assume that $f(a) \geqslant f(b)$, so that $f(b) \leqslant d \leqslant f(a)$. Denote $g = -f$. Then $g(a) \leqslant g(b)$ and $g(a) \leqslant -d \leqslant g(b)$. Therefore, by the above, there exists a point $c \in [a, b]$ such that $g(c) = -d$ and $f(c) = d$.

Corollary. If $f : [a, b] \to \mathbb{R}$ is continuous function on the closed bounded interval $[a, b]$ then its range $f([a, b])$ is the closed interval $[m, M]$ where $m = \inf f$ and $M = \sup f$.

Proof. By the maximum and minimum theorems, there exist points $x_1 \in [a, b]$ and $x_2 \in [a, b]$ such that $f(x_1) = m$ and $f(x - 2) = M$. Applying the intermediate value theorem to the interval $[x_1, x_2]$ we see that f takes all possible values between m and M.

Example. The equation $(x^2 + 1) \sin x = 1$ has a solution in the interval $[0, \pi/2]$. Indeed, $(x^2 + 1)(\sin x) - 1 = -1$ at the point $x = 0$ and $(x^2 + 1)(\sin x) - 1 = \pi^2/4$ at the point $x = \frac{\pi}{2}$. Since the function $(x^2 + 1)(\sin x) - 1$ is continuous, it takes all intermediate values (including 0) on the interval $[0, \pi/2]$.

We know that sums, products and quotients of continuous functions are continuous functions. It turns out that the composition of two continuous functions is also continuous.

Theorem. Let f be a continuous function on a closed interval $[a, b]$, and let $[m, M]$ be its range. If g is a continuous function on $[m, M]$ then the composition $g \circ f(x) = g(f(x))$ is a continuous function on $[a, b]$.

Proof. Let $c \in [a, b]$ and $x_n \in [a, b]$ be an arbitrary sequence which converges to c. Then, since f is continuous, $f(x_n)$ converges to $f(c) \in [m, M]$. Now, since g is continuous, $g(f(x_n))$ converges to $g(f(c))$. Thus we have $g \circ f(x_n) \to g \circ f(c)$ for every sequence $x_n \to c$. In other words, $g \circ f(x) \to g \circ f(c)$ as $x \to c$, which means that the function $g \circ f(x)$ is continuous.

Example. $\sin(1 + x^3)$ is continuous at every point $x \in \mathbb{R}$ because $1 + x^3$ is continuous at all points x and $\sin y$ is continuous at all points y.