FUNCTIONS OF A REAL VARIABLE

Let $\Omega \subseteq \mathbb{R}$ be a nonempty subset of the real line. A (real-valued) function f on Ω is a mapping $\Omega \to \mathbb{R}$, that is, an association $x \mapsto f(x)$ of each element x of Ω to some real number $f(x)$ which is called the value of the function f at the point x. Further on, we shall usually be assuming that Ω is a nondegenerate interval (which may coincide with $\mathbb{R}$).

The set Ω is called the *domain of definition* of the function. The set of all its values $f(x)$ (when x runs over Ω) is said to be the *range* of f. Finally, the set of points $(x,y) \in \mathbb{R}^2$ such that $y = f(x)$ is called the *graph* of the function f. The graph can be thought of as a curve line in the two dimensional space $\mathbb{R}^2$ whose intersection with every vertical straight line passing through $(x,0)$ consists of one point with coordinates $(x, f(x))$. Every function is uniquely defined by its graph.

Some functions are given by "nice" explicit formulae. For example, $f(x) = x^2+x+1$ is a function on the whole real line, $f(x) = (\sqrt{x})^{-1}$ is a real-valued function on the positive half-line $(0,+\infty)$, and so on. But it is not always the case. For instance,

$$f(x) \;=\; \begin{cases} 1\,, & \text{if } x \text{ is rational,} \\ 0\,, & \text{if } x \text{ is irrational,} \end{cases}$$

is a well defined function on the real line (do not try to sketch its graph, it is impossible!). Another important example is the function

$$f(x) \;=\; \begin{cases} 1\,, & \text{if } x \in A \\ 0\,, & \text{if } x \notin A, \end{cases}$$

where A is an arbitrary subset of $\mathbb{R}$. This function is called the *characteristic function* of the set A and is often denoted by χ_A.

BOUNDED FUNCTIONS

Definition. We say that a function is bounded if its range (the set of its values) is a bounded subset of $\mathbb{R}$.

If a function f is bounded then its range has finite g.l.b. m and l.u.b. M and $m \leqslant f(x) \leqslant M$ for all x from the domain of definition. The numbers m and M are usually denoted $\inf f$ and $\sup f$ and are called the greatest lower bound and the least upper bound of the function f, respectively.

If $f(x) = m$ for some x from the domain of definition, one says that f attains its minimum value m. If $f(x) = M$ for some x from the domain of definition, one says that f attains its maximum value M. It is not always the case.

Example. The function $f(x) = x^{-1}$ on the interval $[1,\infty)$ attains the maximum value 1 but does not take the value $\inf f = 0$.

Example. The function $f(x) = (1-x)\sin(1/x)$ is bounded on $(0,1]$ with $\inf f = -1$ and $\sup f = 1$. However, it does not take the values ± 1 anywhere, and there is no obvious way of defining it at $x = 0$.

CONTINUOUS FUNCTIONS

Definition. Let f be a function on an interval I (which may coincide with $\mathbb{R}$), and let $c \in I$.

$\lim_{x\to c} f(x) = y$ means that $\forall\varepsilon > 0\, \exists\delta_\varepsilon > 0 \,:\, 0 < |x-c| < \delta_\varepsilon \Rightarrow |f(x) - y| < \varepsilon$.

$\lim_{x\to c+0} f(x) = u$ means that $\forall\varepsilon > 0\, \exists\delta_\varepsilon > 0 \,:\, c < x < c + \delta_\varepsilon \Rightarrow |f(x) - u| < \varepsilon$.

$\lim_{x\to c-0} f(x) = v$ means that $\forall\varepsilon > 0\, \exists\delta_\varepsilon > 0 \,:\, c - \delta_\varepsilon < x < c \Rightarrow |f(x) - v| < \varepsilon$.

Here y, u, v are some real numbers. The number y is called the limit (or limit value) of f at the point c. The numbers u and v are said to be the right and the left limit, respectively.

Note that the limit of f at a given point c may not exist. Even if the limit exists, it may not coincide with the value of f at the point c. The same is true about the right and left limits.

It is clear from the definition that $\lim_{x\to c} f(x)$ exists and is equal to y if and only if the right and the left limits exist and $u = v = y$. The right limit is not defined at the right end point of I, and the left limit is not defined at the left end point.

Definition of continuity. We say that f is continuous at an interior point c of the interval I if $\lim_{x\to c} f(x) = f(c)$ or equivalently, if
$\forall\varepsilon > 0\, \exists\delta_\varepsilon > 0 \,:\, |x-c| < \delta_\varepsilon \Rightarrow |f(x) - f(c)| < \varepsilon\,.$

If the interval contains the right (left) end point c, we say that f is continuous at c if the left (right) limit of f at the point c exists and coincides with $f(c)$.

A function f defined on an interval I is continuous if $\lim_{x\to c} f(x) = f(c)$ for all $c \in I$. If c is an end point of I and c is included in I then we consider the right or the left limit at c. If c does not belong to I, we do not impose any condition on the right or left limit of $f(x)$ at c.

Remark. Clearly, if f is continuous on I then it is also continuous on any smaller interval $I_1 \subset I$.

Remark. Roughly speaking, continuity means that the graph is a continuous line. But this is not a proper definition, since the notion of a "continuous line" is introduced with the use of the above definition of continuity.

Example. $f(x) = \sqrt{x}$ is a continuous function on the nonnegative half-line $[0,+\infty)$. Indeed, if $c = 0$ then $|f(x) - f(c)| = \sqrt{x} < \varepsilon$ whenever $|x - c| = x < \varepsilon^2$, that is, we can take $\delta = \varepsilon^2$. If $c \neq 0$ then

$$\sqrt{x} - \sqrt{c} = (\sqrt{x} + \sqrt{c})^{-1}(x - c) \leqslant c^{-1/2}(x - c).$$

In this case $|\sqrt{x} - \sqrt{c}| < \varepsilon$ whenever $|x - c| < c^{1/2}\varepsilon$, that is, we can take $\delta = c^{1/2}\varepsilon$.

Example. The function $f(x) = 1/x$ is continuous on the half-open interval $(0, 1]$ but is not defined on $[0, 1]$. It is impossible to extend f to a continuous function on $[0, 1]$

Example. Let $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = a$ where a is some real number. Then, for each $a \in \mathbb{R}$, the function $f(x)$ is not continuous at $x = 0$. Indeed, for any fixed number $b \in [-1, 1]$, any open interval about the origin contains infinitely many points at which f takes the value b.

Theorem. A function f is continuous at c if and only if for every sequence of points $x_n \in I$ such that $\lim_{n\to\infty} x_n = c$ we have $\lim_{n\to\infty} f(x_n) = f(c)$.

Proof. Assume first that f is continuous at c, and let $\{x_n\}$ be a sequence which converges to c. In view of the former, for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta_\varepsilon$. On the other hand, since $x_n \to c$, there exists n_ε such that $|x_n - c| < \delta_\varepsilon$ for all $n \geqslant n_\varepsilon$. Then $|f(x_n) - f(c)| < \varepsilon$ for all $n \geqslant n_\varepsilon$. By definition of convergence, this means that $f(x_n) \to f(c)$.

Assume now that f is not continuous at c. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ we can find x'_δ such that $|x'_\delta - c| < \delta$ but $|f(x'_\delta) - f(c)| \geqslant \varepsilon$. Let us take $\delta = 1/n$ and denote by x_n the corresponding x'_δ. Then $x_n \to c$ because $|x_n - c| < 1/n \to 0$. On the other hand, $|f(x_n) - f(c)| \geqslant \varepsilon$ for all n, which shows that the sequence $f(x_n)$ does not converge to $f(c)$. Thus if $f(x_n) \to f(c)$ for all sequences $\{x_n\}$ such that $x_n \to c$ then the function f must be continuous at c.

The above theorem and the algebraic rules for the limits of sums, products and quotients of sequences immediately imply

Theorem. Let f and g be continuous at a point c functions. Then the functions $f + g$ and $f\,g$ are continuous at c. If $g(c) \neq 0$ then the quotient $\frac{f}{g}$ is also continuous at c.

Example. A polynomial is a continuous function on $\mathbb{R}$. Indeed, in view of the above theorem, it is sufficient to prove that the function $f(x) = x^n$ is continuous for every $n \in \mathbb{N}$. We have $x^n - c^n = (x - c)\,(x^{n-1} + x^{n-2}c + \cdots + x\,c^{n-2} + c^{n-1})$. It follows that

$$|x^n - c^n| \leqslant |x - c|\,(|x|^{n-1} + |x|^{n-2}|c| + \cdots + |x|\,|c|^{n-2} + |c|^{n-1}).$$

If $|x - c| < 1$ then $|x|^{n-1-k}|c|^k < (|c|+1|)^n$ for all $k = 0, \ldots, n-1$ and, consequently, $|x^n - c^n| < n\,(|c| + 1|)^n|x - c|$. Thus we have $|x^n - c^n| < \varepsilon$ whenever $|x - c| < n\,(|c| + 1|)^n\,\varepsilon$, that is, we can take $\delta = n\,(|c| + 1|)^n\,\varepsilon$ in the definition of continuity.

Example. Let $f(x)$ be a function on the interval $[0, 1]$ defined as follows. If x is an irrational number then $f(x) = 0$. If x is rational then it can be uniquely represented as $x = p/q$ with nonnegative coprime integers p and q (that is, integers p and q without a common divisor). In this case we define $f(x) = 1/q$.

The function f is discontinuous at all rational points and is continuous at all irrational points. Indeed, if c is rational then $f(c) \neq 0$ but $f(x_n) = 0$ for any sequence of irrational numbers $x_n \to c$, that is, $f(x_n) \not\to f(c)$. On the other hand, if c is irrational and $\{x_n\}$ is a sequence of rational numbers $x_n = p_n/q_n \to c$ with coprime p_n, q_n then $q_n \to \infty$ (otherwise the sequence would contain infinitely many elements with the same denominator q_n which would not converge to the irrational c). Therefore $f(x_n) \to 0 = f(c)$.