

## TWO CONVERGENCE TESTS

***n*th-root Test Theorem.** Let  $\limsup |a_n|^{1/n} = c$ . If  $c < 1$  then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent. If  $c > 1$  then the series diverges. (If  $c = 1$  nothing can be said.)

*Proof.* Assume that  $c < 1$ . Then there exists a positive number  $b$  such that  $c < b < 1$ . From the definition of the upper limit it follows that  $|a_n|^{1/n} \leq b$  for all sufficiently large  $n$  or, in other words,  $|a_n| \leq b^n$  for all  $n \geq m$ , where  $m$  is some positive integer. Since the series  $\sum_{n=1}^{\infty} b^n$  converges, by the Comparison Theorem the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

Assume now that  $c > 1$ . Then there exists a positive number  $b$  such that  $1 < b < c$ . From the definition of the limit it follows that  $|a_n|^{1/n} \geq b$  for infinitely many values of  $n$ . In other words, there exist positive integers  $n_1 < n_2 < n_3 \dots$  such that  $|a_{n_k}| \geq b^{n_k}$ . Since  $n_k \rightarrow \infty$  and, therefore,  $b^{n_k} \rightarrow \infty$  as  $k \rightarrow \infty$ , we see that  $a_n$  do not converge to 0 as  $n \rightarrow \infty$ . This implies that the series  $\sum_{n=1}^{\infty} a_n$  diverges.

**Ratio Test Theorem.** Assume that  $a_n \neq 0$  for all  $n$ , and that  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n|$  exists and is equal to  $c$ . If  $c < 1$  then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent. If  $c > 1$  then the series diverges. (If  $c = 1$  nothing can be said.)

*Proof.* Assume that  $c < 1$ . Then there exists a positive number  $b$  such that  $c < b < 1$ . From the definition of the limit it follows that  $|a_{n+1}/a_n| \leq b$  for all sufficiently large  $n$  or, in other words,  $|a_{n+1}| \leq b|a_n|$  for all  $n \geq n_0$ , where  $n_0$  is some positive integer depending on  $b$ . Then we have

$$|a_{n_0+1}| \leq b|a_{n_0}|, |a_{n_0+2}| \leq b^2|a_{n_0}|, |a_{n_0+3}| \leq b^3|a_{n_0}|, \dots, |a_{n_0+j}| \leq b^j|a_{n_0}|, \dots,$$

which implies that  $|a_n| \leq |a_{n_0}|b^{-n_0}b^n$  for all  $n \geq n_0$ . Since the series  $\sum_{n=1}^{\infty} b^n$  converges, by the Comparison Theorem the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

Assume now that  $c > 1$ . Then there exists a positive number  $b$  such that  $1 < b < c$ . From the definition of the limit it follows that  $|a_{n+1}/a_n| \geq b$  for all sufficiently large  $n$  or, in other words,  $|a_{n+1}| \geq b|a_n|$  for all  $n \geq n_0$ , where  $n_0$  is some positive integer depending on  $b$ . Then we have

$$|a_{n_0+1}| \geq b|a_{n_0}|, |a_{n_0+2}| \geq b^2|a_{n_0}|, |a_{n_0+3}| \geq b^3|a_{n_0}|, \dots, |a_{n_0+j}| \geq b^j|a_{n_0}|, \dots,$$

which implies that  $a_{n_0+k} \rightarrow +\infty$  as  $k \rightarrow \infty$ . In this case the series  $\sum_{n=1}^{\infty} a_n$  diverges because  $a_n$  do not converge to 0 as  $n \rightarrow \infty$ .

**Remark.** Note that we cannot replace sup with lim sup in the ratio test theorem. For instance, the series  $b_1 + 2b_1 + b_2 + 2b_2 + b_3 + 2b_3 + \dots$  converges whenever the series  $\sum_{n=1}^{\infty} b_n$  converges. But for the first series  $\limsup |a_{n+1}/a_n| \geq 2$  because  $a_{n+1} = 2a_n$  for all odd  $n$ .

## CONDITIONAL CONVERGENCE

**Definition.** If a series converges but is not absolutely convergent, one says that the series converges conditionally.

The following theorem shows that the sum of a conditionally convergent series depends on the order of summation.

**Riemann Series Theorem.** Let  $\sum_{n=1}^{\infty} a_n$  be a conditionally convergent series, and let  $b$  be an arbitrary real number or  $\pm\infty$ . One can always rearrange the terms  $a_n$  in such a way that the new series converges to  $b$ .

**Definition.** A series of the form  $\sum_{n=1}^{\infty} (-1)^n a_n$ , where  $a_n \geq 0$ , is called an alternating series. The alternating series may well converge even if it is not absolutely convergent, that is, even if  $\sum_{n=1}^{\infty} |a_n| = \infty$ .

**Theorem.** If the sequence  $\{a_n\}$  is nonincreasing and  $a_n \rightarrow 0$  then the alternating series  $\sum_{n=1}^{\infty} (-1)^n a_n$  converges.

*Proof.* Let  $s_m = \sum_{n=1}^m (-1)^n a_n$  be the partial sums. Our goal is to show that the sequence  $\{s_m\}$  converges.

Consider first the even values of  $m$ , that is, put  $m = 2k$  where  $k = 1, 2, \dots$ . Since  $s_{2(k+1)} - s_{2k} = a_{2k+2} - a_{2k+1} \leq 0$ , the numbers  $s_{2k}$  form a nonincreasing sequence. This sequence is bounded from below by  $-a_1$  because

$$s_{2k} = -a_1 + a_2 - a_3 + a_4 - a_5 + \dots + a_{2k} = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots + a_{2k}$$

and all the terms in the right hand side with the exception of  $-a_1$  are nonnegative. Now the Monotone Convergence Theorem implies that  $s_{2k}$  converge to a limit  $s \in \mathbb{R}$  as  $k \rightarrow \infty$ .

It remains to show that the whole sequence  $\{s_m\}$  converges to the same limit. In order to do this, we need to prove that for any  $\varepsilon > 0$  there exists  $m_\varepsilon$  such that  $|s - s_m| < \varepsilon$  for all  $m \geq m_\varepsilon$ .

Since  $s_{2k} \rightarrow s$ , we know that there exists a positive integer  $m'_\varepsilon$  such that  $|s - s_m| < \varepsilon/2$  for all even integers  $m \geq m'_\varepsilon$ . (Here we use the definition of convergence with  $\varepsilon/2$  instead of  $\varepsilon$ , which is justified because  $\varepsilon$  is an arbitrary positive number.)

On the other hand, since  $a_m \rightarrow 0$  and  $s_{m+1} - s_m = a_m$ , we have  $|s_{m+1} - s_m| \rightarrow 0$ . From the definition of convergence it follows that there exists a positive integer  $m''_\varepsilon$  such that  $|s_{m+1} - s_m| < \varepsilon/2$  for all  $m \geq m''_\varepsilon$ . Now, if  $m \geq m'_\varepsilon$  and  $m \geq m''_\varepsilon$  and  $m$  is odd, we have

$$|s - s_m| \leq |s - s_{m+1}| + |s_{m+1} - s_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

because  $m+1$  is even and  $m+1 > m'_\varepsilon$ . Thus, if we take  $m_\varepsilon = m'_\varepsilon = m''_\varepsilon$  then the estimate  $|s - s_m| < \varepsilon$  holds for all  $m \geq m_\varepsilon$ . This proves the theorem.

## POWER SERIES

**Definition.** The series of the form  $\sum_{n=1}^{\infty} a_n x^n$  is called a power series. Here  $a_n$  are fixed coefficients and  $x$  is considered as a parameter, so that the sum of the series is a function of  $x$ .

**Theorem.** Let  $\hat{R} = (\limsup |a_n|^{1/n})^{-1}$ . Then the series  $\sum_{n=1}^{\infty} a_n x^n$  is absolutely convergent for  $|x| < \hat{R}$  and is divergent for  $|x| > \hat{R}$ .

Note that  $\hat{R}$  may be  $+\infty$ . The nonnegative “number”  $\hat{R}$  is called the radius of convergence of the power series. The theorem states that the series absolutely converges inside the open interval  $(-\hat{R}, \hat{R})$  and diverges outside the closed interval  $[-\hat{R}, \hat{R}]$ . It does not give any information about the end points, where  $|x| = \hat{R}$ .

*Proof of the Theorem.* Denote  $b_n := a_n x^n$ . We have

$$\limsup |b_n|^{1/n} = \limsup (|a_n|^{1/n} |x|) = c |x|$$

where  $c := \limsup |a_n|^{1/n}$ . Clearly,  $\limsup |b_n|^{1/n} < 1$  if and only if  $|x| < c^{-1}$ , and  $\limsup |b_n|^{1/n} > 1$  if and only if  $|x| > c^{-1}$ . Therefore the Theorem follows from the  $n$ -th Root Test.

**Remark.** Some authors define the radius of convergence  $\hat{R}$  by saying that the series is absolutely convergent for all  $x \in (-\hat{R}, \hat{R})$  and divergent for  $x \notin [-\hat{R}, \hat{R}]$ . The above theorem shows that this definition makes sense, and also give an explicit formula for  $\hat{R}$ .

**Theorem.** Assume that  $a_n \neq 0$  and that the sequence of positive numbers  $|a_n|/|a_{n+1}|$  converges to a limit  $c$ . Then the radius of convergence of the series  $\sum_{n=1}^{\infty} a_n x^n$  coincides with  $c$ .

*Proof.* Denote  $b_n := a_n x^n$ . We have

$$\lim |b_{n+1}/b_n| = \lim |a_{n+1}/a_n| |x| = \lim \frac{|x|}{|a_n/a_{n+1}|} = c^{-1} |x|.$$

By the ratio test, the series converges if and only if  $c^{-1} |x| < 1$ , that is, if and only if  $|x| < c$ . Therefore  $\hat{R} = c$ .

You should memorize the following expansions and the range of  $x$  for which the series converge.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all } x \in \mathbb{R}$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \text{for all } x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad \text{for all } x \in \mathbb{R}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 \dots \quad \text{if } |x| < 1$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{if } |x| < 1$$

**Exercise.** Using a suitable test, show that  $\hat{R} = \infty$  for the first three series, and  $\hat{R} = 1$  for the last two series.