CM221A **ANALYSIS I** **NOTES ON WEEK 3**

CONVERGENT SEQUENCES II

The sandwich theorem. If $x_n \leq y_n \leq z_n$ for all n and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = c$$

then $\lim_{n\to\infty} y_n$ exists and equals c.

Sketch of Proof. Since the sequences $\{x_n\}$ and $\{z_n\}$ converge to c, for each $\varepsilon > 0$ there exists a positive integer n_ε such that $|c - x_n| < \varepsilon$ and $|c - y_n| < \varepsilon$ for all $n \geq n_\varepsilon$. In other words, both points x_n and z_n lie in the open interval $(c-\varepsilon, c+\varepsilon)$. Since y_n lies between x_n and z_n, it also belongs to the interval $(c-\varepsilon, c+\varepsilon)$. This means that $|c - y_n| < \varepsilon$. Thus, for every $\varepsilon > 0$ we have $|c - y_n| < \varepsilon$ whenever $n \geq n_\varepsilon$.

The nested intervals theorem. If $I_n = [a_n, b_n]$ is a sequence of intervals and $I_n \supseteq I_{n+1}$ for all n and the length $(b_n - a_n)$ of I_n converges to 0 as $n \to \infty$, then there exists c such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c.$$

Sketch of Proof. The numbers a_n form a nondecreasing bounded sequence (you must be able to explain why). It converges by the Monotone Convergence Theorem to a limit a. On the other hand, b_n form a nonincreasing bounded sequence, which converges by the Monotone Convergence Theorem 2 to a limit b. Finally, the two limits are equal because $b - a = \lim_{n\to\infty}(b_n - a_n) = 0$.

Remark. In other words, the above theorem says that the intersection $\bigcap_{n=1}^{\infty} I_n$ of all the intervals I_n consists of exactly one point c. Indeed, it is clear from the proof that the point c belong to each interval I_n. On the other hand, any number $a < c$ does not belong to all intervals I_n because a_n converge to c and therefore $a_n > a$ for sufficiently large n. Similarly, any number $b > c$ does not belong to all intervals I_n because b_n converge to c and therefore $b_n < b$ for sufficiently large n.

Definition. We say that c is *accumulation point* of a sequence $\{a_n\}$ if, for any $\varepsilon > 0$, the interval $(c-\varepsilon, c+\varepsilon)$ contains infinitely many elements a_n.

Recall that a_n and a_m with $m \neq n$ are considered as distinct elements of the sequence even if $a_n = a_m$ The above statement means that $a_n \in (c-\varepsilon, c+\varepsilon)$ for infinitely many values of the index n.

Example. The limit of a convergent sequence is an accumulation point.

Theorem. Any bounded sequence has an accumulation point.

Proof. By definition, if a sequence $\{a_n\}$ is bounded then there exists and interval $[a, b]$ such that $a_n \in [a, b]$ for all n. Let us split $[a, b]$ into the union of two closed intervals of length $(b - a)/2$. At least one of these intervals contains infinitely many elements of our sequence (as the number of elements is infinite). Denote this interval by I_1. Next, we split I_1 into the union of two closed intervals of length $(b - a)/4$. At least one of these intervals contains infinitely many elements of our sequence; let us denote it I_2. Repeating this procedure, we obtain a sequence of intervals $I_n = [a_n, b_n]$ satisfying conditions of the nested intervals theorem. Let c be the intersection of these intervals. Then, for any $\varepsilon > 0$, the interval $(c-\varepsilon, c+\varepsilon)$ contains infinitely many elements of our sequence $\{a_n\}$ because $a_n \in (c - \varepsilon, c + \varepsilon)$ and $b_n \in (c - \varepsilon, c + \varepsilon)$ for all sufficiently large n.

Remark. A bounded sequence may have several accumulation points. For example, if a sequence contains all rational numbers then every real number is its accumulation point.

SUBSEQUENCES

Definition. A *subsequence* of a sequence is any sequence obtained by leaving out particular terms from the original sequence. A subsequence of the sequence $\{a_n\}$ is usually denoted by $\{a_{n_k}\}$ where $k \in \mathbb{N}$ is a new index, and n_k are the "numbers" of elements of the original sequence included in the subsequence.

Theorem. Let $\{a_n\}$ be a bounded sequence and c be an accumulation point. Then there is a subsequence which converges to c. If there is only one accumulation point, then all subsequence (including the sequence itself) converge to this point.

Proof. For each $k \in \mathbb{N}$, the interval $(c - \frac{1}{k}, c + \frac{1}{k})$ contains infinitely many elements of our sequence. Let us select elements a_{n_k} with distinct numbers n_k such that $a_{n_k} \in (c - \frac{1}{k}, c + \frac{1}{k})$. These elements form a subsequence which converges to c.

If there is only one accumulation point c then, for each $\varepsilon > 0$, the number of elements of our sequence lying outside the interval $(c - \varepsilon, c + \varepsilon)$ is finite. Indeed, if there were infinitely many elements outside this interval, then subsequence formed by those elements would have an accumulation point which would be an accumulation point for the sequence $\{a_n\}$ itself. This implies that all elements starting from some number n_ε lie in the interval $(c - \varepsilon, c + \varepsilon)$. In other words, for each $\varepsilon > 0$ there is n_ε such that $|c - a_n| < \varepsilon$ for all $n \geq n_\varepsilon$. This means that the sequence converges to c. If $\{a_{n_k}\}$ is a subsequence then $n_m \geq n_\varepsilon$ for some $m \in \mathbb{N}$ because n_k go to infinity as $k \to \infty$. Therefore $|c - a_{n_k}| < \varepsilon$ for all $k \geq m$ (here m depends on ε as it is determined by n_ε). By definition, this means that the subsequence converges to c.

Remark. It may well happen that a subsequence converges to a limit but the sequence itself does not converge. A typical is $\{+1, -1, +1, -1, \ldots\}$. One can show that a sequence converges to a limit c if and only if **every** its subsequence converges to this limit.

CAUCHY SEQUENCES

Definition. We say that $\{a_n\}$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \; \exists n_\varepsilon \in \mathbb{N} \; : \; m, n \geq n_\varepsilon \Rightarrow |a_m - a_n| < \varepsilon$$

or, in usual words, if for all $\varepsilon > 0$ there exists n_ε such that $|a_m - a_n| < \varepsilon$ whenever $m, n \geq n_\varepsilon$.

The sequence $\{a_n\}$ is a Cauchy sequence if the distance $|a_n - a_m|$ between its elements goes to zero as $m, n \to \infty$ (but this is not a rigourous definition).

Theorem. A Cauchy sequence is bounded.

Proof. Let us fix an arbitrary positive ε. By definition, there exist n_ε such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq n_\varepsilon$. This implies that all elements of our sequence with numbers $n \geq n_\varepsilon$ lie in the interval $(a_{n_\varepsilon} - \varepsilon, a_{n_\varepsilon} + \varepsilon)$. The remaining elements $a_1, a_2, \ldots, a_{n_\varepsilon - 1}$ lie in the interval $[x, y]$, where x is the smallest and y is the largest number in the collection $\{a_1, a_2, \ldots, a_{n_\varepsilon - 1}\}$. Thus, all the elements of the sequence lie in the union of the intervals $(a_{n_\varepsilon} - \varepsilon, a_{n_\varepsilon} + \varepsilon)$ and $[x, y]$ which is a subset of another (sufficiently large) finite interval $[a, b]$.

Cauchy's theorem. Every Cauchy sequence converges and, conversely, every convergent sequence is Cauchy.

Proof. Assume first that a sequence $\{a_n\}$ converges to a limit c. Then, for each ε, there exists n_ε such that $|c - a_n| < \varepsilon/2$. Then we have

$$|a_n - a_m| = |(a_n - c) + (c - a_m)| \leq |c - a_n| + |c - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $m, n \geq n_\varepsilon$. This shows that $\{a_n\}$ is a Cauchy sequence.

Assume now that $\{a_n\}$ is a Cauchy sequence. Then it is bounded and, therefore, has an accumulation point c. In order to prove the second part of the theorem, we need to show that the accumulation point is unique. From the definition of a Cauchy sequence it follows that, for any $\varepsilon > 0$ there exists n_ε such that all elements with indices $n \geq n_\varepsilon$ lie in the closed interval I_ε of length 2ε (centred at a_{n_ε}). There are only finitely many elements of our sequence outside I_ε. Therefore every accumulation point must lie in I_ε. If there are two accumulations points c and c_0 then, choosing $\varepsilon < (c - c_0)/4$, we obtain a contradiction, as the points c and c_0 cannot belong to the same interval of length 2ε.