CM221A **ANALYSIS I** **NOTES ON WEEK 2**

You must remember and be able to use all the definitions and theorems stated in this week's notes. Their proofs can be found in the CM115 lecture notes. *The proofs are not examinable.*

CONVERGENT SEQUENCES

Let $\{a_n\}$ be a sequence.

The Crucial Definition. We say that $\{a_n\}$ converges to a number a and write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ if

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \ : \ n \geq n_\varepsilon \Rightarrow |a_n - a| < \varepsilon.$$

In usual words, this means the following: for every positive ε there is a positive integer n_ε such that $|a_n - a| < \varepsilon$ whenever $n \geq n_\varepsilon$. The number a is called the *limit* of $\{a_n\}$. A sequence which has a limit is called a *convergent* sequence.

One can reformulate the above definition in "geometric" terms.

The Crucial Definition: another version. The sequence $\{a_n\}$ converges to a number a if for every $\varepsilon > 0$ there are only finitely many elements a_n lying outside the open interval $(a - \varepsilon, a + \varepsilon)$.

Remark. There are divergent sequences, that is, the sequences which do not converge to a limit. For instance, the sequence $\{+1, -1, +1, -1, +1, \ldots\}$ does not converge.

Definition. We say that $\{a_n\}$ converges to $+\infty$ and write $a_n \to +\infty$ if

$$\forall R > 0 \ \exists n_R \in \mathbb{N} \ : \ n \geq n_R \Rightarrow a_n > R.$$

Similarly, we say that the sequence $\{a_n\}$ converges to $-\infty$ and write $a_n \to -\infty$ if

$$\forall R > 0 \ \exists n_R \in \mathbb{N} \ : \ n \geq n_R \Rightarrow a_n < -R.$$

Example. The sequence $\{1, 2, 3, 4 \ldots\}$ does not converge to a limit in $\mathbb{R}$ but converges to $+\infty$. The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ converges to 0. Both these results follow from the Archimedean Property.

THEOREMS ABOUT CONVERGENT SEQUENCES

Convergence $\Rightarrow$ Boundedness. Every convergent sequence is bounded.

Monotone Convergence Theorem. Every bounded from above nondecreasing sequence a_n converges to its least upper bound (supremum). Every bounded from below nonincreasing sequence converges to its greatest lower bound (infimum).

Comparison Theorem. If $b_n \to 0$ and $|a_n| \le b_n$ for all n then $a_n \to 0$.

The Sandwich Theorem. If $a_n \to c$, $b_n \to c$ and $a_n \le c_n \le b_n$ for all n then $c_n \to c$.

Theorem: algebraic rules for limits. Assume that $a_n \to a$ and $b_n \to b$. Then $a_n + b_n \to a + b$ and $a_n b_n \to ab$. If, in addition, $b \neq 0$ then $a_n/b_n \to a/b$.

The following examples illustrate how one can find limits with the use of above theorems.

Example. Let $a_1 = 2$ and $a_{n+1} = \frac{a_n^2+1}{2a_n}$ for all $n = 1, 2, 3, \ldots$ Then $a_n \to 1$.

Proof. Since $a_1 > 1$ and $\frac{b^2+1}{2b} \ge 1$ for all $b > 0$, we have $a_n \ge 1$ for all n. This implies that $a_{n+1} \le a_n$, that is, the sequence is nonincreasing. By the Monotone Convergence Theorem, it converges to its g.l.b. a. Obviously, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$. Therefore, by the above, $a = \frac{a^2+1}{2a}$. Solving this equation, we obtain $a = 1$.

Example. If $k \in \mathbb{N}$ then $\lim_{n\to\infty} n^k\, 2^{-n} = 0$.

Proof. Note first of all that $\lim_{n\to\infty} \frac{n+1}{n} = 1$. Consequently, $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^k = 1$ for all $k \in \mathbb{N}$. Let $a_n = n^k\, 2^{-n}$. Then $a_{n+1}/a_n = \frac{1}{2}\left(\frac{n+1}{n}\right)^k$. By the above, thee exists n_0 such that $a_{n+1}/a_n < \frac{2}{3}$ for all $n \ge n_0$. But then $a_{n_0+m} \le \left(\frac{2}{3}\right)^m a_{n_0}$ for all $m = 1, 2, \ldots$ This estimate and the Comparison Theorem imply that $\lim_{n\to\infty} a_n = 0$.

Exercise. Let $k \in \mathbb{R}$ and $b > 1$. Prove that $\lim_{n\to\infty} n^k\, b^{-n} = 0$.

SUBSEQUENCES AND ACCUMULATION POINTS

Definition. A *subsequence* of a sequence is any sequence obtained by leaving out particular terms from the original sequence. A subsequence of the sequence $\{a_n\}$ is usually denoted by $\{a_{n_k}\}$ where $k \in \mathbb{N}$ is a new index, and n_k are the "numbers" of elements of the original sequence included in the subsequence.

Definition. A number c is said to be an *accumulation point* (or a *limit point*) of the sequence $\{a_n\}$ if there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = c$.

Exercise. Let $\{a_n\}$ be a convergent sequence and $\lim_{n\to\infty} a_n = a$. Prove that every subsequence of $\{a_n\}$ converges to a.

Corollary. A convergent sequence has a unique accumulation point, which coincides with its limit.

Proof. If a sequence converges to a then, by the above, all its subsequences converges to a. It follows that a is an accumulation point and there are no other accumulation points.

Remark. It may well happen that a subsequence converges to a limit but the sequence itself does not converge. A typical is $\{+1, -1, +1, -1, \ldots\}$. One can show that a sequence converges to a limit c if and only if **every** its subsequence converges to this limit.

Remark. A sequence may have many accumulation points. For example, if a sequence contains all rational numbers then every real number is its accumulation point.

Exercise. Show that c is an accumulation point if and only if any open interval of the form $(c-\varepsilon, c+\varepsilon)$ contains infinitely many elements of the sequence $\{a_n\}$.

The Bolzano–Weierstrass Theorem. Every bounded sequence has a convergent subsequence and, consequently, an accumulation point. If $a_n \in [a,b]$ for all n then the accumulation points of $\{a_n\}$ belong to the closed interval $[a,b]$.

CAUCHY SEQUENCES

Definition. We say that $\{a_n\}$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \ : \ m, n \geq n_\varepsilon \Rightarrow |a_m - a_n| < \varepsilon$$

or, in usual words, if for all $\varepsilon > 0$ there exists n_ε such that $|a_m - a_n| < \varepsilon$ whenever $m, n \geq n_\varepsilon$.

In other words, $\{a_n\}$ is a Cauchy sequence if the distance $|a_n - a_m|$ between its elements goes to zero as $m, n \to \infty$ (but this is not a rigourous definition).

Cauchy's Theorem. Every Cauchy sequence converges and, conversely, every convergent sequence is Cauchy.