n times differentiable functions. We say that f is n times differentiable on (a,b) if each derivative of order up to n exists at every point of the interval (the derivative of order two is the derivative of the first derivative, the derivative of order three is obtained by differentiation the derivative of order three and so on). We say that it is n times continuously differentiable if the final derivative is continuous (the function and its first $(n-1)$ derivatives are automatically continuous). If the interval is $[a,b]$ then we require the one-sided derivatives of all orders up to n to exist at the end-points if the interval. The usual notation for the derivative of order n is $f^{(n)}$, so that $f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(n-1)}(x)$.

TAYLOR'S THEOREM

Theorem (Taylor's formula). If f is n times continuously differentiable on the interval $(a-\varepsilon, a+\varepsilon)$ then, for each $h \in (-\varepsilon, \varepsilon)$, there exists a point c lying between a and $a+h$ (that is, $c \in [a, a+h]$ if h is positive and $c \in [a+h, a]$ if h is negative), such that

$$f(a+h) = f(a) + f'(a)\,h + f^{(2)}(a)\,\frac{h^2}{2!} + f^{(3)}(a)\,\frac{h^3}{3!} + \cdots + f^{(n-1)}(a)\,\frac{h^{n-1}}{(n-1)!} + R_n(a,h)\,,$$

where $R_n(a,h) = f^{(n)}(c)\,\dfrac{h^n}{n!}\,.$

Proof. Let $g(x) = (x-a)^n$ and

$$\tilde f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(a)\,\frac{(x-a)^k}{k!}\,.$$

Note that the functions $\tilde f$ and g and their first $(n-1)$ derivatives vanish at $x=a$. Therefore, applying Cauchy's Mean Value Theorem, we obtain

$$\begin{aligned}\frac{\tilde f(b)}{g(b)} &= \frac{\tilde f(x) - \tilde f(a)}{g(x) - g(a)} = \frac{\tilde f'(c_1)}{g'(c_1)} = \frac{\tilde f'(c_1) - f'(a)}{g'(c_1) - g'(a)} = \frac{\tilde f^{(2)}(c_2)}{g^{(2)}(c_2)} \\ &= \frac{\tilde f^{(2)}(c_2) - \tilde f^{(2)}(a)}{g^{(2)}(c_2) - g^{(2)}(a)} = \frac{\tilde f^{(3)}(c_3)}{g^{(3)}(c_3)} = \cdots = \frac{\tilde f^{(n-1)}(c_{n-1}) - \tilde f^{(n-1)}(a)}{g^{(n-1)}(c_{n-1}) - g^{(n-1)}(a)} = \frac{\tilde f^{(n)}(c_n)}{g^{(n)}(c_n)}\,,\end{aligned}$$

where $c_1 \in (a,b)$, $c_2 \in (a,c_1)$, $c_3 \in (a,c_2)$, …, $c_n \in (a,c_{n-1}) \subset (a,b)$. Since $g^{(n)}(x) = n!$ for all x, we get $\tilde f(b) = \frac{g(b)}{n!}\,f^{(n)}(c_n)$. Now, substituting $b = a+h$, we obtain Taylor's formula.

Definition. If $f(a+h) \;=\; \displaystyle\sum_{k=0}^{\infty} f^{(k)}(a)\,\frac{h^k}{k!}$ (or, equivalently, $R_n(a,h) \to 0$ as $n \to \infty$) for all $h \in (-\varepsilon, +\varepsilon)$ then the function f is said to be *analytic* in the interval $(a-\varepsilon, a+\varepsilon)$.

If the function f is analytic then $f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)\,\frac{(x-a)^k}{k!}$ for all $x \in (a-\varepsilon, a+\varepsilon)$.

Remark. Assume that $|f^{(n)}(x)| \leqslant M_n$ for all $x \in (a-\varepsilon, a+\varepsilon)$, where M_n are some constants. Then $|R_n(a,h)| \leqslant M_n\,\frac{h^n}{n!}$. If the right hand side goes to zero as $n \to \infty$ then f is analytic. The functions $\sin x$ and $\cos x$, for example, are analytic in any open interval.

Remark. It may well happen that a function f has infinitely many derivatives, the Taylor series $\sum_{k=0}^{\infty} f^{(k)}(a)\,\frac{(x-a)^k}{k!}$ converges but the function f is not analytic (see example in the last lecture). The function is analytic if the series converges AND its sum is equal to $f(x)$.

Example. Let $f(x) = \begin{cases} e^{-1/x^2}\,, & x \neq 0\,, \\ 0\,, & x = 0\,. \end{cases}$ For this function, all the derivatives $f^{(n)}$ are equal to zero at the origin. Therefore all terms in Taylor's series with $a = 0$ are equal to zero, and the series does not define the function f.

CONVERGENCE OF POWER SERIES

Theorem (absolute convergence of power series). Assume that the limit $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists (as a finite number or $+\infty$, see Week 2). Then the series

$$\sum_{n=0}^{\infty} a_n x^n, \qquad \sum_{n=1}^{\infty} a_n n x^{n-1}, \qquad \sum_{n=0}^{\infty} a_n x^{n+1}/(n+1)$$

all converge for $|x| < R$ and diverge for $|x| > R$ (the number R is usually called the *radius of convergence.*)

Proof. We have $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}x^{n+1}}{a_n x^n} = \lim_{n\to\infty}\left(x\,\frac{a_{n+1}}{a_n}\right) = x\,\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{x}{R}$ and, similarly,

$$\lim_{n\to\infty} \frac{a_{n+1}(n+1)\,x^n}{a_n n x^{n-1}} = \lim_{n\to\infty}\left(x\,\frac{a_{n+1}}{a_n}\right)\left(\frac{n+1}{n}\right) = x\left(\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\right)\left(\lim_{n\to\infty}\frac{n+1}{n}\right) = \frac{x}{R}\,,$$

$$\lim_{n\to\infty} \frac{a_{n+1}x^{n+2}/(n+2)}{a_n x^{n+1}/(n+1)} = \lim_{n\to\infty}\left(x\,\frac{a_{n+1}}{a_n}\right)\left(\frac{n+1}{n+2}\right) = x\left(\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\right)\left(\lim_{n\to\infty}\frac{n+1}{n+2}\right) = \frac{x}{R}\,.$$

If $|x| < R$ then $\left|\frac{x}{R}\right| < 1$, if $|x| > R$ then $\left|\frac{x}{R}\right| > 1$. Now the required results follow from Ratio Test Theorem (see Week 5).

Note that the series $\sum_{n=1}^{\infty} a_n n x^{n-1}$ is obtained by formal differentiation of $\sum_{n=0}^{\infty} a_n x^n$, and the series $\sum_{n=0}^{\infty} a_n x^{n+1}/(n+1)$ is obtained by formal integration of $\sum_{n=0}^{\infty} a_n x^n$. Assuming that these two operations are justified (they are!), we obtain

Corollary. If $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists and $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for all $|x| < R$.

Recall that (by definition)
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$
$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
By the above theorem, all these series are absolutely convergent for all $x \in \mathbb{R}$. Differentiating them term by term, we obtain

Corollary. $(e^x)' = e^x$, $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$.

Theorem (the product of absolutely convergent series.) If the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely then

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$$

and the series on the right hand side is absolutely convergent.

Proof. Clearly, $c_n = \sum_{i+j=n} a_i\, b_j$. Let $\tilde{c}_n = \sum_{i+j=n} |a_i|\,|b_j|$. Then $|c_n| \leqslant \tilde{c}_n$.

Since the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, $\sum_{n=0}^{m} |a_n| \leqslant C_1$ and $\sum_{n=0}^{m} |b_n| \leqslant C_2$ where C_1 and C_2 are some constants independent of m. We have

$$\sum_{n=0}^{m} \tilde{c}_n = \sum_{n=0}^{m} \sum_{i+j=n} |a_i|\,|b_j| \leqslant \sum_{i\leqslant m\,,\, j\leqslant m} |a_i|\,|b_j| = \left(\sum_{n=0}^{m} |a_n|\right)\left(\sum_{n=0}^{m} |b_n|\right) \leqslant C_1\, C_2$$

for all m. This implies that the series $\sum_{n=0}^{\infty} \tilde{c}_n$ is absolutely convergent. By the comparison test, the series $\sum_{n=0}^{\infty} c_n$ is also absolutely convergent. It remains to notice that

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{i,j=0}^{\infty} a_i\, b_j = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i\, b_j = \sum_{n=0}^{\infty} c_n\,.$$

Corollary. We have $\mathrm{e}^x \mathrm{e}^y = \mathrm{e}^{x+y}$ for all $x, y \in \mathbb{R}$.

Proof. The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ defining the exponential function is absolutely convergent (see above). Therefore, applying the previous theorem with $a_n = \frac{x^n}{n!}$ and $b_n = \frac{y^n}{n!}$, we obtain $\mathrm{e}^x \mathrm{e}^y = \sum_{n=0}^{\infty} c_n$ with $c_n = \sum_{i+j=n} a_i\, b_j = \dfrac{x^i\, y^j}{i!\, j!}$. By the binomial formula, the right hand side is equal to $\frac{(x+y)^n}{n!}$. Thus we have $\mathrm{e}^x \mathrm{e}^y = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \mathrm{e}^{x+y}$.

Remark. More generally, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ then $f(x)\, g(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \sum_{i+j=n} a_i\, b_j$. In particular, if the functions f and g are analytic then

$$f(x)\, g(x) \;=\; \sum_{n=0}^{\infty} \left(\sum_{i+j=n} \frac{f^{(i)}(a) g^{(j)}(a)}{i!j!} \right) x^n = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(a)}{n!}\, x^n\,.$$

which implies that the product fg is also analytic.