NOTATION *(not examinable)*

We shall use the following standard notation

$\mathbb{N}$ is the set of positive integer numbers, $\mathbb{N} = \{1, 2, \ldots\}$.

$\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$\mathbb{Q}$ is the set of rational numbers, $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}\}$.

$\mathbb{R}$ is the set of real numbers.

∞ is a shorthand for "infinity". It is not a proper number.

$\forall$ means "for all" or "for every",

$\exists$ means "there exists" or "there is",

The colon $:$ in a mathematical formula means "such that".

REAL NUMBERS: AXIOMS *(not examinable)*

Real numbers obeys the following axioms.

(**A1**) $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{R}$;

(**A2**) $a + b = b + a$ for all $a, b \in \mathbb{R}$;

(**A3**) there is a unique element in $\mathbb{R}$, denoted 0, such that $a + 0 = a$ for all $a \in \mathbb{R}$;

(**A4**) for every $a \in \mathbb{R}$, there is a unique element in $\mathbb{R}$, denoted $-a$, such that $a + (-a) = 0$;

(**A5**) $(a \times b) \times c = a \times (b \times c)$ for all $a, b, c \in \mathbb{R}$;

(**A6**) $a \times b = b \times a$ for all $a, b \in \mathbb{R}$;

(**A7**) there is a unique element in $\mathbb{R}$, denoted 1, such that $a \times 1 = a$ for all $a \in \mathbb{R}$;

(**A8**) for every nonzero $a \in \mathbb{R}$, there is a unique element in $\mathbb{R}$, denoted a^{-1} or $\frac{1}{a}$, such that $a \times a^{-1} = 1$;

(**A9**) $a \times (b + c) = a \times b + a \times c$ for all $a, b, c \in \mathbb{R}$.

Definition. Subtraction is defined by $a - b = a + (-b)$.

Definition. Division is defined by $\frac{a}{b} = a \times (b^{-1})$.

Remark. 0^{-1} does not exist. The expression $\frac{a}{0}$ has no meaning.

One "orders" two numbers by thinking of the larger as being the higher in order. Formally speaking, there is a relation $<$ between elements of $\mathbb{R}$ obeying the following axioms.

(**A10**) for any $x, y \in \mathbb{R}$, exactly one of the following is true: either $x = y$, or $x < y$, or $y < x$;

(**A11**) if $x < y$ and $y < c$ then $x < c$;

(**A12**) if $x < y$ then $x + c < y + c$ for all $c \in \mathbb{R}$;

(**A13**) if $x < y$ and $c > 0$ then $x\,c < y\,c$.

Definition. We write
$x > y$ if $y < x$,
$x \leqslant y$ if either $x < y$ or $x = y$ (or, in other words, if it is false that $x > y$) and
$x \geqslant y$ if $y \leqslant x$.

Remark. (**A1**)–(**A13**) are axioms and cannot be proved. All other known equalities and inequalities involving the addition and composition can be deduced from the above axioms.

MODULUS

For any $x \in \mathbb{R}$, the modulus (or absolute value) of x is defined by

$$|x| = \begin{cases} x\,, & \text{if } x \geqslant 0, \\ -x\,, & \text{if } x < 0. \end{cases}$$

We have $|a| = |-a|$ end $|ab| = |a|\,|b|$. If $r > 0$ then the inequality $|x| < r$ is equivalent to the pair of inequalities $-r < x$ and $x < r$. The estimate $|a - b| \leqslant |a| + |b|$ is usually called the triangle inequality. The triangle inequality implies that

$$|a_1 + a_2 + \cdots + a_n| \;\leqslant\; |a_1| + |a_2| + \cdots + |a_n|$$

for any collection of real numbers $a_1, a_2, \ldots, a_n$.

All the above results are proved by considering all possible cases of positive and negative a, b and $a-b$ and applying the axioms (**A1**)–(**A13**) (see the online lecture notes for details).

INTERVALS

It is convenient to identify real numbers with points on a straight line (which is usually called the real line). We fix an arbitrary point on the line, called the origin, and assume that this point represents the number 0. Negative numbers lie to the left of the origin and positive numbers lie to the right. The absolute value of a number coincides with the distance from the corresponding point on the line to the origin. The inequality $a < b$ means that b lies to the right of a, and the inequality $|a| < r$ is equivalent to saying that a is closer to the origin than r and $-r$.

Definition. Let $a, b \in \mathbb{R}$. Then

(1) the *open interval* (a, b) is the set of real numbers x such that $a < x$ and $b > x$ (the corresponding points on the line lie strictly between a and b);

(2) the *closed interval* $[a, b]$ is the set of real numbers x such that $a \leqslant x$ and $b \geqslant x$ (the corresponding points lie between a and b with the endpoints included);

(3) the half-open intervals $(a, b]$ and $[a, b)$ are defined in a similar manner, the square bracket indicates that the point is included in the set.

We shall also consider the following infinite intervals.

(4) The open intervals (a, ∞) and $(-\infty, b)$ are the sets of numbers x such that $x > a$ and, respectively, $x < b$.

(5) The closed intervals $[a, \infty)$ and $(-\infty, b]$ are the sets of numbers x such that $x \geqslant a$ and, respectively, $x \leqslant b$.

The open intervals $(0, \infty)$ and $(-\infty, 0)$ are traditionally denoted by $\mathbb{R}_+$ and $\mathbb{R}_-$ (these are the sets of positive and negative numbers).

BOUNDED SETS

A set of real numbers S is said to be

bounded from above if $S \subseteq (-\infty, b]$ for some real number b,

bounded from below if $S \subseteq [a, \infty)$ for some real number a,

bounded if it is bounded from below and from above (or, in other words, if $S \subseteq [a, b]$ for some $a, b \in \mathbb{R}$).

The numbers a and b are said to be the lower and upper bounds for the set S. Note that these numbers are certainly not unique (if they exist).

Definition. The smallest number b for which $S \subseteq (-\infty, b]$ is said to be the *least upper bound* or *supremum* of the set S (abbreviated *l.u.b.* or *sup*). The largest a satisfying $S \subseteq [a, \infty)$ is called the *greatest lower bound* or *infimum* of the set S (abbreviated *g.l.b.* or *inf*).

Remark. Let b be the l.u.b. of a set S. Then, for any $\delta > 0$, we can find a real number x lying in S such that $b - \delta < x \leqslant b$. Indeed, we have $x \leqslant b$ for all $x \in S$ because b is an upper bound. If the inequality $b - \delta < x$ is not true for any $x \in S$ then we have $b - \delta \geq x$ for all $x \in S$. But this means that $b - \delta$ is an upper bound for S, which is not possible because this number is smaller than b.

In a similar way one can prove the following: if a is the g.l.b. of a set S then, for any $\delta > 0$, we can find a real number x lying in S such that $a \leqslant x < a + \delta$.

REAL NUMBERS: COMPLETENESS AXIOM

It is not clear a priori whether every bounded set of real numbers has a l.u.b. or a g.l.b. It turns out that these statements cannot be proved or disproved. The following is an axiom (it is often called the *completeness axiom*).

(**A14**) every bounded from above set of real numbers has a least upper bound.

Roughly speaking, (**A14**) means that there are no holes (gaps) in the real line. Note that the completeness axiom would not be true if we considered only rational numbers. Indeed, the set of rational numbers which are strictly smaller that $\sqrt{2}$ is bounded from above, but its l.u.b. $\sqrt{2}$ is not a rational number.

Proposition. Every bounded from below set $S \subseteq \mathbb{R}$ has a g.l.b.

Proof. Let $-S$ be the set of numbers $-x$ where $x \in S$. This set is bounded from above and, by (**A14**), has a l.u.b. a. Then $-a$ is the g.l.b. of S.

Proposition. The l.u.b. of a set S is unique.

Proof. If there are two distinct upper bounds for S then, by (**A10**), one of them is smaller than the other. But then the larger number is not the smallest upper bound for S and therefore is not a l.u.b.

Remark. The g.l.b and l.u.b. of the set S may not belong to S. For instance, if $S = [1, 2)$ then g.l.b of S is 1, and the l.u.b is 2. Since the number 2 is not included in the interval S, this set does not contain its l.u.b.

If g.l.b. belongs to S then it is the minimal element of the set S, which is denoted by $\min S$. If l.u.b belongs to S then it is the maximal element of the set S, which is denoted by $\max S$.

Example. The open interval $S = (-1, 1)$ does not have the minimal or maximal element. But its g.l.b and the l.u.b. are well defined and are equal to -1 and 1 respectively.

Theorem (Archimedean Property). For every given $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

Proof. If the statement is not true then x is an upper bound for the set $\mathbb{N}$, that is, $\mathbb{N}$ is bounded from above. But then, by (**A14**), the set $\mathbb{N}$ has a l.u.b y. Since y is the l.u.b., $y - 1$ is not an upper bound, which means that $y - 1 < n$ for some positive integer n. But then $y < n + 1$ which contradicts to the fact that y is an upper bound for $\mathbb{N}$.

SEQUENCES

Definition. A *sequence* of real numbers is a "listing" $a_1, a_2, a_3, \ldots$ of numbers $a_n \in \mathbb{R}$, labelled by positive integers $n \in \mathbb{N}$.

The sequence $a_1, a_2, \ldots$ is usually denoted by $\{a_n\}_{n\in\mathbb{N}}$ or just $\{a_n\}$. A sequence can be thought of as a countable ordered set of real numbers; the word "countable" means that elements of the set can be enumerated by positive integers. Traditionally, one assumes that a sequence has infinitely many members. A finite collection of real numbers is sometimes called a *finite* sequence (but then the use of the word "finite" is compulsory).

There is no requirements that the elements of a sequence are distinct numbers. It may well happen that $a_k = a_j$ for two distinct "labels" k and j. For instance, one can consider the constant sequence $a, a, a, \ldots$, where all elements a_n are equal to the same number a. It is also a properly defined sequence.

Example. A sequence with an arbitrary a_1 and other a_n defined by $a_{n+1} = a_1 + n\,d$ with some $d \in \mathbb{R}$ is called an *arithmetic progression.*

Example. A sequence with an arbitrary a_1 and other a_n defined by $a_{n+1} = a_1\,d^n$ with some $d \in \mathbb{R}$ is called a *geometric progression.*

Note that in these two examples the sequences are defined by the simple recurrence relations $a_{n+1} = a_n + d$ and $a_{n+1} = a_n\,d$. One can consider more complicated recurrence relations (that is, equations defining each term of the sequence as a function of the preceding terms).

Example. There is a sequence $\{a_n\}$ that contains all rational numbers as its members. Indeed, the set $\mathbb{Q}$ of rational numbers $\frac{k}{m}$ can be represented as a union of finite sets $\mathbb{Q}_j = \{\frac{k}{m} : |k| + |m| = j\}$, where $j = 1, 2, \ldots$ The required sequence is obtained by enumerating elements of the set $\mathbb{Q}_1$, then elements of the set $\mathbb{Q}_2$(starting from $n_1 + 1$ where n_1 is the number of elements in $\mathbb{Q}_1$), then elements of the set $\mathbb{Q}_2$(starting from $n_2 + 1$ where n_2 is the number of elements in $\mathbb{Q}_1 \bigcup \mathbb{Q}_2$), and so on.

Remark. A sequence cannot contain all real numbers; in other words, the set of real numbers is uncountable. Moreover, every non-degenerate interval is an uncountable set. For the interval $(0, 1)$ this can be proved as follows. Consider an arbitrary sequence of numbers a_n lying in $(0, 1)$. Each of this numbers can be written as a decimal fraction, so that $a_1 = 0.a_{11}a_{12}a_{13}\ldots$, $a_2 = 0.a_{21}a_{22}a_{23}\ldots$ and so on. Choose a decimal faction $0.b_1b_2b_3\ldots$ in such a way that $b_k \neq a_{kk}$ for all $k = 1, 2, \ldots$ Then $b_k \in (0, 1)$ but $b \neq a_n$ for any $n = 1, 2, \ldots$

Definition. We shall say that the sequence $\{a_n\}$ is
nondecreasing (or increasing) if $a_{n+1} \geqslant a_n$ (or $a_{n+1} > a_n$) for all n;
nonincreasing (or decreasing) if $a_{n+1} \leqslant a_n$ (or $a_{n+1} < a_n$) for all n.

PRINCIPLE OF INDUCTION

Let $\{a_n\}$ be a sequence. Suppose that, for each $n \in \mathbb{N}$, we have a statement $P(a_n)$ about the number a_n such that

(1) $P(a_1)$ is true;

(2) for every $k \in \mathbb{N}$, the truth of $P(a_k)$ implies the truth of $P(a_{k+1})$.

Then $P(a_n)$ is true for all a_n.

Example. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

Proof. Consider the above identity as a statement $P(n)$. Clearly, $P(1)$ is true. Assume that $P(k)$ is true. Then

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 .$$

A simple calculation shows that the right hand side coincides with

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} .$$

Thus we have $P(k+1)$. Now, using the induction principle, we see that $P(n)$ is true for all $n \in \mathbb{N}$.