King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

BSC AND MSCI EXAMINATION

5CCM221A (CM221A) REAL ANALYSIS I

SUMMER 2010

TIME ALLOWED: TWO HOURS

THIS PAPER CONSISTS OF TWO SECTIONS, SECTION A AND SECTION B.

SECTION A CONTRIBUTES HALF THE TOTAL MARKS FOR THE PAPER.

ANSWER ALL QUESTIONS IN SECTION A.

ALL QUESTIONS IN SECTION B CARRY EQUAL MARKS, BUT IF MORE THAN TWO ARE ATTEMPTED, THEN ONLY THE BEST TWO WILL COUNT.

NO CALCULATORS ARE PERMITTED.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

TURN OVER WHEN INSTRUCTED

Section A

1. Let $\{a_n\}$, $n = 1, 2, ...$ be a sequence of real numbers.

(a) Using $\varepsilon > 0$, write down the precise definitionof

$$\lim_{n\to\infty} a_n = a.$$

(b) What does it mean to say that the sequence$\{a_n\}$ is nonincreasing?

(c) Write down the definition of the greatest lower bound of a bounded set S of real numbers.

(d) Write down the statement of a theorem about the convergence of a nonincreasing sequence $\{a_n\}$.

In each of the following cases say whether the sequence is convergent or divergent. A rigorous proof is not required, but you should write some words of explanation in each case. If the sequence is convergent, write down the limit.

(e) $$a_n = \frac{1 + (-n)^3}{n + 1}$$

(f) $$a_n = \frac{4^n + 2^n}{5^n}$$

(g) $$a_n = \mathrm{e}^{n-n^2}$$

(h) $$a_n = n^{-1}(2 + \sin n)$$

2. **(a)** Using $\varepsilon > 0$, write down the precise definition of convergence of the series $\sum_{n=1}^{\infty} a_n$ of real numbers.

(b) Write down the precise definition of absolute convergence of the series $\sum_{n=1}^{\infty} a_n$. Explain the relationship between convergence and absolute convergence in both directions.

(c) Write down without proof the integral comparison theorem for a series of the form $\sum_{n=1}^{\infty} f(n)$.

Determine which of the following series converge. You should state which test for convergence you use and explain your answer briefly.

(d) $$\sum_{n=1}^{\infty}(n+1)^{-1}$$

(e) $$\sum_{n=1}^{\infty}(\cos n + \sin n)^{-1}$$

(f) $$\sum_{n=1}^{\infty}\frac{n+2}{n^3+1}$$

(g) $$\sum_{n=1}^{\infty}\frac{(n+1)^2\, 2^n}{n!}$$

3. Let $f : [a, b] \mapsto \mathbb{R}$ be a function.

(a) Give the precise definitions in terms of $\varepsilon$and δ for f to be continuous at a point $x \in (a, b)$.

(b) What does it mean to say that f continuous onthe closed interval $[a, b]$?

(c) State (but do not prove) the intermediate value theorem.

In each of the following cases, determine whether the statement is true or false. You must write some words of explanation or give a counterexample.

(d) If f is continuous on the interval (a, b) then either the left limit $\lim_{x\to b-0} f(x)$ exists and finite, or$|f(x)| \to \infty$ as $x \to b - 0$.

(e) If $f : (a, b) \mapsto \mathbb{R}$ is continuous at every point $x \in (a, b)$ then the function f is bounded.

(f) If $f : (a, b) \mapsto \mathbb{R}$ and $g : (a, b) \mapsto \mathbb{R}$ arecontinuous at a point $x \in (a, b)$ then the quotient $\dfrac{f}{g}$ is continuous at x.

4. (a) Give the precise definitions in terms of $\varepsilon$and δ for a function $f : (a, b) \mapsto \mathbb{R}$ to be differentiable ata point $x \in (a, b)$.

(b) State (but do not prove) the mean value theorem.

In each of the following cases, determine whether the statement is true or false. You must write some words of explanation or give a counterexample.

(c) If f is continuous at a point x then f is differentiable at x.

(d) If f is differentiable at a point x then $f$is continuous at x.

(e) If a function $f : (-1, 1) \mapsto \mathbb{R}$ is differentiable and $xf(x) > 0$ for all $x \neq 0$ then $f(0) = 0$.

(f) If f satisfies the same conditions as in(e) then $f'(0) > 0$.

Section B

5. Let $\{a_n\}$ be a bounded sequence of real numbers.

(a) What does it mean to say that c is an accumulation point of the sequence $\{a_n\}$?

Prove rigorously that(b) c is an accumulation point of the sequence $\{a_n\}$ if and only if there is a subsequence of $\{a_{n_k}\}$ which converges to c;

(c) The sequence $\{a_n\}$ converges if and only if it has only one accumulation point. *You may assume without justification that a bounded sequence has at least one accumulation point.*

6. (a) State (but do not prove) the Bolzano-Weierstrass convergent subsequence theorem.*In the rest of the question the Bolzano-Weierstrass theorem maybe used without justification.*

(b) Prove that a continuous function on a closed bounded interval is bounded.

(c) State the maximum and minimum theorems for acontinuous function and prove one of them. You must explain which axiom for real numbers is used in the proof.

7. (a) Let $f : [a, b] \to \mathbb{R}$ be a piecewise continuous function, and let $F : [a, b] \to \mathbb{R}$ be defined by

$$F(x) := \int_a^x f(t)\,\mathrm{d}t\,.$$

Prove that F is continuous at every $x \in [a, b]$.

(b) Write down (but do not prove) both parts of the fundamental theorem of calculus.

(c) Using the fundamental theorem of calculus and thechain rule, evaluate the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_0^{x^2} \frac{1}{1+t^3}\,\mathrm{d}t\right)\,.$$