CM221A ANALYSIS

SOLUTIONS TO CLASS TEST 1, 2010

1. Prove that every positive real number x, there exists a positive integer n such that $\frac{1}{n} < x^2$. You may NOT use the Archimedean property without proof. **20%**

Multiplying both parts inequalities by positive numbers x^{-2} and n, we see that it is equivalent to the existence of a positive integer n such that $n > x^{-2}$. If such a number n does not exist then x^{-2} is an upper bound for the set of positive integers $\mathbb{N}$. Consequently, $\mathbb{N}$ has a least upper bound b (the completeness axiom). Then $b-1$ is not an upper bound for $\mathbb{N}$, which mean that $b-1 \leqslant n$ for some n. Adding 1 to both parts , we see that $b \leqslant n+1$, that is, b also is not an upper bound for $\mathbb{N}$. Thus the set $\mathbb{N}$ is not bounded from above, which implies the required result.

2. Let $a \in (-1,1)$. Using the principle of induction, show that
$a + a^3 + a^5 + \ldots a^{2n-1} = \dfrac{a - a^{2n+1}}{1-a^2}$ for all positive integers n. **15%**

The equality is true for $n = 1$. Assume that it holds for some n. Then

$$a + a^3 + a^5 + \ldots a^{2n-1} + a^{2n+1} = \frac{a - a^{2n+1}}{1-a^2} + a^{2n+1} = \frac{a - a^{2(n+1)+1}}{1-a^2},$$

that is, the equality holds for $n+1$. By induction, it is true for all n.

3. What does it mean to say that a sequence $\{a_n\}$ converges? Write down a precise definition, using $\varepsilon > 0$. Give an example of a sequence which does not converge.

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The sequence converges if $\forall\varepsilon\ \exists n_\varepsilon$ such that $|c - a_n| < \varepsilon$ whenever $n \geqslant n_\varepsilon$, where c is the limit. For instance, the sequence $1, -1, 1, -1 \ldots$ does not converge.

4. What does it mean to say that a series $\sum_{n=1}^{\infty} a_n$ is
(i) convergent,
(ii) absolutely convergent?
Explain the relation between these two notions. Give an example of a series which converges but is not absolutely convergent. **20%**

The series converges if the sequence of partial sums $\sum_{n=1}^{m} a_n$ converges to a limit as $m \to \infty$. The series is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$. An absolutely convergent series converges, but a convergent series may not be absolutely convergent. An example is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

5. In each of the following cases, determine the range of x for which the series is convergent. Consider all possible values of x and explain what tests and/or theorems you are using.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} (\sin x)^n$. **30%**

(a) *The series converges for all x by the n-th root test because $\left|\frac{x^n}{n^n}\right|^{1/n} = x/n \to 0$ as $n \to \infty$.*

(b) *The series converges for $|x| < 1$ by the ratio test because $\left|\frac{nx^{n+1}}{(n+1)x^n}\right| = x(n+1)/n \to x$ as $n \to \infty$. If $x = 1$ then the series diverges by the integral test. Finally, if $x = -1$ then the series converges by the alternating series theorem.*

(c) *If $|\sin x| < 1$ then the series converges as it is the geometric progression. If $\sin x = \pm 1$ the the series diverges because $(\sin x)^n \not\to 0$.*