CM221A ANALYSIS

SOLUTIONS TO CLASS TEST 3, 2009

1. Let f be a differentiable function on $[a, b]$. What does it mean to say that f has a local maximum at a point $c \in (a, b)$? Prove that $f'(c) = 0$ whenever c is a local maximum. **15%**

f has a local maximum at $c \in (a, b)$ if there exists $\varepsilon > 0$ such that $f(c) \geqslant f(x)$ for all $x \in (c-\varepsilon, c+\varepsilon)$. If c is a local maximum then $\frac{f(x)-f(c)}{x-c} \geqslant 0$ for all $x \in (c-\varepsilon, c)$ and $\frac{f(x)-f(c)}{x-c} \leqslant 0$ for all $x \in (c, c+\varepsilon)$. Letting $x \to c$, we see that $f'(c) = 0$.

2. Find the minimum and the maximum values of the function $f(x) = x + \cos x$ on the interval $[0, \pi]$. **15%**

The derivative $f'(x) = 1 - \sin x$ vanishes at one point $x = \pi/2$. We have $f(\pi/2) = \pi/2$, $f(0) = 1$ and $f(\pi) = \pi - 1$. Thus the maximum value is $\pi - 1$ and the minimum value is 1.

3. State and prove Rolle's theorem (any results about maxima and minima of continuous and differentiable functions may be used without justification). **25%**

<u>Rolle's theorem:</u> *if $f : [a, b] \mapsto \mathbb{R}$ is continuous and it is differentiable at every $x \in (a, b)$ and $f(a) = f(b) = 0$ then there exists a point c in (a, b) at which $f'(c) = 0$.*

<u>Proof:</u> *We know that a continuous function on a bounded closed interval attains its maximum and minimum values. If both these values are zero, the function is identically equal to zero and $f' = 0$ everywhere. If one of these values is not zero and is attained at the point c then $c \in (a, b)$ and $f'(c) = 0$, since the derivative vanishes at the interior points where the function attains its local maxima and minima.*

4. Give an example of a function which is continuous at a point c but is not differentiable at c. **5%**

For instance, $f(x) = |x - c|$.

5. State (but do not prove) Taylor's theorem. **15%**

If f is n times continuously differentiable on the interval $(a-\varepsilon, a+\varepsilon)$ then, for each $h \in (-\varepsilon, \varepsilon)$, there exists a point c lying between a and $a+h$ (that is, $c \in [a, a+h]$ if h is positive and $c \in [a+h, a]$ if h is negative), such that

$$f(a+h) = f(a) + f'(a)\,h + f^{(2)}(a)\,\frac{h^2}{2!} + \cdots + f^{(n-1)}(a)\,\frac{h^{n-1}}{(n-1)!} + R_n(a,h)\,,$$

where $R_n(a,h) = f^{(n)}(c)\,\dfrac{h^n}{n!}$.

6. Use Taylor's theorem with $a = 0$ to write down the first three terms of the power series of the function $f(x) = \dfrac{1}{2 - \mathrm{e}^x}$. **10%**

$f(0) = 1$. *Also* $f'(x) = \mathrm{e}^x(2 - \mathrm{e}^x)^{-2}$ *so* $f'(0) = 1$. *Finally*

$$f''(x) = -2(2 - \mathrm{e}^x)^{-3}(-\mathrm{e}^x)\mathrm{e}^x + (2 - \mathrm{e}^x)^{-2}\mathrm{e}^x$$

so $f''(0) = 3$. *Therefore Taylor's formula yields*

$$f(x) = 1 + x + 3x^2/2 + R_3.$$

7. Apply a suitable theorem to find out the values of x for which the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n! + (2n)!}$ is absolutely convergent. **15%**

We have

$$R = \lim_{n\to\infty} \frac{(n+1)! + (2n+2)!}{n! + (2n)!} = \lim_{n\to\infty} (2n+1)(2n+2) = +\infty.$$

This implies that the series is absolutely convergent for all $x \in \mathbb{R}$.