

THE RELATIVE BREUIL-KISIN CLASSIFICATION OF p -DIVISIBLE GROUPS AND FINITE FLAT GROUP SCHEMES

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ABSTRACT. Assume that $p > 2$, and let \mathcal{O}_K be a p -adic discrete valuation ring with residue field admitting a finite p -basis, and let R be a formally smooth formally finite-type \mathcal{O}_K -algebra. (Indeed, we allow slightly more general rings R .) We construct an anti-equivalence of categories between the categories of p -divisible groups over R and certain semi-linear algebra objects which generalise (φ, \mathfrak{S}) -modules of height ≤ 1 (or Kisin modules). A similar classification result for p -power order finite flat group schemes is deduced from the classification of p -divisible groups. We also show compatibility of various construction of $(\mathbb{Z}_p$ -lattice or torsion) Galois representations, including the relative version of Faltings' integral comparison theorem for p -divisible groups. We obtain partial results when $p = 2$.

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1. INTRODUCTION

One of the main motivations of p -adic Hodge theory, initiated by Fontaine, is to prove comparison isomorphisms between various p -adic cohomologies of varieties over p -adic fields (such as étale, crystalline, and de Rham cohomologies), and we now have very satisfying theory. Recently, the formalism of p -adic Hodge theory (such as period rings and admissible representations) have been generalised to the relative setting, most notably by Brinon. See [Bri10] for a summary of the current status of “relative p -adic Hodge theory”, and [Sch12] for more recent developments built upon the theory of perfectoid spaces [Sch11].

One of the most accessible “test cases” of (absolute) p -adic Hodge theory is p -divisible groups. This case is also important in its own right because it is closely

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related to the study of abelian varieties with good reductions. Therefore, it is natural to ask what we can say about p -divisible groups over a more general base with the formalism of relative p -adic Hodge theory.

In this paper we generalise the classifications of p -divisible group by strongly divisible modules and (φ, \mathfrak{S}) -modules of height ≤ 1 over a p -adic affine formal base which is formally smooth over some p -adic discrete valuation ring (with some reasonable finiteness condition). We also recover, when $p > 2$, the p -adic integral Tate module of a p -divisible group from the corresponding semi-linear algebraic objects.

Let us describe our main results in a simplified setting. (For the actual assumptions on the base ring R , we refer to §2.2.) For simplicity, let k be a perfect field of characteristic $p > 2$. (When $p = 2$ we obtain partial results.) Let W denote its ring of Witt vectors. Let K be a finite totally ramified extension of $\text{Frac } W$, and we let \mathcal{O}_K denote its valuation ring. We fix a uniformiser $\varpi \in \mathcal{O}_K$. Let R be a formally smooth adic \mathcal{O}_K -algebra such that $\widehat{\Omega}_{R/\mathcal{O}_K}$ is finitely generated over R , and R_{red} is finitely generated over k . We further assume that there exists a W -subalgebra $R_0 \subseteq R$ such that $\mathcal{O}_K \otimes_W R_0 = R$. (The last assumption is automatic by Lemma 2.3.4.) Note that all our main results in the “integral theory” depend upon the choice of ϖ and R_0 .

For a p -divisible group G over R , let $\mathbb{D}^*(G)$ denote the contravariant Dieudonné crystal associated to G . We construct (in §3.3) a divided power thickening $S \rightarrow R$ which generalises $S \rightarrow \mathcal{O}_K$.

Theorem 1 (Theorem 3.5). *Let R be as above, and we use the notation from §3.3. Assume that $p > 2$. Then there is an exact anti-equivalence of categories, defined by $G \rightsquigarrow \mathbb{D}^*(G)(S)$, from the category of p -divisible groups over R to the category $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ of Breuil modules (cf., Definition 3.3.2).*

Note that when $R = \mathcal{O}_K$, one can replace Breuil S -modules with certain $W[[u]]$ -modules with simpler structure. In the relative setting, we have a subring $\mathfrak{S} = R_0[[u]] \subseteq S$. We also define Kisin \mathfrak{S} -modules (cf., Definition 6.1.1) in the manner analogous to the case when $R = \mathcal{O}_K$, except that we need to consider a connection in the relative setting in general.

Proposition 2 (Proposition 6.3). *Assume that $p > 2$. The scalar extension $\mathfrak{M} \rightsquigarrow S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ induces an exact equivalence of categories from the category of Kisin \mathfrak{S} -modules to the category of Breuil S -modules.*

Proposition 2 was proved in [CL09, §2.2] when $R = \mathcal{O}_K$ and $p > 2$. We modify this proof in the relative case (which also proves some weaker statement when $p = 2$).

Corollary 3 (Corollary 6.3.1). *Assume that $p > 2$. There exists an exact anti-equivalence of categories from the category of p -divisible groups over R to the category of Kisin \mathfrak{S} -modules.*

We expect Corollary 3 to hold even when $p = 2$, but at the moment we only obtain some partial results such as Corollaries 8.6.1 and 10.3.1.

Note that the equivalences of categories in Theorem 1 and Corollary 3 depend upon the choice of ϖ and R_0 , so they do not behave well under arbitrary base change, but they behave well under étale base change. (Cf. Lemma 2.3.5.)

Finally, the semi-linear algebra objects appearing in Theorem 1 and Corollary 3 carry connections. In some cases when the (anti-)equivalences of categories were constructed without connection, we can remove connections from the statement using Vasiu’s construction of moduli of connections [Vas12, §3]. (See Corollary 10.3.1 for more details).

Let us now discuss the p -adic Tate module representation and integral p -adic comparison isomorphism. Assume that R is a domain, and let \bar{R} denote the normalisation of R in the affine ring of a pro-universal covering of $\mathrm{Spec} R[\frac{1}{p}]$. Choose $\varpi^{(n)} \in \bar{R}$ for $n \geq 0$ so that $\varpi^{(0)} = \varpi$ and $(\varpi^{(n+1)})^p = \varpi^{(n)}$. We construct a “perfectoid subalgebra” $\tilde{R}_\infty \subset \widehat{\bar{R}}$ (in the sense of Scholze [Sch11]), which is constructed, roughly speaking, by adjoining $\varpi^{(i)}$ and compatible p -power roots of lifts of local p -basis of $R/(\varpi)$. (Cf. §7.1.) Set $\mathcal{G}_R := \mathrm{Gal}(\bar{R}[\frac{1}{p}]/R[\frac{1}{p}])$ and $\mathcal{G}_{\tilde{R}_\infty} := \mathrm{Gal}(\widehat{\bar{R}[\frac{1}{p}]}/\tilde{R}_\infty[\frac{1}{p}])$.

We localise R if necessary to ensure that a relative period rings have nice properties¹, and embed S into $A_{\mathrm{cris}}(R)$ using our choice of $\varpi^{(n)}$ in such a way that respects all the relevant structures (cf., §5.1). Using this we can define a \mathbb{Z}_p -lattice \mathcal{G}_R -representation $T_{\mathrm{cris}}^*(\mathcal{M})$ for any Breuil S -module \mathcal{M} (cf., (5.4.1), §5.5). Also, by the theory of (relative) étale φ -modules, one obtains a natural \mathbb{Z}_p -lattice \mathcal{G}_{R_∞} -representation $\mathcal{T}^*(\mathfrak{M})$ for any Kisin \mathfrak{S} -module \mathfrak{M} (cf., Lemma 7.5).

The following theorem is obtained from the same argument as in [Fal99, §6, Thm 7] and the proof of [Kis06, Theorem 2.2.7], respectively.

Theorem 4 (Theorem 5.3, Corollary 5.4.2, Proposition 8.1). *Suppose that R satisfies the “refined almost étaleness” assumption (§2.2.3). Let G be a p -divisible group over R , and $\mathcal{M} := \mathbb{D}^*(G)(S)$. Let \mathfrak{M} be the Kisin \mathfrak{S} -module corresponding to G .*

- (1) *The \mathcal{G}_R -representation $V_p(G)$ is crystalline in the sense of Brinon, and we have $D_{\mathrm{cris}}^*(V_p(G)) \cong D^*(G)$ as filtered (φ, ∇) -modules (cf., Example 4.6.1). Furthermore, there is a natural injective \mathcal{G}_R -map $T_p(G) \rightarrow T_{\mathrm{cris}}^*(\mathcal{M})$, which is an isomorphism if $p > 2$.*
- (2) *For any p , there is a natural injective $\mathcal{G}_{\tilde{R}_\infty}$ -map $\mathcal{T}^*(\mathfrak{M}) \rightarrow T_{\mathrm{cris}}^*(\mathcal{M})$ whose image is $T_p(G)$.*

Note that when $G = A[p^\infty]$ for an abelian scheme A over $\mathrm{Spec} R$, Theorem 4(1) gives a comparison morphism between the first étale homology and the first crystalline homology with constant \mathbb{Q}_p - and \mathbb{Z}_p -coefficients.

Very recently, Peter Scholze proved the comparison isomorphism between relative p -adic étale cohomology and relative de Rham cohomology for proper smooth morphisms $f : X \rightarrow Y$ of varieties over a p -adic field with perfect residue field [Sch12, Theorem 1.9]. Presumably, it should be possible, in the near future, to generalise the proof to handle (log-)crystalline cohomology in place of de Rham cohomology when f has a (log-)smooth integral model, which can be thought of as a “generalisation” of (at least the \mathbb{Q}_p -coefficient statement of) Theorem 4(1).

Finally, we state our main result for commutative finite locally free group schemes²:

Theorem 5 (Theorem 9.4, Proposition 9.5.1). *Assume that $p > 2$. There exists a natural anti-equivalence of categories $H \rightsquigarrow \mathfrak{M}^*(H)$ between the category of p -power order finite flat group schemes H over R such that $H[p^i]$ is flat for each i and the category $(\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\mathrm{Kis}}(\varphi, \nabla^0)$ of certain “torsion Kisin \mathfrak{S} -modules” (Definition 9.2.1). Furthermore, we have a natural $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant isomorphism $H(\bar{R}) \cong \mathcal{T}^*(\mathfrak{M}^*(H))$.*

Let us review previous results when $p > 2$. When $R = \mathcal{O}_K$ with perfect residue field, all our main results are known already. Theorem 4(1) is proved by Faltings [Fal99, §6], while the first assertion was already proved by Fontaine [Fon82,

¹Indeed, we will work with slightly more general rings than Brinon [Bri06, Bri08]. See §4.4 for more details.

²In this paper, we only consider *commutative* finite locally free group schemes, so we will often suppress the adjective “commutative”.

Théorème 6.2]. Theorem 1 was first proved by Breuil [Bre00] and all the remaining statements were proved by Kisin [Kis06].

Brinon and Trihan [BT08] generalised the results of Kisin [Kis06] and Faltings [Fal99, §6] to p -divisible groups over a p -adic discrete valuation ring with imperfect residue field admitting finite p -basis.

When $\mathcal{O}_K = W(k)$ for some perfect field k and R is of topologically finite type over $W(k)$, Theorem 1, Theorem 4(1), and Theorem 5 can be deduced from the result of Faltings [Fal89, §VII]. Theorem 1 can be deduced from an unpublished manuscript by Bloch and Kato [BK] when R is a formal power series over $W(k)$.

When R is a complete regular local ring with perfect residue field (but not necessarily formally smooth over \mathcal{O}_K), Corollary 3 was proved by Eike Lau [Lau10c] generalising Vasiu and Zink [VZ10]. Indeed, they proved a stronger result as they classified p -divisible groups by \mathfrak{S} -modules without connection. On the other hand, our result holds for not necessarily local base rings R such as p -adic completion of a smooth \mathcal{O}_K -algebra.

Let us first outline the proof our results when $p > 2$. The basic idea is to generalise Kisin [Kis06, §A, §2.2, §2.3] and Faltings [Fal99, §6] in a similar way to Brinon and Trihan [BT08]; namely, we prove Theorem 1 by “lifting” A.J. de Jong’s result on Dieudonné crystals [dJ95] via Grothendieck-Messing deformation theory [Mes72], and apply the same argument as [Fal99, §6] to show Theorem 4(1). We then prove Proposition 2 by generalising [CL09, §2.2]. This allows us to avoid the rigid analytic construction (*cf.*, [Kis06, §1] and [BT08, §4]), which is hard to generalise to the relative setting. To prove Theorem 5, we generalise the strategy of [Kis06, §2.3]. Due to the presence of connections, the actual argument is quite elaborate using the theory of moduli of connections [Vas12, §3].

We expect that by working with log Dieudonné crystals we can generalise all the results in this paper to p -divisible groups over some semi-stable bases using an unpublished result of Bloch and Kato [BK], at least when $p > 2$; more precisely, we allow R such that $R/(\varpi)$ satisfies either (0.5.1) or (0.5.2) in [BK]. This case will be studied in the forthcoming paper.

In §2 we prove some lifting result, which is needed for constructing \mathfrak{S} and S . In §3 we introduce various semi-linear algebra objects and prove Theorem 1 when $p > 2$. In §4 we recall (and slightly generalise) the construction and basic properties of relative p -adic period rings. In §5 we prove integral comparison theorem (*cf.*, Theorem 4(1)).

In §6 we introduce generalisations of Kisin modules and prove Proposition 2. In §7, we recall some basic results of perfectoid algebras [Sch11] and the theory of relative étale φ -modules. In §8 we prove Theorem 4(2). In §9 we state our main results on finite locally free group schemes of p -power order. The proof uses the theory of moduli of connection (in a slightly generalised form than the original version in [Vas12, §3]), which will be studied in §10.

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2. SETTINGS, REVIEW, AND SOME COMMUTATIVE ALGEBRA

2.1. General convention. For any ring A , a ring endomorphism $\varphi : A \rightarrow A$, and an A -module M , we write $\varphi^*M := A \otimes_{\varphi, A} M$. We sometimes write φ_A for φ to indicate that it is an endomorphism of A .

For any ideal $J \subset A$, we say that A is J -adic if A is J -adically separated and complete. If $J = (x)$, then we say “ x -adic” instead of “ (x) -adic”.

For any p -adic ring A we let $\widehat{\Omega}_A (= \widehat{\Omega}_{A/\mathbb{Z}_p}) := \varprojlim_n \Omega_{(A/p^n)/\mathbb{Z}_p}$ denote the module of p -adically continuous Kähler differentials. Note that if k_0 is a perfect subfield of k then we have $\widehat{\Omega}_A \cong \widehat{\Omega}_{A/W(k_0)}$. We will work with $\widehat{\Omega}_A$ only when $A/(p)$ locally has a finite p -basis (cf., [dJ95, Lemmas 1.1.3, 1.3.3]), so $\widehat{\Omega}_A$ will always be finitely generated over A .

For any noetherian ring A , we write $A_{\text{red}} := A/\text{rad}(A)$, where $\text{rad}(A)$ is the Jacobson radical. If A is J -adic and A/J is finitely generated over a field, then A_{red} is the maximal reduced quotient of A/J , which justifies the notation.

For any ring A and a divided power ideal $I \subseteq A$, we denote by $s^{[n]}$ the n th divided power of $s \in I$.

2.2. Assumptions on base rings. Throughout the paper, we let R be a p -adic flat \mathbb{Z}_p -algebra with varying technical assumptions depending on the situation. Here, we will list and motivate the assumptions on R for readers’ reference.

2.2.1. p -basis assumption. We assume that $R \cong R_0[u]/E(u)$, where R_0 be a p -adic flat \mathbb{Z}_p -algebra such that $R_0/(p)$ locally admits a finite p -basis, and

$$E(u) = p + \sum_{i=1}^e a_i u^i$$

for some integer $e > 0$, with $a_i \in R_0$ and $a_e \in R_0^\times$. Note that such R is a finite R_0 -algebra which is free of rank e as an R_0 -module. We will let $\varpi \in R$ denote the image of $u \in R_0[u]$.

If there is a Cohen subring $W \subset R_0$ such that $E(u) \in W[u]$, then we set $\mathcal{O}_K := W[u]/E(u)$, which is a finite totally ramified extension of W . In this case, $R := R_0[u]/E(u) = \mathcal{O}_K \otimes_W R_0$. By Lemma 2.3.4 any p -adic flat \mathcal{O}_K -algebra R such that $R/(\varpi)$ locally admits a finite p -basis can be written in this form. Although it is the prototypical case, we do not restrict to the case when $E(u) \in W[u]$ for some Cohen subring W for the following reason, except in the theory of étale φ modules in §7 and §8. (See Remark 7.1.4 for more discussions.)

If R satisfies the p -basis assumption (§2.2.1), then we can define relative Breuil modules in §3.3. In order to relate relative Breuil modules to p -divisible groups, one needs more assumptions on R .

2.2.2. Formally finite-type assumption. In addition to the p -basis assumption (§2.2.1), we assume that R is J_R -adically separated and complete for some finitely generated ideal J_R containing ϖ (defined in §2.2.1), and R/J_R is finitely generated over some field k . Then $R_0/(p) = R/(\varpi)$ is $J_R/(\varpi)$ -adically separated and complete. Note that R_0 is J_{R_0} -adically separated and complete, where $J_{R_0} \subset R_0$ denote the kernel of $R_0 \twoheadrightarrow R/J_R$.

Since $R_0/(p) = R/(\varpi)$ surjects onto R/J_R , it follows that k as above admits a finite p -basis by the p -basis assumption (§2.2.1). By cotangent complex consideration it also follows that $R/(\varpi)$ is formally smooth \mathbb{F}_p -algebra (cf. [dJ95, Lemma 1.1.2] and [Ill71, Ch.III, Corollaire 2.1.3.3]). Therefore, $R/(\varpi)$ satisfies the assumption [dJ95, (1.3.1.1)], and de Jong’s theorem asserts that the crystalline Dieudonné functor over $\text{Spec } R/(\varpi)$ is an equivalence of categories; cf. [dJ95, Main Theorem 1].

Clearly, R_0 is formally smooth over \mathbb{Z}_p with respect to the J_{R_0} -adic topology, and the same holds if \mathbb{Z}_p is replaced by any Cohen subring $W \subset R_0$. If $E(u) \in W[u]$ and $\mathcal{O}_K = W[u]/E(u)$, then R is formally smooth over \mathcal{O}_K . Conversely, any formally smooth formally finite-type \mathcal{O}_K -algebra satisfies the formally finite-type assumption (§2.2.2) by Lemma 2.3.4.

2.2.3. “Refined almost étaleness” assumption. Here, we assume that R is a domain which satisfies the formally finite-type assumption (§2.2.2), such that $R[\frac{1}{p}]$ is finite étale over $R_0[\frac{1}{p}]$ and we have $\widehat{\Omega}_{R_0} = \bigoplus_{i=1}^d R_0 dT_i$ for some finitely many units $T_i \in R_0^\times$. Here, $\widehat{\Omega}_{R_0}$ is the module of p -adically continuous Kähler differentials. Under this condition, we obtain a version of refined almost étaleness (Theorem 4.4.1), which is slightly more general than [And06, §5]. Note that refined almost étaleness is used to show that relative period rings have nice properties (namely, Proposition 4.5).

For example, let W be some Cohen ring with residue field k admitting a finite p -basis, and set $R_0 = W\langle T_1^{\pm 1}, \dots, T_{d'}^{\pm 1} \rangle$ or $R_0 = W[[X_1, \dots, X_{d'}]]$. Then the normal extension $R := R_0[[u]/E(u)]$ satisfies the “refined almost étaleness” assumption (§2.2.3), where $E(u)$ is as in §2.2.1 with the additional assumption that $\frac{\partial}{\partial u} E(\varpi) \in R^\times$. (In the first case, T_i ’s together with a lift of a p -basis of k do the job, and in the second case we use $T_i = 1 + X_i$ instead.)

2.2.4. Normality assumption. In addition to the p -basis assumption (§2.2.1), we assume that $R/(\varpi)$ is a finite product of normal domains. In this case, the crystalline Dieudonné functor over $\mathrm{Spec} R/(\varpi)$ is fully faithful by [BM90, Théorème 4.1.1].

Let us give some examples. Assume that R that satisfies the formally finite-type assumption (§2.2.2). Then R satisfies the normality assumption (§2.2.4), which follows from Lemmas 1.1.3, 1.3.3(d) in [dJ95]. Moduli of connections (cf. Definition 10.2.2) provides examples of R satisfying the normality assumption (§2.2.4) but not necessarily the formally finite-type assumption (§2.2.2). See Lemma 10.2.3 for the precise statement. This example plays an important role in the proof of the classification of finite locally free group schemes (Theorem 9.4).

2.2.5. lci assumption. The definition of relative Breuil modules can be modified for base rings R satisfying a certain lci assumption. (Cf. Remarks 3.3.6 and 3.2.8.) Under additional excellence assumption, we can obtain some full faithfulness result as well. (Cf. Remarks 3.5.6, and 6.3.4.) In order not to over-complicate our notation, we mainly focus on the setting with the p -basis assumption (§2.2.1), and record the results in the lci case in remarks marked with “(lci case)”.

2.3. Formally smooth subalgebra. In this section, we prove some deformation result, which produces many classes of examples of R which satisfy the p -basis assumption (§2.2.1).

Let k be a field of characteristic $p > 0$ which admits a finite p -basis (i.e., $[k : k^p] < \infty$), and we choose a Cohen ring W with residue field k ; i.e., a p -adic discrete valuation ring with fixed isomorphism $W/(p) \cong k$. (Note that W is not a ring of Witt vectors unless k is perfect.) Let us fix a ring endomorphism $\varphi_W : W \rightarrow W$ which lifts the p th power map $\varphi_k : k \rightarrow k$.

Lemma 2.3.1 (Cf. [BM90, §1], [dJ95, Lemma 1.3.3]). *Let \bar{A} be a k -algebra which locally admits a finite p -basis. Then there exists a p -adic flat W -algebra A which lifts \bar{A} , equipped with a lift of Frobenius $\varphi : A \rightarrow A$ over $\varphi_W : W \rightarrow W$. Such A is formally smooth over W under the p -adic topology. If furthermore \bar{A} is formally of finite type over k with ideal of definition \bar{J} (for example, if $\bar{A} = R/(\varpi)$ where R*

satisfies the formally finite-type assumption (§2.2.2)), then any such A is J -adically complete where J is the preimage of \bar{J} , and under this topology A is formally smooth.

Proof. Let us first assume that \bar{A} locally admits a finite p -basis. By [dJ95, Lemma 1.1.2], the augmentation map $L_{\bar{A}/k} \rightarrow \Omega_{\bar{A}/k}$ is a quasi-isomorphism where $L_{\bar{A}/k}$ is a cotangent complex. (Indeed, there is a natural distinguished triangle $\bar{A} \otimes_k^{\mathbb{L}} L_{k/\mathbb{F}_p} \rightarrow L_{\bar{A}/\mathbb{F}_p} \rightarrow L_{\bar{A}/k} \rightarrow [+1]$, and now we apply [dJ95, Lemma 1.1.2] to the first two terms.) Note also that $\Omega_{\bar{A}/k}$ is finite projective over \bar{A} by the existence of local finite p -basis.

Now if A_{W_i} is a lift of \bar{A} over $W_i := W/(p^{i+1})$, then the obstruction class for lifting A_{W_i} over W_{i+1} lies in $\text{Ext}_{\bar{A}}^2(L_{\bar{A}/k}, \bar{A}) = \text{Ext}_{\bar{A}}^2(\Omega_{\bar{A}/k}, \bar{A})$, which is zero. (Cf. [Ill71, Ch.III, Corollaire 2.1.3.3].) This shows the existence of a p -adic flat W -lift A of \bar{A} .

Using the same notation as above, let $\tilde{\varphi}_i : \varphi_{W_i}^* A_{W_i} \rightarrow A_{W_i}$ be a lift of the relative Frobenius map $\varphi : \varphi_k^* \bar{A} \rightarrow \bar{A}$ over k . The obstruction class for lifting $\tilde{\varphi}_i$ to $\tilde{\varphi}_{i+1} : \varphi_{W_{i+1}}^* A_{W_{i+1}} \rightarrow A_{W_{i+1}}$ lies in $\text{Ext}_{\bar{A}}^1(L_{\bar{A}/k}, \bar{A}) = \text{Ext}_{\bar{A}}^1(\Omega_{\bar{A}/k}, \bar{A})$, which is zero. (Cf. [Ill71, Ch.III, Proposition 2.2.2].) This shows the existence of a W -lift $\varphi_W^* A \rightarrow A$ of the relative Frobenius map. Now, we obtain $\varphi : A \rightarrow A$ over φ_W by precomposing this lift with the natural inclusion $A \rightarrow \varphi_W^* A$ (defined by $a \mapsto 1 \otimes a$ for any $a \in A$).

To show that A is formally smooth over W (for the p -adic topology), it suffices to show that $A_{W_i} := A/(p^{i+1})$ is formally smooth over $W_i := W/(p^{i+1})$ for each i . Indeed, by [dJ95, Remark 1.3.4(b)], $\Omega_{A_{W_i}/W_i}$ is finite projective over A_{W_i} . Now one can show inductively that the augmentation map $L_{A_{W_i}/W_i} \rightarrow \Omega_{A_{W_i}/W_i}$ is a quasi-isomorphism, which implies by [Ill71, Ch. III, Proposition 3.1.2] that A_{W_i} is formally smooth over W_i .

Now, assume that \bar{A} is formally of finite type over some field k with ideal of definition \bar{J} . To show that A is J -adically complete (where J is as in the statement), we let A' be the J -adic completion of A . The natural map $A \rightarrow A'$ is clearly injective, and the surjectivity can be checked mod p as both A and A' are p -adic. But $A/(p)$ is already \bar{J} -adically complete. The formal smoothness under the J -adic topology follows from the formal smoothness under the p -adic topology. \square

Remark 2.3.2. The choice of A is unique up to non-unique isomorphism, while the choice of φ is far from unique in general. Therefore, given a continuous map $A \rightarrow A'$ of formally smooth p -adic \mathbb{Z}_p -algebras, it may not be possible in general to choose lifts of Frobenii for A and A' so that f commutes with them. We will see later (Lemma 2.3.5), however, that this is possible (in a unique way) if f is étale.

Remark 2.3.3. Let \bar{A} be an \mathbb{F}_p -algebra locally admitting a finite p -basis, and A its lift over \mathbb{Z}_p . Assume that \bar{A} contains a field k , and we choose a Cohen ring W with residue field k . Then since A is formally smooth over \mathbb{Z}_p by Lemma 2.3.1, there is a lift $W \hookrightarrow A$ of $k \hookrightarrow \bar{A}$. If we choose a lift of Frobenius $\varphi_W : W \rightarrow W$, then we may choose $\varphi_A : A \rightarrow A$ to be compatible with φ_W .

For the following lemma, set $\mathcal{O}_K = W[u]/E(u)$ where $E(u) \in W[u]$ is an Eisenstein polynomial (cf. §2.2.1), and let $\varpi \in \mathcal{O}_K$ be the image of u . Let R be a p -adic flat \mathcal{O}_K -algebra such that $R/(\varpi)$ locally admits a finite p -basis. Let R_0 denote a flat W -lift of R/ϖ (obtained from applying Lemma 2.3.1 to $\bar{A} = R/(\varpi)$).

Lemma 2.3.4. *In the above setting, there exists a W -algebra map $\iota : R_0 \rightarrow R$ which lifts the k -isomorphism $R_0/(p) \xrightarrow{\sim} R/(\varpi)$. Furthermore, the induced \mathcal{O}_K -algebra map $\iota_{\mathcal{O}_K} : R_0 \otimes_W \mathcal{O}_K \rightarrow R$ is an isomorphism of topological \mathcal{O}_K -algebras.*

If furthermore R is formally of finite type over \mathcal{O}_K with radical J_R (in particular, R satisfies assumption §2.2.2), then we may additionally require ι to be continuous for the J_{R_0} -adic topology, where $J_{R_0} \subset R_0$ is the preimage of $J_R/(\varpi)$.

When $R/(\varpi)$ is a finitely generated algebra over k , this lemma is a consequence of [SGA, 1, Exp. III, Corollaire 6.8]. Note that the map $R_0 \rightarrow R$ in the statement is far from unique in general.

Proof. Since R is \mathcal{O}_K -flat, the existence of the isomorphism $\iota_{\mathcal{O}_K}$ follows from the uniqueness of \mathcal{O}_K -lifts of $R/(\varpi)$ (cf., the proof of Lemma 2.3.1).

Now assume that R satisfies the formally finite-type assumption (§2.2.2). It remains to construct a J_{R_0} -adically continuous map ι . By formal smoothness of R_0 , the natural isomorphism $R_0/J_{R_0} \xrightarrow{\sim} R/J_R$ has a continuous lift to $R_0 \rightarrow R/J_R^{i+1}$ for any $i > 0$. By taking limit we get a continuous morphism $\iota' : R_0 \rightarrow R$. Let $\bar{\iota}' : R_0/(p) \rightarrow R/(\varpi)$ denote the map induced by ι' , and let $\bar{\iota} : R_0/(p) \rightarrow R/(\varpi)$ be the isomorphism given by the construction of R_0 in Lemma 2.3.1. By construction, we have $\bar{\iota}^{-1} \circ \bar{\iota}' \equiv \text{id}_{R_0/(p)} \pmod{J_{R_0}/(p)}$. We obtain $\iota : R_0 \rightarrow R$ by modifying ι' by an automorphism of R_0 which lifts $(\bar{\iota}^{-1} \circ \bar{\iota}')^{-1}$. \square

Assume that R satisfies the formally finite-type assumption (§2.2.2), and let (R_0, J_{R_0}) and $E(u) \in R_0[u]$ be as in §2.2.2. Choose a lift of Frobenius $\varphi : R_0 \rightarrow R_0$ as in Lemma 2.3.1. Let $I \subset J_{R_0}$ be a closed ideal containing p . (Often, we will take either $I = J_{R_0}$ or $I = (p)$.) Set $\bar{R} := R_0/I$, and let \bar{R}' be a finitely generated étale \bar{R} -algebra.

Lemma 2.3.5. *With the above setting, there exists a I -adic formally étale R_0 -algebra R'_0 such that $R'_0/IR'_0 \cong \bar{R}'$ as \bar{R} -algebras. Such an R'_0 is unique up to unique isomorphism, and φ_{R_0} uniquely extends to a lift of Frobenius $\varphi_{R'_0} : R'_0 \rightarrow R'_0$.*

Let R' be an I -adic formally étale R -algebra such that R'/IR' is finitely generated over R/IR . Let R'_0 be a lift of $\bar{R}' := R'/(\varpi, I)$. Then there exists a unique R_0 -homomorphism $R'_0 \hookrightarrow R'$ which induces $R'_0[u]/E(u) \cong R'$ as R -algebras.

It is clear that R'_0 is formally étale over R_0 , so R'_0 is necessarily formally smooth over W . Since $\varphi_{R'_0}$ is uniquely determined by φ_{R_0} , there will be no confusion in denoting both by φ in this case.

Proof. The existence and uniqueness of R'_0 follows from standard deformation theory (cf., Proposition 6.1 and Théorème 6.3 in [SGA, 1]). By Proposition 5.8 in [SGA, 1], there exists a unique map $\varphi_{R_0}^* R'_0 \rightarrow R'_0$ which lifts the relative Frobenius $\varphi_{\bar{R}}^* \bar{R}' \rightarrow \bar{R}'$. We set $\varphi_{R'_0} : R'_0 \rightarrow R'_0$ to be the composition $R'_0 \rightarrow \varphi_{R_0}^* R'_0 \rightarrow R'_0$, where the first map is defined by $a \mapsto 1 \otimes a$ for any $a \in R'_0$, and the second map is the unique lift of the relative Frobenius.

The last assertion follows from the uniqueness up to unique isomorphism of R -lift of \bar{R} , which provides a unique R -isomorphism $R'_0[u]/E(u) \cong R'$ that lifts the identity map on \bar{R} . \square

3. CLASSIFICATION OF p -DIVISIBLE GROUPS BY DIEUDONNÉ CRYSTALS

In this section, we introduce a “relative” version of strongly divisible modules, and prove a generalisation of the Breuil classification of p -divisible group over a base ring R which satisfies the formally finite-type assumption (§2.2.2) (and some partial result for more general kind of R). We follow Kisin’s strategy [Kis06, §A] using the main results of [dJ95, BM90] and Grothendieck-Messing deformation theory [Mes72]. Along the way, we give a very brief review of crystalline Dieudonné theory.

3.1. Review: crystalline Dieudonné theory. We follow the notation and convention from [BBM82, dJ95]. Let \mathfrak{X} be a formal scheme over $\mathrm{Spf} \mathbb{Z}_p$, and let $\overline{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf} \mathbb{Z}_p} \mathrm{Spec} \mathbb{F}_p$ denote the closed formal subscheme of \mathfrak{X} defined by the ideal (p) . For example, if $\mathfrak{X} = \mathrm{Spf}(R, (p))$ then $\overline{\mathfrak{X}} = \mathrm{Spec} R/(p)$. Let $(\mathfrak{X}/\mathbb{Z}_p)_{\mathrm{CRIS}}$ and $(\overline{\mathfrak{X}}/\mathbb{Z}_p)_{\mathrm{CRIS}}$ denote the big fppf-crystalline topoi.

Let G be a p -divisible group over \mathfrak{X} and $\overline{G} := G \times_{\mathfrak{X}} \overline{\mathfrak{X}}$. One contravariantly associates to G a crystal $\mathbb{D}^*(G)$ of finite locally free $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ -module in such a way that commutes with base change. See [Mes72], [MM74], or [BBM82] for the construction (cf., [dJ95, Definition 2.4.2(b)]). In particular, since $p = 0$ in $\overline{\mathfrak{X}}$ one obtains $F : \varphi^* \mathbb{D}^*(\overline{G}) \rightarrow \mathbb{D}^*(\overline{G})$ and $V : \mathbb{D}^*(\overline{G}) \rightarrow \varphi^* \mathbb{D}^*(\overline{G})$ from the relative Frobenius and Verschiebung of \overline{G} , and we have $F \circ V = p$ and $V \circ F = p$.

Let $i_{\mathrm{CRIS}} := (i_{\mathrm{CRIS},*}, i_{\mathrm{CRIS}}^*) : (\overline{\mathfrak{X}}/\mathbb{Z}_p)_{\mathrm{CRIS}} \rightarrow (\mathfrak{X}/\mathbb{Z}_p)_{\mathrm{CRIS}}$ be the morphism of topoi induced from the closed immersion $\overline{\mathfrak{X}} \hookrightarrow \mathfrak{X}$. Then $i_{\mathrm{CRIS},*}$ and i_{CRIS}^* induce quasi-inverse exact equivalences of categories between the categories of crystals of finitely presented (respectively, finite locally free) $\mathcal{O}_{\overline{\mathfrak{X}}/\mathbb{Z}_p}$ -modules and $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ -modules. (This follows from [dJ95, Lemma 2.1.4], noting that there is a natural isomorphism $i_{\mathrm{CRIS},*} \mathcal{O}_{\overline{\mathfrak{X}}/\mathbb{Z}_p} \cong \mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ as in [BO78, §5.17.3].) Since the formation of \mathbb{D}^* commutes with base change, we obtain a natural isomorphism $\mathbb{D}^*(G) \cong i_{\mathrm{CRIS},*}(\mathbb{D}^*(\overline{G}))$.

Let $\mathbb{D}^*(G)_{\mathfrak{X}}$ denote the locally free $\mathcal{O}_{\mathfrak{X}}$ -module obtained from the push-forward of $\mathbb{D}^*(G)$ to the Zariski topos. The construction of $\mathbb{D}^*(G)$ also provides the following functorial exact sequence of vector bundles which commutes with base change:

$$(3.1.1) \quad 0 \rightarrow \mathcal{L}ie(G)^* \rightarrow \mathbb{D}^*(G)_{\mathfrak{X}} \rightarrow \mathcal{L}ie(G^{\vee}) \rightarrow 0,$$

where G^{\vee} is the dual p -divisible group. This exact sequence defines the *Hodge filtration* $\mathcal{L}ie(G)^* \subseteq \mathbb{D}^*(G)_{\mathfrak{X}}$ for G . (There are two possible ways to define the Hodge filtration – one via universal vector extension in [Mes72], and the other via [BBM82, Corollaire 3.3.5] – and they coincide by [BM90, Théorème 3.1.7].)

Definition 3.1.2. A *Dieudonné crystal* over \mathfrak{X} is a quadruple $(\mathcal{E}, F, V, \mathrm{Fil}^1 \mathcal{E}_{\mathfrak{X}})$, where

- (1) \mathcal{E} is a crystal of finite locally free $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ -module.
- (2) Let $\overline{\mathcal{E}} := \mathcal{E}|_{\overline{\mathfrak{X}}} (= i_{\mathrm{CRIS}}^* \mathcal{E})$, and let $\varphi : \overline{\mathfrak{X}} \rightarrow \overline{\mathfrak{X}}$ denote the absolute Frobenius morphism. Then $F : \varphi^* \overline{\mathcal{E}} \rightarrow \overline{\mathcal{E}}$ and $V : \overline{\mathcal{E}} \rightarrow \varphi^* \overline{\mathcal{E}}$ are morphisms which satisfy $F \circ V = p$ and $V \circ F = p$.
- (3) Let $\mathcal{E}_{\mathfrak{X}}$ denote the locally free $\mathcal{O}_{\mathfrak{X}}$ -module obtained from the push-forward of \mathcal{E} to the Zariski topos. Then $\mathrm{Fil}^1 \mathcal{E}_{\mathfrak{X}} \subseteq \mathcal{E}_{\mathfrak{X}}$ is a direct factor as an $\mathcal{O}_{\mathfrak{X}}$ -submodule such that $\varphi^*(\mathrm{Fil}^1 \mathcal{E}_{\mathfrak{X}}|_{\overline{\mathfrak{X}}}) = \ker[F : \varphi^*(\mathcal{E}_{\mathfrak{X}}|_{\overline{\mathfrak{X}}}) \rightarrow \mathcal{E}_{\mathfrak{X}}|_{\overline{\mathfrak{X}}}]$. We call $\mathrm{Fil}^1 \mathcal{E}_{\mathfrak{X}} \subseteq \mathcal{E}_{\mathfrak{X}}$ the *Hodge filtration*.

If $\mathfrak{X} = \overline{\mathfrak{X}}$ (i.e., $p\mathcal{O}_{\mathfrak{X}} = 0$) then (3) in Definition 3.1.2 is automatic from the other conditions (cf., [dJ95, Proposition 2.5.2]). In particular, our definition is compatible with [dJ95, Definitions 2.3.4, 2.4.2(b)] in this case.

Lemma 3.1.3. For any p -divisible group G over \mathfrak{X} , the crystal $\mathbb{D}^*(G)$ associated to G has a natural structure of Dieudonné crystal over \mathfrak{X} .

Proof. It suffices to show that $\mathrm{Fil}^1 \mathbb{D}^*(\overline{G})_{\overline{\mathfrak{X}}}$ (the kernel of F as in Definition 3.1.2(3)) coincides with the Hodge filtration for \overline{G} , which follows from [BBM82, Proposition 4.3.10]. \square

Remark 3.1.4. There is an obvious notion of short exact sequences of Dieudonné crystals. It follows from [BBM82, Proposition 4.3.1] that the crystalline Dieudonné functor $G \mapsto \mathbb{D}^*(G)$ is exact over a base where p is locally topologically nilpotent.

Also, one can describe the effect of duality on $\mathbb{D}^*(G)$ [BBM82, §5.3]; namely, $\mathbb{D}^*(G^\vee)$ is the $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ -linear dual of $\mathbb{D}^*(G)$, F (respectively, V) on $\mathbb{D}^*(G^\vee)$ is induced from V (respectively, F) on $\mathbb{D}^*(G)$, and $\text{Fil}^1 \mathbb{D}^*(G^\vee)_{\mathfrak{X}}$ is the annihilator of $\text{Fil}^1 \mathbb{D}^*(G)_{\mathfrak{X}}$ under the natural duality pairing.

Definition 3.1.5. We say that a Dieudonné crystal \mathcal{E} is *F-nilpotent* if $F^n : \varphi^{n*} \mathcal{E} \rightarrow \mathcal{E}$ factors through $p\mathcal{E}$. We similarly define *V-nilpotence*.

Recall that a p -divisible group G over R is called *formal* if $G[p^n]$ are infinitesimal thickenings of $\text{Spec } R$ as a scheme. If $pR = 0$, then G is formal if and only if $G = \varinjlim G[F^n]$ where $G[F^n]$ is the kernel of the n th iterated relative Frobenius morphism $F^n : G \rightarrow \varphi^{n*}G$. A p -divisible group G is called *unipotent* if its dual is formal. Note that a p -divisible group is formal or unipotent if and only if all of its geometric fibres are formal or unipotent, respectively. The following lemma is now straightforward by classical Dieudonné theory:

Lemma 3.1.6. *A p -divisible group G over \mathfrak{X} is formal (respectively, unipotent) if and only if $\mathbb{D}^*(G)$ is *F-nilpotent* (respectively, *V-nilpotent*).*

3.2. Filtered Frobenius modules. One can understand a crystal (over an affine formal scheme) as a suitable module with connection, which allows us to describe Dieudonné crystals by some concrete objects.

Definition 3.2.1 (Cf. [Lau10c, §2.1]). A *frame* is a tuple $(\widehat{D}, \text{Fil}^1 \widehat{D}, R, \varphi, \varphi_1)$ where $R = \widehat{D}/\text{Fil}^1 \widehat{D}$, $\varphi : \widehat{D} \rightarrow \widehat{D}$ is a lift of Frobenius, $\varphi_1 = \varphi/p : \text{Fil}^1 \widehat{D} \rightarrow \widehat{D}$ is such that the ideal generated by $\varphi_1(\text{Fil}^1 \widehat{D})$ is the unit ideal, and $p\widehat{D} + \text{Fil}^1 \widehat{D}$ is contained in the Jacobson radical of \widehat{D} . In this case, we will focus on the case when \widehat{D} is a torsion-free p -adic ring and $\text{Fil}^1 \widehat{D}$ has a (p -adically continuous) divided power structure.

A morphism of frames are morphism of rings that respects all the structures. (There is more general notion of morphisms that allow φ_1 's to differ by a certain unit multiple; cf. [Lau10c, §2.1], we will not use this more general notion.)

Remark 3.2.2. Let R be a p -adic ring, and choose a p -adic \mathbb{Z}_p -flat algebra A , such that $A/(p)$ locally admits a finite p -basis and $R \cong A/I$ for some ideal I . We also choose a lift of Frobenius $\varphi : A \rightarrow A$. Let \widehat{D} denote the p -adically completed divided power envelope of A with respect to I . Let $\text{Fil}^1 \widehat{D}$ denote the ideal topologically generated by the divided powers of elements in I . Clearly, $p\widehat{D} + \text{Fil}^1 \widehat{D}$ is contained in the Jacobson radical, as \widehat{D} is p -adic and the image of $\text{Fil}^1 \widehat{D}$ in $\widehat{D}/(p)$ is a nil-ideal (being a divided power ideal).

We assume that \widehat{D} is \mathbb{Z}_p -flat. This is satisfied if $\widehat{D} = A$ (for example, when $R = A/(p)$). Another example will be given in §3.3. Note that \widehat{D} may fail to be \mathbb{Z}_p -flat – such an example is given in the proof of [BO83, Proposition A.2].

It easily follows that $\varphi : A \rightarrow A$ uniquely extends to $\varphi : \widehat{D} \rightarrow \widehat{D}$, and $\varphi(\text{Fil}^1 \widehat{D}) \subset p\widehat{D}$. (To see that $\varphi : \widehat{D} \rightarrow \widehat{D}$ is well defined, observe that \widehat{D} is also a p -adically completed divided power envelope for $(p, I) \subset A$, and $\varphi(p, I) \subset (p, I)$. To see $\varphi(\text{Fil}^1 \widehat{D}) \subset p\widehat{D}$, just note that $\varphi : \widehat{D}/(p) \rightarrow \widehat{D}/(p)$ is the p -th power map so it annihilates $\text{Fil}^1 \widehat{D}/(p)$.) We set $\varphi_1 := \frac{\varphi}{p} : \text{Fil}^1 \widehat{D} \rightarrow \widehat{D}$, which is well-defined as \widehat{D} is \mathbb{Z}_p -flat. Although $\varphi_1(\text{Fil}^1 \widehat{D})$ may not generate the unit ideal in general, this is often satisfied in practice – indeed, one can often find an element $x \in \text{Fil}^1 \widehat{D}$ such that $\varphi_1(x)$ is a unit. If all of these are satisfied, we obtain a frame $(\widehat{D}, \text{Fil}^1 \widehat{D}, R, \varphi, \varphi_1)$.

For \widehat{D} constructed from A as above, let $d_{\widehat{D}} : \widehat{D} \rightarrow \widehat{D} \otimes_A \widehat{\Omega}_A$ denote the p -adically continuous derivation obtained by naturally extending the universal continuous

derivation $d_A : A \rightarrow \widehat{\Omega}_A$ by setting $d_{\widehat{D}}(s^{[n]}) := s^{[n-1]}d_A(s)$ for any $s \in I$ and n . (Here, $s^{[n]} := \frac{s^n}{n!}$.)

Definition 3.2.3. For a frame \widehat{D} , let $\mathbf{MF}_{\widehat{D}}(\varphi)$ denote the category of tuples $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_{\mathcal{M}}, \varphi_1)$ where

- (1) \mathcal{M} is a finite projective \widehat{D} -module;
- (2) $\text{Fil}^1 \mathcal{M} \subset \mathcal{M}$ is a \widehat{D} -submodule containing $(\text{Fil}^1 \widehat{D})\mathcal{M}$, and such that $\mathcal{M}/\text{Fil}^1 \mathcal{M}$ is projective over $R = \widehat{D}/\text{Fil}^1 \widehat{D}$;
- (3) $\varphi_{\mathcal{M}}$ is a φ -linear endomorphism of \mathcal{M} such that $(1 \otimes \varphi_{\mathcal{M}})(\varphi^* \text{Fil}^1 \mathcal{M}) = p\mathcal{M}$;
- (4) $\varphi_1 = \varphi_{\mathcal{M}}/p : \text{Fil}^1 \mathcal{M} \rightarrow \mathcal{M}$;

If \widehat{D} , constructed as in Remark 3.2.2, is \mathbb{Z}_p -flat, then we let $\mathbf{MF}_{\widehat{D}}(\varphi, \nabla)$ denote the category of tuples $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_{\mathcal{M}}, \varphi_1, \nabla_{\mathcal{M}})$ where $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_{\mathcal{M}}, \varphi_1) \in \mathbf{MF}_{\widehat{D}}(\varphi)$ and $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_A \widehat{\Omega}_A$ is a topologically quasi-nilpotent integrable connection over $d_{\widehat{D}}$ which commutes with $\varphi_{\mathcal{M}}$. (Cf. [dJ95, Remarks 2.2.4].)

Remark 3.2.4. Since \widehat{D} is \mathbb{Z}_p -flat, one can find a unique injective morphism $\psi_{\mathcal{M}} : \mathcal{M} \rightarrow \varphi^* \mathcal{M}$ such that $(1 \otimes \varphi_{\mathcal{M}}) \circ \psi_{\mathcal{M}} = p \text{id}_{\mathcal{M}}$ and $\psi_{\mathcal{M}} \circ (1 \otimes \varphi_{\mathcal{M}}) = p \text{id}_{\varphi^* \mathcal{M}}$.

Let \mathcal{E} be a Dieudonné crystal over $\text{Spf}(R, (p))$, and set $\mathcal{M} := \mathcal{E}(\widehat{D})$. We define:

- (1) The linearisation $1 \otimes \varphi_{\mathcal{M}}$ of $\varphi_{\mathcal{M}}$ is induced from $F : \varphi^*(\overline{\mathcal{E}}) \rightarrow \overline{\mathcal{E}}$, where $\overline{\mathcal{E}}$ is the pull-back of \mathcal{E} over $\text{Spec } R/(p)$;
- (2) $\text{Fil}^1 \mathcal{M} \subset \mathcal{M}$ is the preimage of the Hodge filtration $\text{Fil}^1 \mathcal{E}(R) \subset \mathcal{E}(R)$ by the natural projection $\mathcal{M} \rightarrow \mathcal{M}/(\text{Fil}^1 \widehat{D})\mathcal{M} \cong \mathcal{E}(R)$;

Proposition 3.2.5. For \mathcal{M} associated to a Dieudonné crystal \mathcal{E} as above, we have $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M}}/p) \in \mathbf{MF}_{\widehat{D}}(\varphi)$. If furthermore \widehat{D} is constructed as in Remark 3.2.2 and is \mathbb{Z}_p -flat, then there exists a natural connection $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_A \widehat{\Omega}_A$, so that $\mathcal{E} \rightsquigarrow (\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{M}}/p, \nabla_{\mathcal{M}})$ induces an exact equivalence of categories from the category of Dieudonné crystals over $\text{Spf}(R, (p))$ to $\mathbf{MF}_{\widehat{D}}(\varphi, \nabla)$.

Proof. Let us first settle the proposition when p is nilpotent in R (i.e., when $\text{Spf}(R, (p)) = \text{Spec } R$). Note that $V : \overline{\mathcal{E}} \rightarrow \varphi^* \overline{\mathcal{E}}$ induces a \widehat{D} -linear map $\psi_{\mathcal{M}} : \mathcal{M} \rightarrow \varphi^* \mathcal{M}$ such that $(1 \otimes \varphi_{\mathcal{M}}) \circ \psi_{\mathcal{M}} = p \text{id}_{\mathcal{M}}$ and $\psi_{\mathcal{M}} \circ (1 \otimes \varphi_{\mathcal{M}}) = p \text{id}_{\varphi^* \mathcal{M}}$. By Definition 3.1.2(3) we have $\varphi^* \text{Fil}^1 \mathcal{E}(R/p) = \ker(F) = \text{im}(V)$, so it follows that $(1 \otimes \varphi_{\mathcal{M}})(\varphi^* \text{Fil}^1 \mathcal{M}) \subseteq p\mathcal{M}$. In particular, the image of $\psi_{\mathcal{M}}$ is contained in the image of $\varphi^* \text{Fil}^1 \mathcal{M}$ in $\varphi^* \mathcal{M}$. So from $(1 \otimes \varphi_{\mathcal{M}}) \circ \psi_{\mathcal{M}} = p \text{id}_{\mathcal{M}}$, it follows that $(1 \otimes \varphi_{\mathcal{M}})(\varphi^* \text{Fil}^1 \mathcal{M}) = p\mathcal{M}$. The rest of the conditions are clear.

Now let \widehat{D} be the p -adically completed PD envelope of $A \rightarrow R$, where $A/(p)$ locally has a finite p -basis. Then, $\mathcal{E} \rightsquigarrow \mathcal{E}(\widehat{D})$ induces a natural equivalence of categories between crystals and certain modules with connection. (Indeed, by the proof of [dJ95, Proposition 2.2.2] it suffices to handle the case when $R = A/(p)$, in which case we may apply [BM90, Proposition 1.3.3] for some Zariski cover of $\text{Spec } A/(p)$.) We also assume that \widehat{D} is \mathbb{Z}_p -flat. Then one can naturally view $\mathcal{M} := \mathcal{E}(\widehat{D})$ as an object of $\mathbf{MF}_{\widehat{D}}(\varphi, \nabla)$ for a Dieudonné crystal \mathcal{E} over R . Conversely, by inverting the above equivalence of categories $\mathcal{E} \rightsquigarrow \mathcal{E}(\widehat{D})$, given $\mathcal{M} \in \mathbf{MF}_{\widehat{D}}(\varphi, \nabla)$ one obtains a crystal $\mathcal{E}_{\mathcal{M}}$ over R with a horizontal isomorphism $\mathcal{M} \cong \mathcal{E}_{\mathcal{M}}(\widehat{D})$, and we have $F : \varphi^* \mathcal{E}_{\mathcal{M}} \rightarrow \mathcal{E}_{\mathcal{M}}$ and $V : \mathcal{E}_{\mathcal{M}} \rightarrow \varphi^* \mathcal{E}_{\mathcal{M}}$ which are induced from $1 \otimes \varphi_{\mathcal{M}}$ and $\psi_{\mathcal{M}}$. (Here $\psi_{\mathcal{M}}$ is defined in Remark 3.2.4.) The Hodge filtration is given by $\text{Fil}^1 \mathcal{E}_{\mathcal{M}}(R) = \text{Fil}^1 \mathcal{M}/(\text{Fil}^1 \widehat{D})\mathcal{M}$, which satisfies Definition 3.1.2(3). So we obtain a quasi-inverse $\mathcal{M} \rightsquigarrow \mathcal{E}_{\mathcal{M}}$ of the functor $\mathcal{E} \rightsquigarrow \mathcal{E}(\widehat{D})$. This proves the proposition when p is locally nilpotent in R .

Now, let R be any p -adic ring, and set $\mathcal{M} := \mathcal{E}(\widehat{D})$ for a Dieudonné crystal \mathcal{E} over $\mathrm{Spf}(R, (p))$. Note that all the extra structures on \mathcal{M} except $\mathrm{Fil}^1 \mathcal{M}$ depend only on $\overline{\mathcal{E}}$, which is a Dieudonné crystal over $\mathrm{Spec} R/(p)$. Set $\mathcal{M}_n := \mathcal{E}|_{R/(p^n)}(\widehat{D})$ equipped with all the extra structure as in the proposition. Then the natural reduction map $\mathcal{M} \rightarrow \mathcal{M}_n$ is an \widehat{D} -isomorphism compatible with Frobenius structures and connections if they are defined, and identifies $\mathrm{Fil}^1 \mathcal{M}_n$ with $p^n \mathcal{M} + \mathrm{Fil}^1 \mathcal{M}$. Therefore, we have $\mathcal{M} \xrightarrow{\sim} \varprojlim_n \mathcal{M}_n$ respecting all the extra structures, and the proposition for R is deduced from the proposition for $R/(p^n)$. \square

Definition 3.2.6. Let \widehat{D} be a frame (Definition 3.2.1), and let \mathcal{M} be an object of either $\mathbf{MF}_{\widehat{D}}(\varphi)$ or $\mathbf{MF}_{\widehat{D}}(\varphi, \nabla)$. Then \mathcal{M} is called φ -nilpotent if $\varphi^n_{\mathcal{M}}(\mathcal{M}) \subset p\mathcal{M}$ for some $n \gg 0$. Similarly, \mathcal{M} is called ψ -nilpotent if $\psi^n_{\mathcal{M}}(\mathcal{M}) \subset p(\varphi^{n*} \mathcal{M})$ for $n \gg 0$ where $\psi_{\mathcal{M}} = p(1 \otimes \varphi_{\mathcal{M}})^{-1} : \mathcal{M} \rightarrow \varphi^* \mathcal{M}$. Let $\mathbf{MF}_{\widehat{D}}(\varphi)^{\varphi \text{ nilp}}$ and $\mathbf{MF}_{\widehat{D}}(\varphi)^{\psi \text{ nilp}}$ respectively denote the full subcategories of φ -nilpotent and ψ -nilpotent objects. We similarly define $\mathbf{MF}_{\widehat{D}}(\varphi, \nabla)^{\varphi \text{ nilp}}$ and $\mathbf{MF}_{\widehat{D}}(\varphi, \nabla)^{\psi \text{ nilp}}$.

Note that if \mathcal{E} is an F -nilpotent (respectively, V -nilpotent) Dieudonné crystal, then $\mathcal{E}(\widehat{D})$ is φ -nilpotent (respectively, ψ -nilpotent). The converse holds if \widehat{D} is constructed as in Remark 3.2.2.

Example 3.2.7. We give an example where we can make “nice” choices of A and \widehat{D} . Let \bar{A} be a \mathbb{F}_p -algebra which locally admits a finite p -basis. Then by Lemma 2.3.1 there is a p -adic \mathbb{Z}_p -flat lift A with a lift of Frobenius φ . In this case, we may set $\widehat{D} = A$ and $\mathrm{Fil}^1 \widehat{D} = (p)$. Then it follows from [dJ95, Proposition 2.5.2] that the objects in $\mathbf{MF}_{\widehat{D}}(\varphi, \nabla)$ are precisely *crystalline Dieudonné modules* defined in [dJ95, Definition 2.3.4]. Note that $p \in \mathrm{Fil}^1 \widehat{D}$ and $\varphi_1(p) = 1$, so $(R_0, pR_0, R_0/(p), \varphi, \varphi_1)$ is a frame and $\mathbf{MF}_A(\varphi, \nabla)$ makes sense.

Remark 3.2.8 (lci case). We present more examples when \widehat{D} is a p -adic flat \mathbb{Z}_p -algebra. Assume that R is a noetherian \mathbb{F}_p -algebra, and all the local rings are complete intersection (i.e., R is locally complete intersection). Assume also that there exists a p -adic \mathbb{Z}_p -flat noetherian ring A such that $A/(p)$ locally admits a finite p -basis and there is a surjective map $A \twoheadrightarrow R$. Let \widehat{D} be the p -adic PD envelop of $A \twoheadrightarrow R$ as constructed in Remark 3.2.2.

We now claim that such \widehat{D} is \mathbb{Z}_p -flat. For any maximal ideal $\mathfrak{m} \subset R$, let $\widehat{A}_{\mathfrak{m}}$ denote the completion of A with respect to the kernel of $A \twoheadrightarrow R/\mathfrak{m}$. Since A is noetherian, $\widehat{A}_{\mathfrak{m}}$ is a flat A -algebra. It then follows from [BO78, Proposition 3.21] that $\widehat{D}_{\mathfrak{m}} := \widehat{A}_{\mathfrak{m}} \widehat{\otimes}_A \widehat{D}$ is the p -adically completed PD envelop of $\widehat{A}_{\mathfrak{m}} \twoheadrightarrow \widehat{R}_{\mathfrak{m}}$, where $\widehat{\otimes}_A$ is the p -adic completion of the usual tensor product.

By the p -basis assumption on $A/(p)$, the completion $\widehat{A}_{\mathfrak{m}}/(p)$ is a complete local noetherian ring with finite p -basis by [dJ95, Lemma 1.1.2]. By cotangent complex consideration via [dJ95, Lemma 1.1.1], $\widehat{A}_{\mathfrak{m}}/(p)$ is formally smooth over \mathbb{F}_p so it is regular by [Mat86, Theorems 28.7]. As $\widehat{R}_{\mathfrak{m}}$ is assumed to be complete intersection, the kernel of $\widehat{A}_{\mathfrak{m}} \twoheadrightarrow \widehat{R}_{\mathfrak{m}}$ is generated by a regular sequence. In such a case, it is known that the p -adically completed PD envelop $\widehat{D}_{\mathfrak{m}}$ is \mathbb{Z}_p -flat. (Cf. the proof of [dJM99, Lemma 4.7].)

Note that we have $\mathrm{MaxSpec} R \xrightarrow{\sim} \mathrm{MaxSpec} \widehat{D}$ because $\mathrm{Fil}^1 \widehat{D}$ is contained in the Jacobson radical (cf. Remark 3.2.2). This shows that $\{\mathrm{Spf}(\widehat{D}_{\mathfrak{m}}, (p))\}_{\mathfrak{m} \in \mathrm{MaxSpec} R}$ is an fpqc covering of $\mathrm{Spf}(\widehat{D}, (p))$. As $\widehat{D}_{\mathfrak{m}}$ is \mathbb{Z}_p -flat, we have \mathbb{Z}_p -flatness of \widehat{D} .

Remark 3.2.9. Since the definition of $\mathbf{MF}_{\widehat{D}}(\varphi)$ depends upon non-canonical choices, one cannot expect to have a notion of base change for arbitrary map $f : R \rightarrow R'$ of p -adic rings. If, on the other hand, one can find (A, φ) and (A', φ) which induce

“frames” for R and R' , respectively (cf., §3.2), and there is a φ -compatible map $\tilde{f} : A \rightarrow A'$ which reduces to f , then \tilde{f} extends to a map of p -adically completed divided power envelopes $\widehat{D} \rightarrow \widehat{D}'$, which respects all the extra structures. Note that Lemma 2.3.5 provides some examples of f where there is \tilde{f} with desired properties. See §3.6 (Ex1) for more details.

Now the scalar extension $\mathcal{M} \rightsquigarrow \mathcal{M}' := \widehat{D}' \otimes_{\widehat{D}} \mathcal{M}$ induces a functor $\mathbf{MF}_{\widehat{D}}(\varphi) \rightarrow \mathbf{MF}_{\widehat{D}'}(\varphi)$ as follows: we set $\varphi_{\mathcal{M}'} := \varphi_{\widehat{D}'} \otimes \varphi_{\mathcal{M}}$, and $\mathrm{Fil}^1 \mathcal{M}' := \ker[\mathcal{M}' \rightarrow R' \otimes_R (\mathcal{M}/\mathrm{Fil}^1 \mathcal{M})]$, and

$$(3.2.10) \quad \nabla_{\mathcal{M}'}(s' \otimes m) = m \otimes d_{\widehat{D}'}(s') + s' \cdot \nabla_{\mathcal{M}}(m)$$

for any $s' \in \widehat{D}'$ and $m \in \mathcal{M}$.

3.3. Ring S . We use the notation from §2.2.1, such as $R, R_0, E(u), \varpi \in R$. We also fix a lift of Frobenius $\varphi : R_0 \rightarrow R_0$, which exists by Lemma 2.3.1. We set $\mathfrak{S} := R_0[[u]]$ and extend φ by $\varphi(u) = u^p$. Now, let S be the p -adically completed divided power envelope of \mathfrak{S} with respect to the kernel of $\mathfrak{S} \twoheadrightarrow R$, and define $\mathrm{Fil}^1 S, \varphi, d_S$ as in Remark 3.2.2 (for $A = \mathfrak{S}$ and $\widehat{D} = S$). We give the p -adic topology to all of these rings.

Recall that $R = \mathfrak{S}/E(u)$. From the assumption on $E(u)$ as in §2.2.1, it is straightforward to check the following (using binomial coefficients):

$$(3.3.1) \quad S = \left\{ \sum_{n \geq 0} f_n \frac{u^n}{[n/e]!}; \text{ where } f_n \in R_0 \text{ and } f_n \rightarrow 0 \text{ } p\text{-adically} \right\},$$

where e is the degree of $E(u)$. The equality takes place in $R_0[\frac{1}{p}][[u]]$. It follows that S is \mathbb{Z}_p -flat and $\mathrm{Fil}^1 S$ is topologically generated by the divided powers of $E(u)$.

Let $c := \varphi(E(u))/p$. One can check by direct computation that $c \in S^\times$, so we have $\langle \frac{c}{p}(\mathrm{Fil}^1 S) \rangle = S$. (Indeed, the map $S \twoheadrightarrow R_0$, induced from $u \mapsto 0$, maps c to 1.) This shows that $(S, \mathrm{Fil}^1 S, R, \varphi, \varphi_1)$ is a frame in the sense of Definition 3.2.1, and we can apply the discussions in §3.2. In particular, $\mathbf{MF}_S(\varphi, \nabla)$ is in equivalence with the category of Dieudonné crystals over $\mathrm{Spf}(R, (\varpi))$ by Proposition 3.2.5.

In addition to d_S , we define another connection $d_S^u : S \rightarrow S \otimes_{R_0} \widehat{\Omega}_{R_0}$ by $d_S^u(s) := d_S(s) \bmod du$ for any $s \in S$; more concretely, we have $d_S^u(\sum_n \frac{u^n}{[n/e]!} f_n) = \sum_n \frac{u^n}{[n/e]!} d_{R_0}(f_n)$ for $f_n \in R_0$ which p -adically tends to 0.

We define a derivation $N : S \rightarrow S$ by $N := -u \frac{\partial}{\partial u}$. For any $s \in S$ we have $d_S(s) = (-1/u)N(s)du + d_S^u(s)$, and $N \circ \varphi = p\varphi \circ N$.

In addition to $\mathbf{MF}_S(\varphi, \nabla)$ and $\mathbf{MF}_S(\varphi)$ (cf., Definition 3.2.3), we make the following definitions:

Definition 3.3.2. Let $\mathbf{MF}_S^w(\varphi, \nabla^u)$ be the category of $(\mathcal{M}, \mathrm{Fil}^1 \mathcal{M}, \varphi_1) \in \mathbf{MF}_S(\varphi)$ equipped with a topologically quasi-nilpotent integrable connection $\nabla_{\mathcal{M}}^u : \mathcal{M} \rightarrow \mathcal{M} \otimes_{R_0} \widehat{\Omega}_{R_0}$ over d_S^u which commutes with $\varphi_{\mathcal{M}}$.

Let $\mathbf{MF}_S^{\mathrm{Br}}(\varphi, \nabla^0)$ be the category of $(\mathcal{M}, \mathrm{Fil}^1 \mathcal{M}, \varphi_1) \in \mathbf{MF}_S(\varphi)$ equipped with a topologically quasi-nilpotent integrable connection $\nabla_{\mathcal{M}_0}$ on $\mathcal{M}_0 := R_0 \otimes_S \mathcal{M}$ which commutes with $\varphi_{\mathcal{M}_0} = \varphi_{R_0} \otimes \varphi_{\mathcal{M}}$. We call an object in $\mathbf{MF}_S^{\mathrm{Br}}(\varphi, \nabla^0)$ a *Breuil S -module*.

Remark 3.3.3. For $\mathcal{M} \in \mathbf{MF}_S(\varphi)$, the giving of $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathfrak{S}} \widehat{\Omega}_{\mathfrak{S}}$ that makes \mathcal{M} an object in $\mathbf{MF}_S(\varphi, \nabla)$ is equivalent to the giving of $\nabla_{\mathcal{M}}^u : \mathcal{M} \rightarrow \mathcal{M} \otimes_{R_0} \widehat{\Omega}_{R_0}$ and a differential operator $N_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$ over N (i.e., $N_{\mathcal{M}}(sm) = N(s)m + sN_{\mathcal{M}}(m)$ for any $s \in S$ and $m \in \mathcal{M}$), such that

$$(1) \quad (\mathcal{M}, \nabla_{\mathcal{M}}^u) \in \mathbf{MF}_S^w(\varphi, \nabla^u);$$

- (2) $N_{\mathcal{M}}(\mathcal{M}) \subset u\mathcal{M}$; and $-\nabla_{\mathcal{M}}^u \circ (u^{-1}N_{\mathcal{M}}) = (u^{-1}N_{\mathcal{M}} \otimes 1_{\widehat{\Omega}_{R_0}}) \circ \nabla_{\mathcal{M}}^u$;
(3) We have $N_{\mathcal{M}} \circ \varphi_{\mathcal{M}} = p \cdot \varphi_{\mathcal{M}} \circ N_{\mathcal{M}}$.

Given $N_{\mathcal{M}}$ and $\nabla_{\mathcal{M}}^u$ as above, let $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathfrak{S}} \widehat{\Omega}_{\mathfrak{S}}$ be the connection defined by $\nabla_{\mathcal{M}}(m) = -u^{-1}N_{\mathcal{M}}(m)du + \nabla_{\mathcal{M}}^u(m)$ for any $m \in \mathcal{M}$. One can check that $(\mathcal{M}, \nabla_{\mathcal{M}}) \in \mathbf{MF}_S(\varphi, \nabla)$ if and only if $N_{\mathcal{M}}$ and $\nabla_{\mathcal{M}}^u$ satisfy the above conditions. (Note that the connection $\nabla_{\mathcal{M}}$ is integrable if and only if $\nabla_{\mathcal{M}}^u$ is integrable and anti-commutes with $u^{-1}N_{\mathcal{M}}$.) Using this description, one can define a “forgetful functor” $\mathbf{MF}_S(\varphi, \nabla) \rightarrow \mathbf{MF}_S^w(\varphi, \nabla^u)$ by forgetting $N_{\mathcal{M}}$.

Let $I_0 \subset S$ be the ideal topologically generated by u and $\frac{u^j}{j!}$ for $j > 0$. Then we have $S/I_0 \cong R_0$, and the natural projection $S \rightarrow R_0$ commutes with φ 's. Now one can define functors $\mathbf{MF}_S(\varphi, \nabla) \rightarrow \mathbf{MF}_{R_0}^{\text{Br}}(\varphi, \nabla^0)$ and $\mathbf{MF}_S^w(\varphi, \nabla^u) \rightarrow \mathbf{MF}_{R_0}^{\text{Br}}(\varphi, \nabla^0)$ by “reducing the connection modulo I_0 .” (Note that these functors are compatible with the forgetful functor $\mathbf{MF}_S(\varphi, \nabla) \rightarrow \mathbf{MF}_S^w(\varphi, \nabla^u)$ that we constructed in the previous paragraph.)

If $\mathcal{M} = \mathcal{E}(S) \in \mathbf{MF}_S(\varphi, \nabla)$ for some Dieudonné crystal \mathcal{E} over $\text{Spf}(R, (\varpi))$, then we have $R_0 \otimes_S \mathcal{M} \cong \mathcal{E}_{R/(\varpi)}(R_0)$ as objects in $\mathbf{MF}_{R_0}(\varphi, \nabla)$. (To see the isomorphism is horizontal, interpret a connection as a HPD stratification in the sense of [BO78, Theorem 6.6], and work out the effect of base change. We leave the details to readers.)

We can construct the functors from $\mathbf{MF}_S(\varphi, \nabla)$, $\mathbf{MF}_S^w(\varphi, \nabla^u)$, and $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ into $\mathbf{MF}_{R_0}(\varphi, \nabla)$ by $\mathcal{M} \rightsquigarrow \mathcal{M}_0 := R_0 \otimes_S \mathcal{M}$ and reducing the extra structures modulo I_0 . (Cf. Remark 3.2.9.) These mod- I_0 reduction functors are compatible with the functors among $\mathbf{MF}_S(\varphi, \nabla)$, $\mathbf{MF}_S^w(\varphi, \nabla^u)$, and $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ defined above.

Finally, by working with $\text{Fil}^r S$ for $r \leq p-1$ we can study “higher weight” cases. Also by allowing $\nabla_{\mathcal{M}}$ to have a “logarithmic pole” at $u=0$ and $\mathcal{N}_{\mathcal{M}} \bmod I_0$ to be not necessarily zero, we can handle “log crystals”.

Lemma 3.3.4. *The functors $\mathbf{MF}_S(\varphi, \nabla) \rightarrow \mathbf{MF}_S^w(\varphi, \nabla^u)$ and $\mathbf{MF}_S^w(\varphi, \nabla^u) \rightarrow \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$, defined in Remark 3.3.3, are fully faithful.*

Proof. The full faithfulness of the functor $\mathbf{MF}_S^w(\varphi, \nabla^u) \rightarrow \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ follows from Lemma 3.3.5. To show the full faithfulness of $\mathbf{MF}_S(\varphi, \nabla) \rightarrow \mathbf{MF}_S^w(\varphi, \nabla^u)$, we want to show that any morphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ of objects in $\mathbf{MF}_S^w(\varphi, \nabla^u)$ commutes with the differential operators $N_{\mathcal{M}}$ and $N_{\mathcal{M}'}$.

Consider $\delta_f := N_{\mathcal{M}'} \circ f - f \circ N_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}'$. By assumption we have $\delta_f(\mathcal{M}) \subseteq I_0 \cdot \mathcal{M}'$ where $I_0 := \ker(S \rightarrow R_0)$. But since δ_f also commutes with Frobenius structures, for any $m \in \text{Fil}^1 \mathcal{M}$ we have $\delta_f(\varphi_1(m)) = c^{-1}\varphi_1(E(u)\delta_f(m))$ where $c := \frac{\varphi}{p}(E(u))$. In particular, we have $\delta_f(\mathcal{M}) \subseteq \varphi(I_0) \cdot \mathcal{M}'$ as $\varphi_1(\text{Fil}^1 \mathcal{M})$ generates \mathcal{M} . By repeating this, we obtain $\delta_f(\mathcal{M}) \subseteq \bigcap_n \varphi^n(I_0) \cdot \mathcal{M}' = \{0\}$. \square

Lemma 3.3.5. *Let $\mathcal{M} \in \mathbf{MF}_S^w(\varphi, \nabla^u)$, and set $\mathcal{M}_0 := R_0 \otimes_S \mathcal{M} \in \mathbf{MF}_{R_0}(\varphi, \nabla)$. Then there exists a unique φ -compatible section $s : \mathcal{M}_0[\frac{1}{p}] \rightarrow \mathcal{M}[\frac{1}{p}]$. Furthermore, s is horizontal and the map $1 \otimes s : S[\frac{1}{p}] \otimes_{R_0} \mathcal{M}_0 \xrightarrow{\sim} \mathcal{M}[\frac{1}{p}]$ is an isomorphism.*

Proof. Let us choose an arbitrary section $s_0 : \mathcal{M}_0 \rightarrow \mathcal{M}$, and we would like to show that the following “formula” gives a well-defined morphism $s : \mathcal{M}_0[\frac{1}{p}] \rightarrow \mathcal{M}[\frac{1}{p}]$:

$$\begin{aligned} s &:= \lim_{n \rightarrow \infty} (1 \otimes \varphi_{\mathcal{M}}^n) \circ (\varphi^{*n} s_0) \circ (1 \otimes \varphi_{\mathcal{M}_0}^{-n}) \\ &= s_0 + \sum_{n=0}^{\infty} \left[(1 \otimes \varphi_{\mathcal{M}}^{n+1}) \circ (\varphi^{*n+1} s_0) \circ (1 \otimes \varphi_{\mathcal{M}_0}^{-n-1}) - (1 \otimes \varphi_{\mathcal{M}}^n) \circ (\varphi^{*n} s_0) \circ (1 \otimes \varphi_{\mathcal{M}_0}^{-n}) \right]. \end{aligned}$$

Note that $(1 \otimes \varphi_{\mathcal{M}_0})[\frac{1}{p}] : \varphi^* \mathcal{M}_0[\frac{1}{p}] \rightarrow \mathcal{M}_0[\frac{1}{p}]$ is an isomorphism, and the preimage of \mathcal{M}_0 is contained in $p^{-1}(\varphi^* \mathcal{M}_0)$.

Set $s_{n+1} := (1 \otimes \varphi_{\mathcal{M}}^{n+1}) \circ (\varphi^{*n+1} s_0) \circ (1 \otimes \varphi_{\mathcal{M}_0}^{-n-1}) - (1 \otimes \varphi_{\mathcal{M}}^n) \circ (\varphi^{*n} s_0) \circ (1 \otimes \varphi_{\mathcal{M}_0}^{-n})$. Clearly $s_1(\mathcal{M}_0) \subseteq p^{-1}u\mathcal{M}$, because s_0 is a section. By iterating this, one obtains that $s_{n+1}(\mathcal{M}_0) \subseteq p^{-n-1}u^{p^n}\mathcal{M}$ for any $n \geq 0$. Note that $p^{-n-1}u^{p^n} = p^{-n-1}(q_n!)u^{p^n - q_n e}(u^e)^{[q_n]}$ where $q_n := \lfloor \frac{p^n}{e} \rfloor$, so we have

$$\text{ord}_p(p^{-n-1}(q_n!)) \geq -n-1 + p^{n-1}/e - 1 \xrightarrow{n \rightarrow \infty} \infty$$

This proves that $s : \mathcal{M}_0[\frac{1}{p}] \rightarrow \mathcal{M}[\frac{1}{p}]$ is well-defined, and by construction s commutes with φ 's. The map $1 \otimes s : S[\frac{1}{p}] \otimes_{R_0} \mathcal{M}_0 \xrightarrow{\sim} \mathcal{M}[\frac{1}{p}]$ is a surjective map of projective $S[\frac{1}{p}]$ -modules of same rank (by Nakayama lemma), so it is an isomorphism.

Now, if there are two sections s and s' , then $(s - s')(\mathcal{M}_0[\frac{1}{p}]) \subseteq I_0\mathcal{M}[\frac{1}{p}]$. If both s and s' commute with φ 's then we obtain

$$\begin{aligned} (s - s')(\mathcal{M}_0[1/p]) &\subseteq (s - s')((1 \otimes \varphi_{\mathcal{M}_0})^n(\varphi^{*n} \mathcal{M}_0[1/p])) \\ &\subseteq (1 \otimes \varphi_{\mathcal{M}}^n) \left(\varphi^{*n}((s - s')(\mathcal{M}_0[1/p])) \right) \subseteq \varphi^n(I_0)\mathcal{M}[1/p]. \end{aligned}$$

Therefore, we have $\text{im}(s - s') \subseteq \bigcap_n \varphi^n(I_0)\mathcal{M}[\frac{1}{p}] = \{0\}$, which establishes the uniqueness of s .

It is left to show that s is horizontal. Consider $\delta_s := (s \otimes 1) \circ \nabla_{\mathcal{M}_0} - \nabla_{\mathcal{M}}^u \circ s : \mathcal{M}_0[\frac{1}{p}] \rightarrow \mathcal{M}[\frac{1}{p}] \otimes_{R_0} \widehat{\Omega}_{R_0}$. By construction, δ_s commutes with φ 's and its image is divisible by u . So by the same argument that shows the uniqueness of s , we conclude that $\delta_s = 0$. \square

Remark 3.3.6 (lci case). We can extend the construction of S and Lemma 3.3.4 in the following variant as well. Let R be a p -adic flat \mathbb{Z}_p -algebra which is locally complete intersection. Assume that there exists a p -adic noetherian flat \mathbb{Z}_p -algebra A such that

- $A/(p)$ locally admits a finite p -basis;
- there exists a surjective map $A[[u]] \twoheadrightarrow R$, where u maps to a regular element $\varpi \in R$;
- the kernel of $A[[u]] \twoheadrightarrow R$ includes an element $E(u) = p + \sum_{i=1}^e a_i u^i$ with $a_i \in A$ and $a_e \in A^\times$.

Let S_0 be the p -adically completed PD envelop of $A \twoheadrightarrow R/(\varpi)$, and S be the p -adically completed PD envelop of $A[[u]] \twoheadrightarrow R$. By Remark 3.2.8, S_0 defines a frame. We also have the description of S similar to (3.3.1) with R_0 replaced by S_0 , so S is also \mathbb{Z}_p -flat and $\varphi_1(E(u)) \in S^\times$; in other words, S defines a frame.

Now, by simply replacing R_0 with S_0 , we can define $\mathbf{MF}_S(\varphi, \nabla)$, $\mathbf{MF}_S^w(\varphi, \nabla^u)$, and $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$, and the proof of Lemma 3.3.4 generalises to this setting.

3.4. Review of results by Berthelot-Messing and de Jong. Let \bar{A} be an \mathbb{F}_p -algebra locally admitting a finite p -basis, and choose (A, φ) lifting \bar{A} and the Frobenius morphism (which exists by Lemma 2.3.1). For any p -divisible group G_0 over \bar{A} we set $\mathcal{M}_0^*(G_0) := \mathbb{D}^*(G_0)(A)$, which belongs to $\mathbf{MF}_A(\varphi, \nabla)$ by Example 3.2.7. We recall the following results on the functors \mathbb{D}^* and \mathcal{M}_0^* :

- (1) Suppose that \bar{A} is formally smooth and formally of finite type over some field k which admits a finite p -basis. (For example, $\bar{A} = R/(\varpi)$ where R satisfies the formally finite-type assumption (§2.2.2).) Then \mathcal{M}_0^* is an anti-equivalence of categories. If $\bar{\mathfrak{X}}$ is a scheme that has a Zariski covering by such $\text{Spec } \bar{A}$'s, then \mathbb{D}^* over $\bar{\mathfrak{X}}$ is an anti-equivalence of categories. (Cf. [dJ95, Main Theorem 1].)

- (2) Assume that $\bar{\mathfrak{X}}$ is a normal \mathbb{F}_p -scheme which is locally irreducible and locally admits a finite p -basis. Then \mathbb{D}^* over $\bar{\mathfrak{X}}$ is fully faithful. (Cf. [BM90, Théorème 4.1.1].) The same statement holds for \mathcal{M}_0^* if $\bar{\mathfrak{X}}$ is affine (for example, $\bar{\mathfrak{X}} = \text{Spec } R/(\varpi)$ where R satisfies the normality assumption (§2.2.4)).

Let us set up the notation for the the main theorem of this section below. Assume that \mathfrak{X} is a formal scheme which has a Zariski covering $\{\text{Spf}(R_\alpha, (\varpi))\}$ where R_α satisfies the condition in §3.3. Consider the exact functor

$$\{p\text{-divisible groups over } R\} \xrightarrow{\mathbb{D}^*} \{\text{Dieudonné crystals over } \mathfrak{X} \text{ (Definition 3.1.2)}\}.$$

If $\mathfrak{X} = \text{Spf}(R, (p))$ where R satisfies the condition in §3.3, then consider the exact functor

$$\mathcal{M}^* : \{p\text{-divisible groups over } R\} \rightarrow \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$$

defined by $\mathcal{M}^*(G) := \mathbb{D}^*(G)(S)$ for S as chosen in §3.3. Note that $\mathcal{M}^*(G)$ is a priori an object in $\mathbf{MF}_S(\varphi, \nabla)$ by Proposition 3.2.5, but we view it as an object in $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ via the fully faithful functor $\mathbf{MF}_S(\varphi, \nabla) \rightarrow \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ constructed in Remark 3.3.3. (Cf. Lemma 3.3.4.)

Theorem 3.5. *Assume that $p > 2$.*

If \mathfrak{X} is Zariski locally covered by $\text{Spf}(R, (\varpi))$ where each R satisfies the formally finite-type assumption (§2.2.2), then \mathbb{D}^ and \mathcal{M}^* (if the latter is defined) are anti-equivalences of categories.*

If \mathfrak{X} is Zariski locally covered by $\text{Spf}(R, (\varpi))$ where each R satisfies the normality assumption (§2.2.4), then \mathbb{D}^ and \mathcal{M}^* (if the latter is defined) are fully faithful.*

If $p = 2$, then in both cases above the functors \mathbb{D}^ and \mathcal{M}^* (if the latter is defined) are fully faithful up to isogeny.*

When $p = 2$, we can obtain a better result when restricted to formal or unipotent p -divisible groups (cf. Theorem 3.5.4, Corollary 10.3.1).

Proof. To prove the theorem it suffices to prove the statement for \mathcal{M}^* when $\mathfrak{X} = \text{Spf}(R, (\varpi))$ (by Proposition 3.2.5 and Lemma 3.3.4), so we assume this from now on.

Let us outline the idea. First, assume that $p > 2$. Then we define a functor G^* on $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)^{\mathbb{D}}$ and show that it is a quasi-inverse of \mathcal{M}^* if \mathcal{M}_0^* (as defined in §3.4) is fully faithful (respectively, quasi-inverse up to isogeny if \mathcal{M}_0^* is fully faithful up to isogeny). This strategy can be modified to show our claim when $p = 2$. The construction of the functor G^* is motivated by the proof of [Kis06, Proposition A.6], while we start with the results of de Jong and Berthelot-Messing (which is recalled in §3.4) instead of classical Dieudonné theory.

Let $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)^{\mathbb{D}}$ be the full subcategory of $\mathcal{M} \in \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ such that $R_0 \otimes_S \mathcal{M}$ is in the essential image of \mathcal{M}_0^* (defined in §3.4). Clearly, $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)^{\mathbb{D}}$ contains the essential image of \mathcal{M}^* . (Note that if R satisfies the formally finite-type assumption (§2.2.2) then $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)^{\mathbb{D}} = \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ by de Jong's theorem.)

Suppose $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_1, \nabla_{\mathcal{M}_0}) \in \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)^{\mathbb{D}}$, and set $\mathcal{M}_0 := R_0 \otimes_S \mathcal{M} \in \mathbf{MF}_{R_0}(\varphi, \nabla)$. By assumption, there exists a p -divisible group G_0 over $R/(\varpi)$ such that $\mathcal{M}_0 \cong \mathbb{D}^*(G_0)(R_0)$. To construct G corresponding to \mathcal{M} , we will lift G_0 over R in a functorial way using Grothendieck-Messing deformation theory [Mes72].

For any non-negative integer i , let $I_i \subset S$ be the ideal topologically generated by u^{i+1} and $\frac{u^{e_j}}{j!}$ for any $e_j > i$. By applying the discussion of Remark 3.2.9 to $f : R \twoheadrightarrow R/(\varpi^{i+1})$, $A' = \mathfrak{S}/(u^{i+1})$ and $\widehat{D}' = S_i := S/I_i$, it follows that $\mathcal{M}/I_i \mathcal{M}$ has a natural structure of $\mathbf{MF}_{S_i}(\varphi)$. Recall that for any p -divisible group G_i over $R/(\varpi^{i+1})$, we have $\mathbb{D}^*(G_i)(S_i) \in \mathbf{MF}_{S_i}(\varphi)$ by Proposition 3.2.5,

Claim 3.5.1. *For any $0 \leq i < e$, there exist a p -divisible group G_i over $R/(\varpi^{i+1})$ and an isomorphism $\mathbb{D}^*(G_i)(S_i) \cong S_i \otimes_S \mathcal{M}$ in $\mathbf{MF}_{S_i}(\varphi)$ which lifts the isomorphism $\mathbb{D}^*(G_i)(R_0) \cong R_0 \otimes_S \mathcal{M}$. This construction $\mathcal{M} \rightsquigarrow G_i$ is functorial in G_0 and \mathcal{M} .*

The case when $i = 0$ is obvious. Now assuming the claim for $i - 1$ for some $0 < i < e$, and we will prove the claim for i . Consider $\mathfrak{S}_i := \mathfrak{S}/(u^{i+1}) \twoheadrightarrow R/(\varpi^i)$, and let \widehat{D}_i be the p -adically completed divided power envelope of \mathfrak{S}_i with respect to the kernel. Then we have $\widehat{D}_i \cong S_i$ with $\mathrm{Fil}^1 \widehat{D}_i = I_{i-1} + \mathrm{Fil}^1 S_i$. The natural projection $\mathfrak{S} \twoheadrightarrow \mathfrak{S}/(u^{i+1}) = \mathfrak{S}_i$ induces a map $S \twoheadrightarrow \widehat{D}_i$ respecting all the extra structure, so we obtain the scalar extension functor $\mathbf{MF}_S(\varphi) \rightarrow \mathbf{MF}_{\widehat{D}_i}(\varphi)$. Note that $\mathrm{Fil}^1(\widehat{D}_i \otimes_S \mathcal{M})$ is generated by $I_{i-1}\mathcal{M}/I_i\mathcal{M}$ and the image of $\mathrm{Fil}^1 \mathcal{M}$. Similarly, we obtain the scalar extension functor $\mathbf{MF}_{\widehat{D}_i}(\varphi) \rightarrow \mathbf{MF}_{S_{i-1}}(\varphi)$.

By Proposition 3.2.5 we have $\mathbb{D}^*(G_{i-1})(\widehat{D}_i) \in \mathbf{MF}_{\widehat{D}_i}(\varphi)$. Applying Lemma 3.5.2 to $\widehat{D} = \widehat{D}_i$ and $\widehat{J} := I_{i-1}\widehat{D}_i$, we deduce that there is a unique φ -compatible isomorphism $\mathbb{D}^*(G_{i-1})(\widehat{D}_i) \cong \widehat{D}_i \otimes_S \mathcal{M}$ which lifts the isomorphism $\mathbb{D}^*(G_{i-1})(S_{i-1}) \cong S_{i-1} \otimes_S \mathcal{M}$ given by the induction hypothesis.

Consider the divided power thickening $R/(\varpi^{i+1}) \twoheadrightarrow R/(\varpi^i)$ with the ‘‘trivial’’ divided power structure (i.e., $(\varpi^i)^{[j]} = 0$ for any $j > 1$). This is compatible with the canonical divided power structures on $p\mathbb{Z}_p$. Choose a lift G_i of G_{i-1} which corresponds to the filtration defined by the image of $\mathrm{Fil}^1 \mathcal{M}$ in $R/(\varpi^{i+1}) \otimes_S \mathcal{M}$ via Grothendieck-Messing deformation theory. Then by construction we have natural φ -compatible isomorphisms $\mathbb{D}^*(G_i)(S_i) \cong \mathbb{D}^*(G_{i-1})(\widehat{D}_i) \cong S_i \otimes_S \mathcal{M}$, which takes $\mathrm{Fil}^1 \mathbb{D}^*(G_i)(S_i)$ to the image of $\mathrm{Fil}^1 \mathcal{M}$ in $S_i \otimes_S \mathcal{M}$. Note also that the formation of G_i is functorial in G_{i-1} and the filtration. (See [Mes72, Ch.V, Theorem 1.6] for the precise statement.) This proves Claim 3.5.1.

Consider $\mathfrak{S} \twoheadrightarrow R/(p) = R/(\varpi^e)$ and let \widehat{D}_e be the p -adically completed divided power envelope of \mathfrak{S} with respect to the kernel. Then we have $\widehat{D}_e \cong S$ with $\mathrm{Fil}^1 \widehat{D}_e = I_{e-1} + \mathrm{Fil}^1 S$. As in the proof of Claim 3.5.1, we have scalar extension functors $\mathbf{MF}_S(\varphi) \rightarrow \mathbf{MF}_{\widehat{D}_e}(\varphi)$ which sends $(\mathcal{M}, \mathrm{Fil}^1 \mathcal{M})$ to $\widehat{D}_e \otimes_S \mathcal{M} \cong (\mathcal{M}, I_{e-1}\mathcal{M} + \mathrm{Fil}^1 \mathcal{M})$. Applying Lemma 3.5.2 to $\widehat{D} = \widehat{D}_e$ and $\widehat{J} = I_{e-1}$ and proceeding similarly to the proof of Claim 3.5.1, we deduce that there is a unique φ -compatible isomorphism $\mathbb{D}^*(G_{e-1})(\widehat{D}_e) \cong \mathcal{M}$ which lifts the isomorphism

$$\mathbb{D}^*(G_{e-1})(S_{e-1}) \cong S_{R,e-1} \otimes_S \mathcal{M}$$

in $\mathbf{MF}_{S_{e-1}}(\varphi)$.

Now assume that $p > 2$, in which case pR is a topologically nilpotent divided power ideal. Let $G_{\mathcal{M}}$ be the lift of G_{e-1} which corresponds to the filtration

$$\mathrm{Fil}^1 \mathcal{M}/(\mathrm{Fil}^1 S)\mathcal{M} \subset \mathcal{M}/(\mathrm{Fil}^1 S)\mathcal{M}.$$

(Note that $G_{\mathcal{M}}$ functorially depends on \mathcal{M} and G_0 .) As in the proof of Claim 3.5.1, we obtain a natural isomorphism $\mathcal{M}^*(G_{\mathcal{M}}) \cong \mathcal{M}$ in $\mathbf{MF}_S^{\mathrm{Br}}(\varphi, \nabla^0)$, which proves essential subjectivity of \mathcal{M}^* for $p > 2$.

If \mathcal{M}_0^* , defined in §3.4, is fully faithful (which is the case if R satisfies the normality assumption (§2.2.4)), then we define a functor G^* from $\mathbf{MF}_S^{\mathrm{Br}}(\varphi, \nabla^0)^{\mathbb{D}}$ to the category of p -divisible groups over R by choosing a quasi-inverse $R_0 \otimes_S \mathcal{M} \rightsquigarrow G_0$ of \mathcal{M}_0^* and setting $G^*(\mathcal{M}) := G_{\mathcal{M}}$ to be the lift of G_0 produced by the procedure described above. Then by construction we have a natural isomorphism $(\mathcal{M}^* \circ G^*)(\mathcal{M}) \cong \mathcal{M}$ for any $\mathcal{M} \in \mathbf{MF}_S^{\mathrm{Br}}(\varphi, \nabla^0)^{\mathbb{D}}$. We also get a natural isomorphism $(G^* \circ \mathcal{M}^*)(G) \cong G$ for any p -divisible group G over R from the uniqueness of each deformation step. This settles the theorem when $p > 2$.

Assume that $p = 2$ and we show that the functor \mathcal{M}^* is fully faithful up to isogeny; indeed, we show that for any p -divisible groups G and G' over R and a morphism $\alpha : \mathcal{M}^*(G') \rightarrow \mathcal{M}^*(G)$ there exists a unique morphism $f' : G \rightarrow G'$ such that $\mathcal{M}^*(f') = p^2\alpha$. As in the case when $p > 2$, α gives rise to a morphism $f_{R/p} : G_{R/p} \rightarrow G'_{R/p}$. By [Kat81, Lemma 1.1.3], it follows that $p^2 f_{R/p}$ lifts to a unique morphism $f'_{R/p^2} : G_{R/p^2} \rightarrow G'_{R/p^2}$. Applying the Grothendieck-Messing deformation theory to the (topologically nilpotent!) divided power thickening $R \rightarrow R/p^2$, there exists a unique morphism $f' : G \rightarrow G'$ which lifts f'_{R/p^2} and satisfies $\mathcal{M}^*(f') = p^2\alpha$. \square

Lemma 3.5.2. *Let \widehat{D} be a frame (cf., Definition 3.2.1) such that $\text{Fil}^1 \widehat{D}$ is a divided power ideal, and $\widehat{J} \subseteq \text{Fil}^1 \widehat{D}$ be a divided power sub-ideal such that $\varphi^n(\widehat{J}) \subset p^{n+j_n} \widehat{J}$ for some $\{j_n\}$ with $j_n \rightarrow \infty$ as $n \rightarrow \infty$. (In particular, \widehat{D}/\widehat{J} is automatically \mathbb{Z}_p -flat so \widehat{D}/\widehat{J} has a natural frame structure induced from \widehat{D} , and the reduction modulo \widehat{J} induces a functor $\mathbf{MF}_{\widehat{D}}(\varphi) \rightarrow \mathbf{MF}_{\widehat{D}/\widehat{J}}(\varphi)$.)*

Then for $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{MF}_{\widehat{D}}(\varphi)$, any isomorphism $\bar{\theta} : \mathcal{M}_1/\widehat{J}\mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2/\widehat{J}\mathcal{M}_2$ in $\mathbf{MF}_{\widehat{D}/\widehat{J}}(\varphi)$ uniquely lifts to a φ -compatible \widehat{D} -linear isomorphism $\theta : \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2$.

Proof. The proof is identical to the proof of [Kis06, Lemma A.4]. Let us first show the uniqueness. Let θ and θ' be two lifts as in the statement. Then since $\theta - \theta'$ commutes with φ 's, we have $(\theta - \theta')(\mathcal{M}_1) \subseteq \bigcap_{n \geq 0} \varphi^n(\widehat{J})(\mathcal{M}_2) = \{0\}$.

Let us show the existence. Note first that $1 \otimes \varphi_{\mathcal{M}_i}$ is injective; indeed, \widehat{D} is p -torsion free by assumption, and $(1 \otimes \varphi_{\mathcal{M}_i})[\frac{1}{p}]$ is an isomorphism because it is a surjective map of finite projective $\widehat{D}[\frac{1}{p}]$ -modules of the same rank. Therefore $(1 \otimes \varphi_{\mathcal{M}_1})/p : \text{Fil}^1(\varphi^* \mathcal{M}_1) \rightarrow \mathcal{M}_1$ is an isomorphism, where $\text{Fil}^1(\varphi^* \mathcal{M}_1) \subset \varphi^* \mathcal{M}_1$ is the image of $\varphi^*(\text{Fil}^1 \mathcal{M}_1)$. (Note that φ is not necessarily flat.)

Pick an arbitrary lift $\theta_0 : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ of $\bar{\theta}$. As θ in the statement is not required to respect Fil^1 's, we may replace $\text{Fil}^1 \mathcal{M}_i$ by $\widehat{J}\mathcal{M}_i + \text{Fil}^1 \mathcal{M}_i$ to assume that $\theta_0(\text{Fil}^1 \mathcal{M}_1) \subset \text{Fil}^1 \mathcal{M}_2$. Let us recursively define lifts θ_n of $\bar{\theta}$, such that θ_0 is the chosen lift (when $n = 0$), and for any $n \in \mathbb{Z}_{\geq 0}$ we have the following diagram:

$$(3.5.3) \quad \begin{array}{ccc} \text{Fil}^1(\varphi^* \mathcal{M}_1) & \xrightarrow{\varphi^* \theta_n} & \text{Fil}^1(\varphi^* \mathcal{M}_2) \\ (1 \otimes \varphi)/p \downarrow \cong & & \cong \downarrow (1 \otimes \varphi)/p \\ \mathcal{M}_1 & \xrightarrow{\theta_{n+1}} & \mathcal{M}_2 \end{array}$$

Now we show by induction that $(\theta_{n+1} - \theta_n)(\mathcal{M}_1) \subseteq (\varphi/p)^n(\widehat{J})\mathcal{M}_2$ as follows. For any $x \in \mathcal{M}_1$, we can find $y \in \text{Fil}^1(\varphi^* \mathcal{M}_1)$ such that $(1 \otimes \varphi_{\mathcal{M}_1})(y) = x$. When $n = 0$, we have by construction:

$$(\theta_1 - \theta_0)(x) = [(1 \otimes \varphi_{\mathcal{M}_2}/p) \circ (\varphi^* \theta_0) - \theta_0 \circ (1 \otimes \varphi_{\mathcal{M}_1}/p)](y) \in \widehat{J}\mathcal{M}_2.$$

Now assume that $(\theta_n - \theta_{n-1})(\mathcal{M}_1) \in (\varphi/p)^{n-1}(\widehat{J})\mathcal{M}_2$ for some $n > 0$. Then we have

$$(\theta_{n+1} - \theta_n)(x) = [(1 \otimes \varphi_{\mathcal{M}_2}/p) \circ \varphi^*(\theta_n - \theta_{n-1})](y) \in (\varphi/p)^n(\widehat{J})\mathcal{M}_2.$$

By the assumption on \widehat{J} , the series $\theta := \text{“}\lim_{n \rightarrow \infty} \theta_n\text{”} = \theta_0 + \sum_{n=0}^{\infty} (\theta_{n+1} - \theta_n)$ converges, and clearly θ satisfies all the requirements in the statement. \square

The following theorem can be obtained from the same proof as Theorem 3.5:

Theorem 3.5.4. *Let p be any prime, including $p = 2$, and assume that R satisfies the formally finite-type assumption (§2.2.2). Then \mathcal{M}^* induces an anti-equivalence of*

categories from the category of formal (respectively, unipotent) p -divisible groups over R to $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)^{\varphi \text{ nilp}}$ (respectively, $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)^{\psi \text{ nilp}}$). A similar result holds for \mathbb{D}^* over a formal scheme base \mathfrak{X} which can be covered by $\text{Spf}(R, (\varpi))$ for such R .

Proof. For formal or unipotent p -divisible groups, the Grothendieck-Messing deformation theory holds for $R \twoheadrightarrow R/(p)$ (cf. [Zin02, §3.3, Corollary 97]), hence the proof of Theorem 3.5 works for any p . Lemma 3.1.6 shows that the essential image is as desired; cf. the discussion following Definition 3.2.6. \square

Remark 3.5.5. In general, we cannot replace $\mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$ in Theorem 3.5 with $\mathbf{MF}_S(\varphi)$. On the other hand, there are cases when we can “forget the connection” – see Corollary 10.3.1 for more details.

Remark 3.5.6 (lci case). Let R, ϖ, S_0, S , and $E(u)$ be as in Remark 3.3.6. We additionally assume that R is *excellent* (as well as locally complete intersection); excellence is satisfied, for example, if $A/(p)$ is formally finitely generated over some field, by [Val75, Theorem 4]. Then for a p -divisible group G_0 over $R/(\varpi)$, let $\mathcal{M}_0^*(G_0) := \mathbb{D}^*(G_0)(S_0) \in \mathbf{MF}_{S_0}(\varphi, \nabla)$. Then the functor \mathcal{M}_0^* is fully faithful by [dJM99, Theorem 4.6].

For a p -divisible group G over R , let $\mathcal{M}^*(G) := \mathbb{D}^*(G)(S) \in \mathbf{MF}_S^{\text{Br}}(\varphi, \nabla^0)$. By precisely the same proof of Theorem 3.5, we can obtain that \mathcal{M}^* is fully faithful if $p > 2$, and it is fully faithful up to isogeny if $p = 2$.

3.6. Base Change. Let $f : R \rightarrow R'$ be a map where both R and R' satisfying the p -basis assumption (§2.2.1). We choose (\mathfrak{S}, φ) and (\mathfrak{S}', φ) for R and R' as in §3.3, respectively. As we have observed in Remark 3.2.9, the “base change” of the functor \mathcal{M}^* under f can be defined if there exists a φ -compatible morphism $\mathfrak{S} \rightarrow \mathfrak{S}'$ which reduces to f .

We have already observed (in Remark 3.2.9), the assumption is satisfied in the following cases:

(Ex1) Assume that R is J -adic and $p \in J$. Let R' be a J -adic formally étale algebra such that $R/J \rightarrow R'/JR'$ is finitely generated (i.e., the morphism $\text{Spf}(R', JR') \rightarrow \text{Spf}(R, J)$ is étale). Set $\tilde{J} := \ker(R_0 \twoheadrightarrow R/J)$ and assume that R_0 is J_0 -adically complete (by replacing R_0 with the J_0 -adic completion if necessary). Then by Lemma 2.3.5 one can find a unique pair (R'_0, φ) so that $R \rightarrow R'$ lifts to $\mathfrak{S} \rightarrow \mathfrak{S}' := R'_0[[u]]$ respecting φ .

We list some other cases where this assumption is satisfied:

(Ex2) R is a discrete valuation ring with residue field k (with finite p -basis) and R' is a p -adic flat R -algebra such that $R'/(\varpi)$ locally admits a finite p -basis (cf. Lemma 2.3.4);

(Ex3) if $R' = R[[x_i]][y_j]$, then we may take $R'_0 = R_0[[x_i]][y_j]$;

(Ex4) Let $W \rightarrow W'$ be a map of Cohen rings with residue fields admitting finite p -bases. By Lemma 2.3.1, there is a lift of Frobenius $\varphi' : W' \rightarrow W'$ which leaves W invariant. Assume that R and R_0 are W -algebras, and choose $\varphi : R_0 \rightarrow R_0$ over $\varphi : W \rightarrow W$. Now, we set $R' := R \widehat{\otimes}_W W'$, $R'_0 := R_0 \widehat{\otimes}_W W'$, and $\varphi_{R'_0} = \varphi_{R_0} \otimes \varphi$.

(Ex5) Let $\mathfrak{p} \subset R$ be a prime ideal containing ϖ , and set $\mathfrak{p}_0 := \ker(R_0 \twoheadrightarrow R/\mathfrak{p})$. (For example, if $\mathfrak{p} = (\varpi)$ then $\mathfrak{p}_0 = (p)$.) If R' is the ϖ -adic completion of the localisation $R_{\mathfrak{p}}$ then we may take R'_0 to be the ϖ -adic completion of $(R_0)_{\mathfrak{p}_0}$. Note that R'_0 and R' are actually p -adic without noetherian-ness assumption on R (cf., Corollaire 2 in [Bou06, Ch. III, §2, no. 12, page 228]), and φ_{R_0} extends to R'_0 because $\varphi^{-1}(\mathfrak{p}_0) = \mathfrak{p}_0$. (Note that φ induces the identity map on the underlying topological space of $\text{Spec } R/(\varpi)$.)

The same construction works if \mathfrak{p} and \mathfrak{p}_0 are finitely generated prime ideals, and R' is the \mathfrak{p} -adic completion of $R_{\mathfrak{p}}$ and R'_0 is the \mathfrak{p}_0 -adic completion of $(R_0)_{\mathfrak{p}_0}$.

(Ex6) Let R'_0 be the p -adic completion of $\varinjlim_{\varphi} (R_0)_{(p)}$, and $R' := R'_0 \otimes_{R_0} R$.

Note that in (Ex6) case we have a φ -compatible isomorphism $R'_0 \cong W(k')$ by the universal property of Witt vectors over perfect rings, where $k' := \varinjlim_{\varphi} \text{Frac}(R/(\varpi))$ is the perfect closure of $\text{Frac}(R/(\varpi))$.

4. REVIEW: RELATIVE p -ADIC HODGE THEORY

In this section, we recall (and slightly generalise) the construction and basic properties of relative p -adic period rings. We refer to [Bri10] for a brief, but more complete, overview of relative p -adic Hodge theory.

4.1. Period rings. Let R be a *normal domain* which satisfies the p -basis assumption (§2.2.1). (We may soon assume further that R satisfies the “refined almost étaleness” assumption (§2.2.3) later.) Choose a separable closure E of $\text{Frac}(R)$, and define \overline{R} to be the union of normal R -subalgebras $R' \subset E$ such that $R'[\frac{1}{p}]$ is finite étale over $R[\frac{1}{p}]$. Note that $\text{Spec } \overline{R}[\frac{1}{p}]$ is a pro-universal covering of $\text{Spec } R[\frac{1}{p}]$ and \overline{R} is an integral closure of R in $\overline{R}[\frac{1}{p}]$. Set $\widehat{\overline{R}} := \varprojlim_n \overline{R}/(p^n)$ and $\mathcal{G}_R := \text{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi_1^{\text{ét}}(\text{Spec } R[\frac{1}{p}], \overline{\eta})$ where $\overline{\eta} : \text{Spec } R[\frac{1}{p}] \rightarrow \text{Spec } E$. (The “correct” notation for \mathcal{G}_R should be $\mathcal{G}_{R[\frac{1}{p}]}$, but we suppress this for the typographical reason. When $R = \mathcal{O}_K$ is a discrete valuation ring we allow both notations $\mathcal{G}_{\mathcal{O}_K} = \mathcal{G}_K$.) When $R = \mathcal{O}_K$, we have $\overline{R} = \mathcal{O}_{\overline{K}}$, $\widehat{\overline{R}} = \mathcal{O}_{\mathbb{C}_K}$, and $\mathcal{G}_R = \text{Gal}(\overline{K}/K)$.

Let $\overline{R}^{\flat} := \varprojlim_{x \mapsto x^p} \overline{R}/(p)$, which is a perfect ring equipped with a natural \mathcal{G}_R -action.

(We follow the notation of Scholze [Sch11, Lemma 6.2]; perhaps, $\widehat{\overline{R}}^{\flat}$ would be a more precise notation as $(\widehat{\overline{R}}[\frac{1}{p}], \widehat{\overline{R}})$ is a perfectoid affinoid $\mathcal{O}_{\mathbb{C}_K}$ -algebra, but there is no danger of confusion for using \overline{R}^{\flat} . See §7.1 for (slightly) more discussions on perfectoid algebras.) As in the classical case (i.e., $R = \mathcal{O}_K$ with perfect residue field), there is a natural multiplicative bijection $\overline{R}^{\flat} \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} \widehat{\overline{R}}$, defined by the component-wise reduction modulo p . To see that it is an isomorphism, we construct its inverse as follows. For any $(x_n)_{n \in \mathbb{Z}_{\geq 0}} \in \overline{R}^{\flat}$, define

$$(4.1.1) \quad x^{(n)} := \lim_{m \rightarrow \infty} (\tilde{x}_{m+n})^{p^m}$$

for any lift $\tilde{x}_{m+n} \in \widehat{\overline{R}}$ of $x_{m+n} \in \overline{R}/(p)$. Note that $x^{(n)}$ is independent of all the choices, and $(x_n)_{n \in \mathbb{Z}_{\geq 0}} \mapsto (x^{(n)})_{n \in \mathbb{Z}_{\geq 0}}$ is the desired inverse.

For any $a \in \widehat{\overline{R}}$, there exists an element $\tilde{a} := (a^{(n)}) \in \overline{R}^{\flat}$ with $a^{(0)} = a$. (Note that \tilde{a} is not uniquely determined by a .) Another useful element is $\epsilon = (\epsilon^{(n)})$ where $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. The choice of ϵ is equivalent to the choice of a \mathbb{Z}_p -basis of $\mathbb{Z}_p(1)$.

Consider the following map:

$$(4.1.2) \quad \theta : W(\overline{R}^{\flat}) \rightarrow \widehat{\overline{R}}, \quad \theta(a_0, a_1, \dots) := \sum_{n=0}^{\infty} p^n a_n^{(n)}.$$

The same proof as [Bri08, Propositions 5.1.1, 5.1.2] shows that θ is a surjective ring homomorphism with kernel principally generated by $\xi := p - [\tilde{p}]$, where $[\cdot]$ denotes the Teichmüller lift.

Let $B_{\mathrm{dR}}^{\nabla,+}(R)$ be the $\ker(\theta)$ -adic completion of $W(\overline{R}^{\flat})[\frac{1}{p}]$. Then $t := \log[\epsilon]$ makes sense as an element of $B_{\mathrm{dR}}^{\nabla,+}(R)$ (and indeed, it actually lies in the classical de Rham period ring $B_{\mathrm{dR}}^+(\mathbb{Z}_p)$). Set $B_{\mathrm{dR}}^{\nabla}(R) := B_{\mathrm{dR}}^{\nabla,+}(R)[\frac{1}{t}]$. These rings carry natural \mathcal{G}_R -actions and filtrations $\mathrm{Fil}^r B_{\mathrm{dR}}^{\nabla}(R) = t^r B_{\mathrm{dR}}^{\nabla,+}(R)$, which coincides with the $\ker(\theta)$ -adic filtration.

Consider the R -linear extension $\theta_R : R \otimes_{\mathbb{Z}_p} W(\overline{R}^{\flat}) \rightarrow \widehat{R}$ of θ , and set $A_{\mathrm{inf}}(R) := \varprojlim_n (R \otimes_{\mathbb{Z}_p} W(\overline{R}^{\flat})) / (\theta_R^{-1}(p\widehat{R}))^n$. Define $B_{\mathrm{dR}}^+(R)$ to be the $\ker(\theta_R)$ -adic completion of $A_{\mathrm{inf}}(R)[\frac{1}{p}]$, and $B_{\mathrm{dR}}(R) := B_{\mathrm{dR}}^+(R)[\frac{1}{t}]$, where $t := \log[\epsilon]$, which makes sense in $B_{\mathrm{dR}}^+(R)$. These rings carry natural \mathcal{G}_R -actions. For filtration, we set $\mathrm{Fil}^r B_{\mathrm{dR}}^+(R) := (\ker \theta_R)^r B_{\mathrm{dR}}^+(R)$ for $r \geq 0$ and extend it to $\mathrm{Fil}^r B_{\mathrm{dR}}(R) := \sum_{n \geq -r} \frac{1}{t^n} \mathrm{Fil}^{n+r} B_{\mathrm{dR}}^+(R)$. When $R = \mathcal{O}_K$ with perfect residue field, $B_{\mathrm{dR}}^{\nabla}(\mathcal{O}_K) = B_{\mathrm{dR}}(\mathcal{O}_K)$ is the usual de Rham period ring constructed by Fontaine [Fon94]. If it is possible to define a connection on $B_{\mathrm{dR}}(R)$ (cf., Proposition 4.5), then we have $B_{\mathrm{dR}}^{\nabla}(R) = (B_{\mathrm{dR}}(R))^{\nabla=0}$, which explains the notation.

We define $A_{\mathrm{cris}}^{\nabla}(R)$ to be the p -adically completed divided power envelope of $W(\overline{R}^{\flat})$ with respect to $\ker(\theta)$. The Witt vector Frobenius extends to φ on $A_{\mathrm{cris}}^{\nabla}(R)$. We let $\mathrm{Fil}^1 A_{\mathrm{cris}}^{\nabla}(R)$ denote the ideal topologically generated by the divided powers of $\xi := p - [\widehat{p}]$, and $\mathrm{Fil}^r A_{\mathrm{cris}}^{\nabla}(R)$ its r th divided power ideal. Note that $A_{\mathrm{cris}}^{\nabla}(R)$ only depends on \overline{R} .

Consider the R_0 -linear extension $\theta_{R_0} : R_0 \otimes_{\mathbb{Z}_p} W(\overline{R}^{\flat}) \rightarrow \widehat{R}$ of θ , and define $A_{\mathrm{cris}}(R)$ to be the p -adically completed divided power envelope of $R_0 \otimes_{\mathbb{Z}_p} W(\overline{R}^{\flat})$ with respect to $\ker(\theta_{R_0})$. (Note that $A_{\mathrm{cris}}(R)$ depends on the choice of \overline{R} and R_0 , which we suppress from the notation.)

As before, $A_{\mathrm{cris}}(R)$ is equipped with a Frobenius endomorphism φ which extends φ on R_0 and the Witt vector Frobenius on $W(\overline{R}^{\flat})$. Let $\mathrm{Fil}^1 A_{\mathrm{cris}}(R)$ be the ideal topologically generated by the divided powers of $\ker(\theta_{R_0})$, and $\mathrm{Fil}^r A_{\mathrm{cris}}(R)$ its r th divided power ideal. In addition, we define a connection $\nabla : A_{\mathrm{cris}}(R) \rightarrow A_{\mathrm{cris}}(R) \widehat{\otimes}_{R_0} \widehat{\Omega}_{R_0}$ by $W(\overline{R}^{\flat})$ -linearly extending the universal continuous derivation of R_0 so that $\nabla(f^{[j]}) := f^{[j-1]} \nabla(f)$ for any $f \in \mathrm{Fil}^1 A_{\mathrm{cris}}(R)$ and $j > 0$. One can directly check that φ and \mathcal{G}_R -action on $A_{\mathrm{cris}}(R)$ is horizontal. Also note that $A_{\mathrm{cris}}^{\nabla}(R)$ is naturally embedded in $A_{\mathrm{cris}}(R)$, and coincides with $A_{\mathrm{cris}}(R)^{\nabla=0}$, hence the notation.

The elements $[\epsilon] \in W(\overline{R}^{\flat})$ and the formal power series $t = \log[\epsilon]$ can be viewed in $A_{\mathrm{cris}}^{\nabla}(R)$ and $A_{\mathrm{cris}}(R)$, and have all the expected properties. (Indeed, they all lie in the ‘‘classical period ring’’ $A_{\mathrm{cris}}(\mathbb{Z}_p)$, constructed by Fontaine [Fon94].) We define $B_{\mathrm{cris}}^{\nabla}(R) := A_{\mathrm{cris}}^{\nabla}(R)[\frac{1}{t}]$ and $B_{\mathrm{cris}}(R) := A_{\mathrm{cris}}(R)[\frac{1}{t}]$. Note that p is invertible in these rings since p divides t^{p-1} . (Note that p divides t^{p-1} in $A_{\mathrm{cris}}(\mathbb{Z}_p)$, which is well-known.) For any $r \in \mathbb{Z}$, we define $\mathrm{Fil}^r B_{\mathrm{cris}}(R) = \sum_{n \geq -r} \frac{1}{t^n} \mathrm{Fil}^{n+r} A_{\mathrm{cris}}(R)[\frac{1}{p}]$ and similarly define $\mathrm{Fil}^r B_{\mathrm{cris}}^{\nabla}(R)$.

Definition 4.2. A filtered (φ, ∇) module over R (relative to R_0) is defined to be a quadruple $(D, \varphi_D, \nabla_D, \mathrm{Fil}^{\bullet} D_R)$, where

- (1) D is a finite projective $R_0[\frac{1}{p}]$ -module;
- (2) $\varphi_D : D \rightarrow D$ is a φ -linear endomorphism such that $1 \otimes \varphi_D$ is an isomorphism;
- (3) $\nabla_D : D \rightarrow D \otimes_{R_0} \widehat{\Omega}_{R_0}$ is an integrable topologically quasi-nilpotent connection (i.e., there exists a R_0 -lattice $\mathcal{M}_0 \subset D$ on which ∇_D induces a topologically quasi-nilpotent connection);

- (4) $\text{Fil}^\bullet D_R$ is a decreasing separated exhaustive R -filtration on $D_R := R \otimes_{R_0} D$, which satisfies Griffiths transversality with respect to ∇_D .

We denote by $\mathbf{MF}_{R/R_0}(\varphi, \nabla)$ the category of filtered (φ, ∇) -modules over R relative to R_0 . For any $D \in \mathbf{MF}_{R/R_0}(\varphi, \nabla)$, the *Hodge-Tate weights* of D are $w \in \mathbb{Z}$ such that $\text{gr}^w D_R \neq 0$.

One can naturally define short exact sequences, direct sums, \otimes -products, duals, etc.

4.3. Crystalline representations. For any p -adic \mathcal{G}_R -representation V and $D \in \mathbf{MF}_{R/R_0}(\varphi, \nabla)$, we define:

$$(4.3.1) \quad D_{\text{cris}}^*(V) := \text{Hom}_{\mathcal{G}_R}(V, B_{\text{cris}}(R))$$

$$(4.3.2) \quad D_{\text{dR}}^*(V) := \text{Hom}_{\mathcal{G}_R}(V, B_{\text{dR}}(R))$$

$$(4.3.3) \quad V_{\text{cris}}^*(D) = \text{Hom}_{R_0[1/p], \varphi, \nabla, \text{Fil}^\bullet}(D, B_{\text{cris}}(R)).$$

Note that $D_{\text{cris}}^*(V)$ is an $R_0[1/p]$ -module equipped with a φ -semilinear endomorphism and a connection coming from $B_{\text{cris}}(R)$, $D_{\text{dR}}^*(V)$ is an $R[1/p]$ -module equipped with a filtration coming from $B_{\text{dR}}(R)$, and $V_{\text{cris}}^*(D)$ is a \mathbb{Q}_p -vector space equipped with a continuous \mathcal{G}_R -action.

Note that in order to show that these constructions yield finitely generated objects (over some suitable rings), let alone prove some natural properties, one needs to make some extra assumptions on R , which we explain now.

4.4. Remark on Faltings' purity and refined almost étaleness. For the rest of the section, we assume that R satisfies the “refined almost étaleness” assumption (§2.2.3), which we recall now:

Assumption §2.2.3: The (irreducible) normal extension R of R_0 satisfies the formally finite-type assumption (§2.2.2), $R[1/p]$ is finite étale over $R_0[1/p]$, and we have $\widehat{\Omega}_{R_0} = \bigoplus_{i=1}^d R_0 dT_i$ for some finitely many units $T_i \in R_0^\times$.

See §2.2.3 for examples.

This condition is slightly more general than the conditions on R under which the relative crystalline period rings $A_{\text{cris}}^\nabla(R)$ and $A_{\text{cris}}(R)$ are studied in [Bri06] and [Bri08] (cf., [Bri10, §1, §3]); the cases covered in the literature are when R is a discrete valuation ring with the residue field k admitting a finite p -basis [Bri06], or when the residue field k is perfect and R is obtained by some combination of finite étale extension, localisation, and completion starting from $\mathcal{O}_K[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$ [Bri08].

The point of the “refined almost étaleness” assumption (§2.2.3) is, as the name suggests, the property called refined almost étaleness (i.e., the condition (RAE) in [And06, §5]), which plays the key technical role in ensuring that relative period rings are well-behaved. Suppose that the assumption §2.2.3 is satisfied, and let A be a normal R -algebra such that $A[1/p]$ is finite étale over $R[1/p]$. Set $R_{(n)} := R[\zeta_{p^n}; T_1^{\pm 1/p^n}, \dots, T_d^{\pm 1/p^n}]$, where ζ_{p^n} is a primitive p^n th root of unity and T_i 's are as in §2.2.3. We similarly define $R_{0,(n)}$. (Note that $R_{(n)}$ is a finite normal extension of R , and is defined over R_0 in the sense that $R_{(n)} = R_{0,(n)} \otimes_{R_0} R$.) We let $A_{(n)} := R_{(n)} \otimes_R A$, which is normal $R_{(n)}$ -algebra which becomes étale after inverting p . Therefore, there exists an idempotent $\epsilon_{A,n} \in (A_{(n)} \otimes_{R_{(n)}} A_{(n)})[1/p]$ which corresponds to the splitting of the natural map $(A_{(n)} \otimes_{R_{(n)}} A_{(n)})[1/p] \rightarrow A_{(n)}[1/p]$. For $\epsilon \in \frac{1}{p-1}\mathbb{Z}[1/p]$, let $p^\epsilon \in \bigcup_n K(\zeta_{p^n})$ denote any element with $\text{ord}_p(p^\epsilon) = \epsilon$, where ord_p is the valuation normalised so that $\text{ord}_p(p) = 1$.

Theorem 4.4.1. *Under the above setting, there exists an integer l (only depending on A) such that $p^{lp^{-n}} \epsilon_{A,n} \in A_{(n)} \otimes_{R_{(n)}} A_{(n)}$ for any n .*

Proof. This theorem is essentially due to Faltings [Fal02, Theorem 4, §2b] and Andreatta [And06, Theorem 5.1], and we indicate how to deduce this theorem from the aforementioned results.

It suffices to handle the case when $R = R_0$ by viewing A as a normal extension of R_0 ; indeed, via the natural projection $A_{(n)} \otimes_{R_{0,(n)}} A_{(n)}[\frac{1}{p}] \twoheadrightarrow A_{(n)}[\frac{1}{p}] \otimes_{R_{(n)}} A_{(n)}[\frac{1}{p}]$, the claim for an idempotent $\epsilon_{A,n} \in A_{(n)} \otimes_{R_{0,(n)}} A_{(n)}[\frac{1}{p}]$ implies the claim for its image in $A_{(n)} \otimes_{R_{(n)}} A_{(n)}[\frac{1}{p}]$. From now on, we assume that R is formally finitely generated over a Cohen ring W with residue field k , and $R/(p)$ locally admits a finite p -basis.

Let us first assume that k is algebraically closed. Choose a W -algebra map $\iota : W[X_1^{\pm 1}, \dots, X_r^{\pm 1}] \rightarrow R_0$ with $r \geq d$, such that $\iota(X_i) = T_i$ for $i = 1, \dots, d$, and the composition $k[X_1^{\pm 1}, \dots, X_r^{\pm 1}] \xrightarrow{k \otimes \iota} R/(\varpi) \twoheadrightarrow R/J_R$ is surjective. This is possible by Bertini's theorem (as k is assumed to be algebraically closed). We set $R'_0 := \varprojlim_s W[X_1^{\pm 1}, \dots, X_r^{\pm 1}]/(J')^s$ where $J' := \ker(W[X_i^{\pm 1}] \twoheadrightarrow R/J_R)$, and let $R' := \mathcal{O}_K \otimes_W R'_0$.

The map ι extends to a quotient map $R' \twoheadrightarrow R$, and it admits a continuous section since R is formally smooth over W . (Recall that we assumed $R = R_0$.) In particular, there exists a finitely generated projective R -module M such that $R' \cong R[[M]] := \prod_{i \geq 0} \text{Sym}_R^i M$ by [dJ95, Lemma 1.3.3]. We set $R'_{(n)} := R'[\zeta_{p^n}; X_1^{\pm 1/p^n}, \dots, X_r^{\pm 1/p^n}]$, then the same argument as above shows that $R'_{(n)} \cong R_{(n)}[[M_n]]$ for some finitely generated projective $R_{(n)}$ -module M_n .

Recall that the lemma is known for any finite normal R' -algebra A' such that $A'[\frac{1}{p}]$ is étale over $R'[\frac{1}{p}]$ by Faltings' purity theorem [Fal02, Theorem 4] (cf., [And06, Theorem 5.11]) and [And06, Theorem 5.1]. Now, for any normal R -algebra A such that $A[\frac{1}{p}]$ is finite étale over $R[\frac{1}{p}]$, we can apply the result to $A' := R' \otimes_R A$ and obtain an integer l such that

$$p^{lp^{-n}} \epsilon_{A',n} \in A'_{(n)} \otimes_{R'_{(n)}} A'_{(n)} \cong R_{(n)}[[M_n]] \otimes_{R_{(n)}} (A_{(n)} \otimes_{R_{(n)}} A_{(n)})$$

for any $n \geq 0$. The image of $p^{lp^{-n}} \epsilon_{A',n}$ under the projection $R_{(n)}[[M_n]] \otimes_{R_{(n)}} (A_{(n)} \otimes_{R_{(n)}} A_{(n)}) \twoheadrightarrow A_{(n)} \otimes_{R_{(n)}} A_{(n)}$ (defined by quotienting out M_n) is exactly $p^{lp^{-n}} \epsilon_{A,n}$, so the lemma follows when k is algebraically closed.

When k is not algebraically closed, we choose a map $W \rightarrow W(\bar{k})$, which always exists. Since the lemma is known for $W(\bar{k}) \otimes_W R$, we can deduce the lemma for R by repeating the argument in the previous paragraph. \square

Let us list a few useful properties of these relative period rings. Although the statement is slightly more general than those found in the literature [Bri06, Bri08], it is not hard to extend it to our setting.

Proposition 4.5. *Assume that R satisfies the “refined almost étaleness” assumption (§2.2.3). For $i = 1, \dots, d$, let $u_i \in A_{\text{cris}}(R) \subset B_{\text{dR}}(R)$ denote the image of $T_i - [\hat{T}_i]$, where T_i 's are chosen as in §4.4.*

- (1) *We have $B_{\text{dR}}^+(R) = B_{\text{dR}}^{\nabla,+}(R)[[u_1, \dots, u_d]]$, and the associated graded rings are $\text{gr}^\bullet B_{\text{dR}}^{\nabla,+} \cong \widehat{R}[\frac{1}{p}][t]$ and $\text{gr}^\bullet B_{\text{dR}}^+(R) \cong \widehat{R}[\frac{1}{p}][t, u_1, \dots, u_d]$.*
- (2) *The $B_{\text{dR}}^{\nabla}(R)$ -linear connection defined by $\nabla(u_i) = 1 \otimes dT_i$ satisfies the Griffiths transversality (i.e., $\nabla(\text{Fil}^r B_{\text{dR}}(R)) \subseteq \text{Fil}^{r-1} B_{\text{dR}}(R) \otimes_{R_0} \widehat{\Omega}_{R_0}$).*
- (3) *We have $A_{\text{cris}}(R) = A_{\text{cris}}^{\nabla}(R)\langle u_1, \dots, u_d \rangle^{PD}$; i.e., the p -adically completed divided power polynomial in u_i 's over $A_{\text{cris}}^{\nabla}(R)$.*

- (4) The ideal $\text{Fil}^r A_{\text{cris}}(R)$ is topologically generated by $(p - [\tilde{p}])^{[j_0]} u_1^{[j_1]} \dots u_d^{[j_d]}$ for $\sum_{n=0}^d j_n \geq r$, and we have $\text{Fil}^r A_{\text{cris}}^\nabla(R) = \text{Fil}^r A_{\text{cris}}(R) \cap A_{\text{cris}}^\nabla(R)$. In particular, $t \in \text{Fil}^1 A_{\text{cris}}^\nabla(R) \subset \text{Fil}^1 A_{\text{cris}}(R)$.
- (5) The connection ∇ on $A_{\text{cris}}(R)$ is the unique $A_{\text{cris}}^\nabla(R)$ -linear connection such that $\nabla(u_i^{[n]}) = u_i^{[n-1]} \otimes dT_i$, and $\varphi : A_{\text{cris}}(R) \rightarrow A_{\text{cris}}(R)$ is horizontal.
- (6) Both $B_{\text{dR}}^+(R)$ and $A_{\text{cris}}(R)$ have no non-zero t -torsion. (In particular, $A_{\text{cris}}(R)$ has no non-zero p -torsion.) The rings $\widehat{R}[\frac{1}{p}]$, $B_{\text{dR}}^+(R)$ and $B_{\text{dR}}(R)$ are faithfully flat over $R[\frac{1}{p}]$, and $B_{\text{cris}}(R)$ is faithfully flat over $R_0[\frac{1}{p}]$.
- (7) The natural map $R_0 \otimes_{\mathbb{Z}_p} W(\overline{R}^\flat) \hookrightarrow R \otimes_{\mathbb{Z}_p} W(\overline{R}^\flat) \hookrightarrow A_{\text{cris}}(R) \rightarrow B_{\text{dR}}(R)$, which is injective, filtered, horizontal, and \mathcal{G}_R -equivariant. Furthermore, the natural map $R \otimes_{R_0} A_{\text{cris}}(R) \rightarrow B_{\text{dR}}(R)$ is injective.
- (8) As an $R[\frac{1}{p}]$ -algebra $B_{\text{dR}}^+(R)$ contains $\widehat{R}[\frac{1}{p}]$ as a \mathcal{G}_R -stable subring, and we have $B_{\text{dR}}(R)^{\mathcal{G}_R} = R[\frac{1}{p}]$. Let $\widehat{R}_0^{\text{ur}}$ denote the closure of maximal ind-étale R_0 -subalgebra in \widehat{R} . Then $A_{\text{cris}}(R)$ contains $\widehat{R}_0^{\text{ur}}$ as a \mathcal{G}_R -stable subring, and $B_{\text{cris}}(R)^{\mathcal{G}_R} = R_0[\frac{1}{p}]$.
- (9) The sequence $0 \rightarrow \mathbb{Q}_p \rightarrow B_{\text{cris}}^\nabla(R)^{\varphi=1} \rightarrow B_{\text{dR}}^\nabla(R)/B_{\text{dR}}^{\nabla,+}(R) \rightarrow 0$ is exact. (This sequence is called the fundamental exact sequence).

Proof. The proofs in [Bri06] and [Bri08] work in our setting (since we have all the ingredients for the proof; especially, Theorem 4.4.1), so we content with giving references for the proofs.

For (1) and (3), the same proof for the case when R is a discrete valuation ring carries over if we work with R_0 instead of \mathcal{O}_{K_0} (cf., Propositions 2.9, 2.19, and 2.39 in [Bri06]); note that R_0 is formally smooth over \mathbb{Z}_p , which suffices for the proof to work. A direct computation using (1) shows (2). (Cf. [Bri06, Proposition 2.23], [Bri08, Proposition 5.3.9].)

The statement (4) follows from (3) and [Bri08, Proposition 5.1.2], which asserts that the kernel of $\theta : W(\overline{R}^\flat) \rightarrow \widehat{R}$ is principally generated by $p - [\tilde{p}]$. The statement (5) follows from a direct computation using (3)–(4) (cf., Proposition 6.2.5 of [Bri08]).

The p - and t -torsion statement in (6) follows from the proofs of Propositions 5.1.5 and 6.1.10 of [Bri08]. The faithful flatness statement in (6) follows from the proofs of Théorèmes 3.2.3, 5.4.1, and 6.3.8 of [Bri08], which uses refined almost étaleness (Theorem 4.4.1). The proofs of Propositions 6.2.1 and 6.2.7 of [Bri08] show the injectivity statements of (7) while the rest of (7) can be directly checked.

To see that there is a natural embedding $\widehat{R}_0^{\text{ur}} \hookrightarrow A_{\text{cris}}(R)$, note that θ_{R_0} induces a nilpotent thickening $A_{\text{cris}}(R)/(p^n) \rightarrow \overline{R}/(p^n)$, so by ind-étaleness there is a unique map $R_0^{\text{ur}} \rightarrow A_{\text{cris}}(R)/(p^n)$ lifting the natural map $R_0^{\text{ur}} \rightarrow \overline{R}/(p^n)$ for any n . The same argument shows the embedding $\widehat{R}[\frac{1}{p}] \hookrightarrow B_{\text{dR}}^+(R)$ (cf., [Bri08, Proposition 5.2.3]).

The proofs of Propositions 5.2.12 and 6.2.9 of [Bri08] shows (8). For (9) see the proof of [Bri08, Proposition 6.2.23]. \square

4.6. (Relative) crystalline \mathcal{G}_R -representations. Assume that R satisfies the “refined almost étaleness” assumption (§2.2.3). One can deduce (rather formally from Proposition 4.5) the following properties for D_{cris} , D_{dR} , and V_{cris} (4.3.1–4.3.3) in the same manner as [Bri08, §8]:

- (1) For any p -adic \mathcal{G}_R -representation V , the following natural maps (which respects all the structure)

$$\begin{aligned}\alpha_{\text{cris}} &: B_{\text{cris}}(R) \otimes_{R_0[\frac{1}{p}]} D_{\text{cris}}^*(V) \rightarrow B_{\text{cris}}(R) \otimes_{\mathbb{Q}_p} V \\ \alpha_{\text{dR}} &: B_{\text{dR}}(R) \otimes_{R[\frac{1}{p}]} D_{\text{dR}}^*(V) \rightarrow B_{\text{dR}}(R) \otimes_{\mathbb{Q}_p} V\end{aligned}$$

are injective (cf., [Bri08, Propositions 8.2.4 and 8.2.6]). If α_{cris} is an isomorphism then we say that V is *crystalline*. In this case, $D_{\text{cris}}^*(V)$ is finitely generated projective over $R_0[\frac{1}{p}]$ by Proposition 4.5(6), and the natural map $R \otimes_{R_0} D_{\text{cris}}^*(V) \rightarrow D_{\text{dR}}^*(V)$ is an isomorphism by Proposition 4.5(7). In particular, we may naturally view $D_{\text{cris}}^*(V) \in \mathbf{MF}_{R/R_0}(\varphi, \nabla)$ (cf., [Bri08, §8.3]). We call $D \in \mathbf{MF}_{R/R_0}(\varphi, \nabla)$ *admissible* if there exists V such that $D \cong D_{\text{cris}}^*(V)$.

- (2) Let $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\mathcal{G}_R)$ denote the category of crystalline \mathcal{G}_R -representations, and let $\mathbf{MF}_{R/R_0}^{\text{a}}(\varphi, \nabla) \subset \mathbf{MF}_{R/R_0}(\varphi, \nabla)$ denote the full subcategory of admissible objects. Then D_{cris}^* and V_{cris}^* are quasi-inverse anti-equivalences of Tannakian categories between $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\mathcal{G}_R)$ and $\mathbf{MF}_{R/R_0}^{\text{a}}(\varphi, \nabla)$ (cf., Théorèmes 8.4.2 and 8.5.1 of [Bri08]).

Example 4.6.1 (Case of p -divisible groups). For any p -divisible group R , one can define $D^*(G) := (\mathbb{D}^*(G)(R_0[\frac{1}{p}], \text{Fil}^1 \mathbb{D}^*(G)(R)[\frac{1}{p}])$, and this is clearly a filtered (φ, ∇) -module with Hodge-Tate weights in $\{0, 1\}$. (Note that Griffiths transversality imposes no condition.) We will show later (Corollary 5.4.2) that under the “refined almost étaleness” assumption (§2.2.3) we have $V_{\text{cris}}^*(D^*(G)) \cong V_p(G)$ (i.e., $V_p(G)$ is crystalline and $D^*(G)$ is admissible). We deduce this from a finer statement about the integral lattice $T_p(G) \subset V_p(G)$ and $\mathbb{D}^*(G)$.

5. RELATIVE INTEGRAL p -ADIC COMPARISON ISOMORPHISM

In this section, we prove the integral p -adic comparison theorem for p -divisible groups over R , which directly follows from the proof of Faltings [Fal99, §6]. A similar approach to ours can be found in [BT08] over a p -adic discrete valuation ring with residue field admitting a finite p -basis, and perhaps the main result of this section was already well known to experts, but we include this section for completeness.

We continue to assume that R is a normal domain which satisfies the p -basis assumption (§2.2.1), and will specify when we make a stronger assumption on R (namely, the “refined almost étaleness” assumption (§2.2.3)).

5.1. The rings S and $A_{\text{cris}}(R)$. Choose $\varpi^{(n)} \in \overline{R}$ for $n \geq 0$ so that $\varpi^{(0)} = \varpi$ and $(\varpi^{(n+1)})^p = \varpi^{(n)}$. Note that $\tilde{\varpi} := (\varpi^{(n)})$ defines an element in \overline{R}^{\flat} . Set $R_{\infty} := \bigcup_n R[\varpi^{(n)}] \subset \overline{R}$, and $\mathcal{G}_{R_{\infty}} := \text{Gal}(\overline{R}[\frac{1}{p}]/R_{\infty}[\frac{1}{p}])$.

We define an R_0 -algebra map $\mathfrak{S} \rightarrow R_0 \otimes_{\mathbb{Z}_p} W(\overline{R}^{\flat})$ by sending $u \mapsto [\tilde{\varpi}]$, and this naturally extends to a $\mathcal{G}_{R_{\infty}}$ -invariant (but *not* \mathcal{G}_R -invariant) injective map $S \rightarrow A_{\text{cris}}(R)$ which respects φ, ∇ , filtrations, and divided power structure.

5.2. Integral comparison isomorphism. Recall that for a p -divisible group G over R , the (contravariant) Dieudonné crystal $\mathbb{D}^*(G)$ is obtained from the Lie algebra of the universal vector extension for (lifts of) G^{\vee} (cf., [Mes72, Ch.VI]). Using this definition, one can easily obtain $\mathbb{D}^*(\mathbb{Q}_p/\mathbb{Z}_p)(S) \cong (S, \text{Fil}^1 S, \varphi/p, d_S)$ and $\mathbb{D}^*(\widehat{\mathbb{G}}_m)(S) \cong (S, S, \varphi, d_S)$.

As in the proof of Theorem 3.5, let us denote $\mathcal{M}^*(G) := \mathbb{D}^*(G)(S)$ for any p -divisible group G over R . Then we have $\mathbb{D}^*(G_{\overline{R}})(A_{\text{cris}}(R)) \cong A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G)$

which respects all the extra structures, except the \mathcal{G}_R -action. (Indeed the isomorphism is only \mathcal{G}_{R_∞} -equivariant as $S \subset A_{\text{cris}}(R)$ is not \mathcal{G}_R -invariant but \mathcal{G}_{R_∞} -invariant.) Let us write $\mathbf{MF}_{A_{\text{cris}}(R)}(\varphi, \nabla)$ for the category of $A_{\text{cris}}(R)$ -module equipped with Fil^1 , φ_1 and ∇ in the exactly same way as $\mathbf{MF}_S^w(\varphi, \nabla^u)$.

As \mathcal{G}_R -modules we may naturally identify $T_p(G) \cong \text{Hom}_{\overline{R}}(\mathbb{Q}_p/\mathbb{Z}_p, G_{\overline{R}})$. So we obtain a pairing

$$T_p(G) \times (A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G)) \rightarrow A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(\mathbb{Q}_p/\mathbb{Z}_p) = A_{\text{cris}}(R)$$

by $(x, m) \mapsto x^*m$ for any $x : \mathbb{Q}_p/\mathbb{Z}_p \rightarrow G_{\overline{R}}$ and $m \in \mathbb{D}^*(G_{\overline{R}})(A_{\text{cris}}(R))$. Therefore we obtain the following integral comparison morphism

$$(5.2.1) \quad \rho_G : A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G) \rightarrow A_{\text{cris}}(R) \otimes_{\mathbb{Z}_p} T_p(G)^*.$$

With the naturally defined extra structures on the both sides, ρ_G can be naturally viewed as a morphism in $\mathbf{MF}_{A_{\text{cris}}(R)}(\varphi, \nabla)$.

Remark 5.2.2. There is a natural \mathcal{G}_R -action on $\mathbb{D}^*(G_{\overline{R}})(A_{\text{cris}}(R))$ induced from the natural \mathcal{G}_R -action on $A_{\text{cris}}(R)$, so we have a natural \mathcal{G}_R -action on $A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G)$. With this \mathcal{G}_R -action on the target, ρ_G is \mathcal{G}_R -equivariant. Note that the \mathcal{G}_R -action on $A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G)$ does not fix $\mathcal{M}^*(G)$; indeed, only \mathcal{G}_{R_∞} fixes $\mathcal{M}^*(G)$.

Theorem 5.3. *The map ρ_G is injective with cokernel annihilated by t . In particular, each of the morphisms below*

$$B_{\text{cris}}(R) \otimes_{R_0[\frac{1}{p}]} D^*(G) \rightarrow B_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G) \xrightarrow{\rho_G[\frac{1}{p}]} B_{\text{cris}}(R) \otimes_{\mathbb{Q}_p} V_p(G)^*$$

is an isomorphism, where $D^*(G)$ is defined in Example 4.6.1 and the first map is induced by the unique section $D^*(G) \rightarrow \mathcal{M}^*(G)[\frac{1}{p}]$ as in Lemma 3.3.5. Furthermore, the composition $B_{\text{cris}}(R) \otimes_{R_0[\frac{1}{p}]} D^*(G) \xrightarrow{\sim} B_{\text{cris}}(R) \otimes_{\mathbb{Q}_p} V_p(G)^*$ is \mathcal{G}_R -equivariant.

Proof. The theorem follows if we show that ρ_G (5.2.1) is injective with cokernel killed by t ; indeed, the last assertion on \mathcal{G}_R -equivariance follows from the \mathcal{G}_R -equivariance of ρ_G (Remark 5.2.2) and the \mathcal{G}_R -invariance of the section $s_{A_{\text{cris}}(R)} : D^*(G) \rightarrow A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G)$.

The proof is exactly the same as [Fal99, §6]. Let us first make $\rho_{\widehat{\mathbb{G}}_m}$ explicit when $G = \widehat{\mathbb{G}}_m$. Let $\beta : \mathbb{Z}_p(1) \rightarrow \text{Fil}^1 A_{\text{cris}}(R)$ be the map that sends $\varepsilon = (\varepsilon^{(n)}) \in \varprojlim_n \mu_{p^n}(\overline{R}) \subset \overline{R}^\times$ to $\log[\varepsilon]$. Then, one can verify that the morphism

$$\rho_{\widehat{\mathbb{G}}_m} : (A_{\text{cris}}(R), A_{\text{cris}}(R), \varphi, \nabla) \rightarrow \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1), A_{\text{cris}}(R))$$

sends 1 to β , as explained in [Fal99, §6]. (One way to see this is by applying [Mes72, Ch. VI, Theorem 2.2] to the sections over the PD completion of $A_{\text{cris}}(R)$. See [Mes72, Ch. VI, §2.5] for the construction of the morphism of Dieudonné crystals corresponding to a morphism of p -divisible groups.) Therefore if we naturally identify $\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1), A_{\text{cris}}(R))$ with $t^{-1}A_{\text{cris}}(R)$, then $\rho_{\widehat{\mathbb{G}}_m}$ can be identified with the natural inclusion $A_{\text{cris}}(R) \hookrightarrow t^{-1}A_{\text{cris}}(R)$. This shows that $\rho_{\widehat{\mathbb{G}}_m}$ is injective and its cokernel is killed by t .

Let us now handle the general case. For any $y \in \text{Hom}_{\overline{R}}(G_{\overline{R}}, \widehat{\mathbb{G}}_{m, \overline{R}})$, one can check that the following diagram commutes:

$$(5.3.1) \quad \begin{array}{ccc} A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G) & \xrightarrow{\rho_G} & A_{\text{cris}}(R) \otimes_{\mathbb{Z}_p} T_p(G)^* \\ \uparrow y^* & & \uparrow \text{id}_{A_{\text{cris}}(R)} \otimes T_p(y)^* \\ A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(\widehat{\mathbb{G}}_m) & \xrightarrow{\rho_{\widehat{\mathbb{G}}_m}} & A_{\text{cris}}(R) \otimes_{\mathbb{Z}_p} T_p(\widehat{\mathbb{G}}_m)^* \end{array}$$

Recall that we have a natural \mathcal{G}_R -equivariant isomorphism $\text{Hom}_{\overline{R}}(G_{\overline{R}}, \widehat{\mathbb{G}}_{m, \overline{R}}) \xrightarrow{\sim} T_p(G)^*(1)$ defined by sending $y : G_{\overline{R}} \rightarrow \widehat{\mathbb{G}}_{m, \overline{R}}$ to $x \mapsto y \circ x$ for any $x \in T_p(G) \cong \text{Hom}_{\overline{R}}(\mathbb{Q}_p/\mathbb{Z}_p, G_{\overline{R}})$. Now choose a \mathbb{Z}_p -basis $\varepsilon \in T_p(\widehat{\mathbb{G}}_m)$ so that $t = \beta(\varepsilon)$, and let $\eta \in T_p(G)^*$ be such that y corresponds to $\eta \otimes \varepsilon \in T_p(G)^*(1)$ under the natural isomorphism (i.e., for any $x \in T_p(G)$, we have $y \circ x = \eta(x)\varepsilon$). Recall that $\rho_{\widehat{\mathbb{G}}_m}(1 \otimes 1) = \beta$, and one can compute

$$(\text{id} \otimes T_p(y)^*)(\beta) : x \mapsto \beta(y \circ x) = \beta(\varepsilon)\eta(x) = t\eta(x);$$

i.e., $(\text{id} \otimes T_p(y)^*)(\beta) = t \otimes \eta$. By the commutative diagram (5.3.1), it follows that $t \otimes \eta$ is in the image of ρ_G . Since $\eta \in T_p(G)^*$ can be arbitrary as we vary y , the theorem follows. \square

5.4. Galois-stable lattices. If R satisfies the ‘‘refined almost étaleness’’ assumption (§2.2.3) (so that we have refined almost étaleness and the period rings have nice properties: Proposition 4.5) then one can define $T_{\text{cris}}^*(\mathcal{M})$ for any $\mathcal{M} \in \mathbf{MF}_S^w(\varphi, \nabla^u)$ as follows:

$$(5.4.1) \quad T_{\text{cris}}^*(\mathcal{M}) := \text{Hom}_{S, \text{Fil}^1, \varphi^1, \nabla}(\mathcal{M}, A_{\text{cris}}(R)),$$

where $A_{\text{cris}}(R)$ is viewed as an S -algebra as in §5.1, and is given $\text{Fil}^1 A_{\text{cris}}(R)$, $\varphi_1 := \frac{\varphi}{p} : \text{Fil}^1 A_{\text{cris}}(R) \rightarrow A_{\text{cris}}(R)$, and $\nabla : A_{\text{cris}}(R) \rightarrow A_{\text{cris}}(R) \otimes_{R_0} \widehat{\Omega}_{R_0}$.

Clearly, $T_{\text{cris}}^*(\mathcal{M})$ is p -adic, and has a natural continuous \mathcal{G}_{R_∞} -action induced from the \mathcal{G}_{R_∞} -action on $A_{\text{cris}}(R)$. It is not *a priori* obvious if $T_{\text{cris}}^*(\mathcal{M})$ is finite free over \mathbb{Z}_p , but this follows from Corollary 5.4.2 below. (Indeed, we also show that $\text{rank}_{\mathbb{Z}_p} T_{\text{cris}}^*(\mathcal{M}) = \text{rank}_S \mathcal{M}$.)

Note that the map $S \rightarrow A_{\text{cris}}(R)$ is only \mathcal{G}_{R_∞} -invariant, so we only obtain \mathcal{G}_{R_∞} -action on $T_{\text{cris}}^*(\mathcal{M})$. Using the differential operator $N_{\mathcal{M}}$, however, one can define a \mathcal{G}_R -action on $T_{\text{cris}}^*(\mathcal{M})$ that extends its natural \mathcal{G}_{R_∞} -action. See §5.5 for more details.

Corollary 5.4.2. *Suppose that R satisfies the ‘‘refined almost étaleness’’ assumption (§2.2.3). Then ρ_G (as in (5.2.1)) induces a \mathcal{G}_{R_∞} -equivariant injective morphism*

$$T_p(G) \rightarrow \text{Hom}_{\mathbf{MF}_{A_{\text{cris}}(R)}(\varphi, \nabla)}(A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G), A_{\text{cris}}(R)) \cong T_{\text{cris}}^*(\mathcal{M}^*(G)),$$

which is isomorphism if $p > 2$, and has cokernel annihilated by p if $p = 2$. In particular, it induces an isomorphism $D^*(G) \xrightarrow{\sim} D_{\text{cris}}^*(V_p(G))$ as filtered isocrystals.

Proof. It is straightforward from Theorem 5.3, noting that p divides t if and only if $p = 2$, in which case $p = 2$ divides t exactly once. \square

This corollary in particular proves that when R satisfies the ‘‘refined almost étaleness’’ assumption (§2.2.3), the \mathcal{G}_R -representation $V_p(G)$ is crystalline and $D^*(G)$ is admissible for any p -divisible group G over R .

Remark 5.4.3. It follows from the proof of Theorem 5.3 that $T_p(\widehat{\mathbb{G}}_m) = 2 \cdot T_{\text{cris}}^*(\widehat{\mathbb{G}}_m)$ in $V_p(\widehat{\mathbb{G}}_m)$ (if $p = 2$).

5.5. Galois Action on the \mathbb{Z}_p -lattice. Suppose that R satisfies the ‘‘refined almost étaleness’’ assumption (§2.2.3). It follows from Theorem 5.3 that $T_{\text{cris}}^*(\mathcal{M}^*(G))$ can be viewed as a \mathcal{G}_{R_∞} -stable \mathbb{Z}_p -lattice in $V_p(G)$, and is \mathcal{G}_R -stable if $p > 2$. In this section, we define a natural \mathcal{G}_R -action directly on $T_{\text{cris}}^*(\mathcal{M}^*(G))$ for any p , in such a way that the natural \mathcal{G}_R -equivariant map $T_{\text{cris}}^*(\mathcal{M}^*(G)) \hookrightarrow V_p(G)$ is \mathcal{G}_R -stable. We generalise the construction in [Liu08, §5] (cf., [Bre02, §2.2]).

Let us fix some notation. For any $n \geq 0$ define a cocycle $\epsilon^{(n)} : \mathcal{G}_R \rightarrow \widehat{R}^\times$ as follows:

$$(5.5.1) \quad \epsilon^{(n)}(g) := g \cdot \varpi^{(n)} / \varpi^{(n)}, \text{ for any } g \in \mathcal{G}_R.$$

Set $\epsilon(g) := (\epsilon^{(n)}(g)) \in \overline{R}^\flat$, and $t_g := \log[\epsilon(g)] \in A_{\text{cris}}^\nabla(R)$. Note that for any $g \in \mathcal{G}_R$, t_g is a \mathbb{Z}_p -multiple of $t \in \text{Fil}^1 A_{\text{cris}}(R)$ (where t is as in §4.1), and $t_g = 0$ if and only if $g \in \mathcal{G}_{R_\infty}$.

For $\mathcal{M} \in \mathbf{MF}_S(\varphi, \nabla)$, let us modify the \mathcal{G}_R -action on $A_{\text{cris}}(R) \otimes_S \mathcal{M}$ using the differential operator $N_{\mathcal{M}}$ as follows:

$$(5.5.2a) \quad g \cdot (a \otimes x) := g(a) \sum_{i=0}^{\infty} (t_g)^{[i]} \otimes N_{\mathcal{M}}^i(x),$$

for $g \in \mathcal{G}_R$, $a \in A_{\text{cris}}(R)$, and $x \in \mathcal{M}$. Here, $(t_g)^{[i]}$ is the standard i th divided power; i.e., $(t_g)^{[i]} := t_g^i/i!$ if $i > 0$ and $(t_g)^{[0]} := 1$ (even when $t_g = 0$).

To see that the sum (5.5.2a) converges, since $N_{\mathcal{M}}(\mathcal{M}) \subset u\mathcal{M}$, it suffices to show that $(t_g)^{[i]} \rightarrow 0$ as $i \rightarrow \infty$. But this follows from [Fon94, §5.2.4] as $(t_g)^{[i]} \in A_{\text{cris}}(\mathbb{Z}_p)$. By the proof of [Liu08, Lemma 5.1.1], equation (5.5.2a) gives a \mathcal{G}_R -action which respects φ and the natural filtration on $A_{\text{cris}}(R) \otimes_S \mathcal{M}$. When $g \in \mathcal{G}_{R_\infty}$ we recover the natural \mathcal{G}_{R_∞} -action on $A_{\text{cris}}(R) \otimes_S \mathcal{M}$.

For any $f \in T_{\text{cris}}^*(\mathcal{M})$, we $A_{\text{cris}}(R)$ -linearly extend f to $A_{\text{cris}}(R) \otimes_S \mathcal{M} \rightarrow A_{\text{cris}}(R)$. Now the following formula clearly defines a continuous action of $g \in \mathcal{G}_R$ on $T_{\text{cris}}^*(\mathcal{M})$:

$$(5.5.2b) \quad g \cdot f : x \mapsto g \cdot (f(g^{-1}(1 \otimes x))), \text{ for } x \in \mathcal{M}.$$

If $\mathcal{M} := \mathbb{D}^*(G)(S)$ for some p -divisible group G over R , then the image of the natural map $D^*(G) \hookrightarrow \mathcal{M}[\frac{1}{p}]$ lies in the kernel of $N_{\mathcal{M}}$. It then follows from Corollary 5.4.2 that the natural injective \mathcal{G}_{R_∞} -map $T_{\text{cris}}^*(\mathcal{M}^*(G)) \rightarrow V_p(G)$ is indeed \mathcal{G}_R -equivariant for the \mathcal{G}_R -action on $T_{\text{cris}}^*(\mathcal{M}^*(G))$ defined as in (5.5.2a, 5.5.2b). In particular, the natural injective map $\rho_G : T_p(G) \hookrightarrow T_{\text{cris}}^*(\mathcal{M})$ is \mathcal{G}_R -equivariant, and ρ_G is a \mathcal{G}_R -isomorphism when $p > 2$.

5.6. Base change. Let R and R' be normal domains which satisfy the p -basis assumption (§2.2.1), and consider a map $f : R \rightarrow R'$ which restricts to a φ -compatible map $R_0 \rightarrow R'_0$ for some suitable choices. (Cf. §3.6.) We also let $f : S \rightarrow S'$ denote the map extending $f|_{R_0}$ by sending $\overline{u} \mapsto u$.

Choose a separable closure E' of $\text{Frac } R'$ and define \overline{R}' to be the union of normal finite R' -subalgebras of E' which is only ramified at (ϖ) , as in §4.1. Set $R'_\infty := \bigcup_n R'[\varpi^{(n)}]$, $\mathcal{G}_{R'} = \text{Gal}(\overline{R}'[\frac{1}{p}]/R'[\frac{1}{p}])$, and $\mathcal{G}_{R'_\infty} := \text{Gal}(\overline{R}'[\frac{1}{p}]/R'_\infty[\frac{1}{p}])$. Let \overline{R}' , R'_∞ , $\mathcal{G}_{R'}$, and $\mathcal{G}_{R'_\infty}$ denote the obvious objects for R' . Choose $\overline{f} : \overline{R} \rightarrow \overline{R}'$ over $f : R \rightarrow R'$ (which is possible, and the choice is unique up to the actions by \mathcal{G}_R and $\mathcal{G}_{R'}$), and consider the map $\mathcal{G}_{R'} \rightarrow \mathcal{G}_R$ of profinite groups induced by it. Under these choices, we obtain a map $A_{\text{cris}}(\overline{f}) : A_{\text{cris}}(R) \rightarrow A_{\text{cris}}(R')$ respecting all the extra structures. (In particular, $A_{\text{cris}}(\overline{f})$ is $\mathcal{G}_{R'}$ -equivariant if we let $\mathcal{G}_{R'}$ act on $A_{\text{cris}}(R)$ via the map $\mathcal{G}_{R'} \rightarrow \mathcal{G}_R$.)

Now one can easily see that the formation of ρ_G commutes with the base change which satisfies the above assumption; in other words, for any p -divisible group G over R , we have the following cartesian diagram

$$(5.6.1) \quad \begin{array}{ccc} A_{\text{cris}}(R) \otimes_S \mathcal{M}^*(G) & \xrightarrow{\rho_G} & A_{\text{cris}}(R) \otimes_{\mathbb{Z}_p} T_p(G)^* \\ \downarrow & & \downarrow A_{\text{cris}}(\overline{f}) \otimes 1 \\ A_{\text{cris}}(R') \otimes_{S'} \mathcal{M}^*(G_{R'}) & \xrightarrow{\rho_{G_{R'}}} & A_{\text{cris}}(R') \otimes_{\mathbb{Z}_p} T_p(G_{R'})^* \end{array}$$

where the left vertical arrow is induced by the map $A_{\text{cris}}(\overline{f})$ and the isomorphism $S' \otimes_S \mathcal{M}^*(G) \cong \mathcal{M}^*(G_{R'})$ constructed in §3.6. If both R and R' satisfy the “refined almost étaleness” assumption (§2.2.3), from the left vertical arrow we obtain a

$\mathcal{G}_{R'_\infty}$ -isomorphism $T_{\text{cris}}^*(\mathcal{M}^*(G)) \xrightarrow{\sim} T_{\text{cris}}^*(\mathcal{M}^*(G_{R'}))$, which is an isomorphism because it has a saturated image and $T_{\text{cris}}^*(\mathcal{M}^*(G_{R'}))$ and $T_{\text{cris}}^*(\mathcal{M}^*(G))$ have the same \mathbb{Z}_p -rank (by the diagram above). If $p > 2$, this recovers the natural identification $T_p(G) \cong T_p(G_{R'})$.

6. KISIN MODULES: EQUIVALENCE OF CATEGORIES

The notion of Kisin modules (i.e., \mathfrak{S} -modules of height ≤ 1) was generalised to the case when the base is a complete regular local base with perfect residue field by Vasu and Zink [VZ10] and Lau [Lau10b, Lau10c], and they also constructed a natural equivalence of categories between Kisin modules and p -divisible groups using display theory.

We generalise the notion of Kisin modules so that it can be applied to some non-local base (cf. Definition 6.1.1), and construct a natural functor from the category of p -divisible groups into the category of Kisin modules. The main result of this section is the construction of natural equivalence of categories between Kisin modules and Breuil modules, generalising [CL09, Theorem 2.2.1]. Combining this with Theorem 3.5, we obtain a classification theorem of p -divisible group when $p > 2$ (Corollary 6.3.1), which works over not necessarily local base rings, but is weaker in the intersecting case (even when $p > 2$).

In this section, we assume that R satisfies the p -basis assumption (§2.2.1). Readers are welcome to work under the normality assumption (§2.2.4) though, since in the application to p -divisible groups and finite locally free group schemes it suffices to consider the base rings R satisfying the normality assumption (§2.2.4).

6.1. Definitions and basic properties.

Definition 6.1.1. Let R satisfies the p -basis assumption (§2.2.1), and we use the notation from §2.2.1 and §3.3, such as (\mathfrak{S}, φ) and E . A *quasi-Kisin \mathfrak{S} -module* is a pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where

- (1) \mathfrak{M} is a finitely generated projective \mathfrak{S} -module;
- (2) $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -linear map such that $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by E .

Let $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$ denote the category of quasi-Kisin \mathfrak{S} -modules.

Let $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ denote the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where \mathfrak{M} is a quasi-Kisin \mathfrak{S} -module, and $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathfrak{S}} \widehat{\Omega}_{\mathfrak{S}}$ on $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ is a topologically quasi-nilpotent integrable connection which commutes with $\varphi_{\mathcal{M}} := \varphi_S \otimes \varphi_{\mathfrak{M}}$.

Let $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$ denote the category of pairs $(\mathfrak{M}, \nabla_{\mathcal{M}_0})$ where $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ and $\nabla_{\mathcal{M}_0}$ is a connection on $\mathcal{M}_0 := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ which makes \mathcal{M}_0 into an object in $\mathbf{MF}_{R_0}(\varphi, \nabla)$. We call an object $(\mathfrak{M}, \nabla_{\mathcal{M}_0}) \in \mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$ a *Kisin \mathfrak{S} -module*.

Remark 6.1.2. The recipe in Remark 3.3.3 defines a fully faithful functor $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla) \rightarrow \mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$ by Lemma 3.3.4, which is an equivalence of categories when R satisfies the formally finite-type assumption (§2.2.2) and $p > 2$, by Theorem 3.5.

Generalising the construction in [Bre98], for any $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ we can make $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ into an object in $\mathbf{MF}_S(\varphi)$ as follows:

$$(6.1.3) \quad \text{Fil}^1 \mathcal{M} := \{x \in \mathcal{M} \mid 1 \otimes \varphi_{\mathfrak{M}}(x) \in (\text{Fil}^1 S) \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M}\}$$

$$(6.1.4) \quad \varphi_1 : \text{Fil}^1 \mathcal{M} \xrightarrow{1 \otimes \varphi_{\mathfrak{M}}} (\text{Fil}^1 S) \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_1 \otimes 1} S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \mathcal{M}.$$

One can directly check that the above construction satisfies the definition of $\mathbf{MF}_S(\varphi)$.

We set $\varphi_{\mathcal{M}} = \varphi_S \otimes \varphi_{\mathfrak{M}}$, and then $\varphi_1 = \varphi_{\mathcal{M}}/p$ defines a map $\text{Fil}^1 \mathcal{M} \rightarrow \mathcal{M}$. Furthermore, if $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ then $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ is an object in $\mathbf{MF}_S(\varphi, \nabla)$.

Lemma 6.1.5. *Let $\mathfrak{M}^\bullet := [0 \rightarrow \mathfrak{M}_1 \rightarrow \mathfrak{M}_2 \rightarrow \mathfrak{M}_3 \rightarrow 0]$ be a sequence of maps of finite projective \mathfrak{S} -modules. Then \mathfrak{M}^\bullet is exact if and only if $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\bullet$ is exact.*

Proof. The “only if” direction is clear from \mathfrak{S} -flatness of \mathfrak{M}_3 . Assume that $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\bullet$ is exact. Since \mathfrak{M}_i are \mathfrak{S} -projective the natural maps $\mathfrak{M}_i \rightarrow S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i$ are injective for any $i = 1, 2, 3$, so \mathfrak{M}^\bullet is left exact. By Nakayama lemma and faithful flatness of $\varphi : R_0 \rightarrow R_0$, it suffices to show that $R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\bullet$ is right exact, which follows since $R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\bullet \cong R_0 \otimes_S (S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^\bullet)$. \square

Definition 6.1.6. Let $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$. Since $1 \otimes \varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is injective, we have a unique injective \mathfrak{S} -linear map

$$\psi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$$

such that $(1 \times \varphi_{\mathfrak{M}}) \circ \psi_{\mathcal{M}} = E \text{id}_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}} \otimes (1 \otimes \varphi_{\mathfrak{M}}) = E \text{id}_{\varphi^* \mathfrak{M}}$.

We say that $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ is φ -nilpotent if for some $n \gg 1$ we have $\varphi_{\mathfrak{M}}^n(\mathfrak{M}) \subset (p, u)\mathfrak{M}$. We extend this definition to $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ and $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$, and use the superscript $\varphi \text{ nilp}$ for the full subcategories of φ -nilpotent objects (for example, $\mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\varphi \text{ nilp}}$). We similarly define ψ -nilpotent objects in $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$, $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$, and $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$, and use the superscript $\psi \text{ nilp}$ for the full subcategories of ψ -nilpotent objects

Let $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$, and we set $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \in \mathbf{MF}_S(\varphi)$ and

$$\mathcal{M}_0 := \mathfrak{S}/(u) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cong R_0 \otimes_S \mathcal{M} \in \mathbf{MF}_{R_0}(\varphi),$$

where $\mathbf{MF}_{R_0}(\varphi)$ is defined using the frame as in Example 3.2.7.

Lemma 6.1.7. *Under the notation as above, the following are equivalent:*

- (1) \mathfrak{M} is φ -nilpotent (respectively, ψ -nilpotent) in the sense of Definition 6.1.6.
- (2) \mathcal{M} is φ -nilpotent (respectively, ψ -nilpotent) in the sense of Definition 3.2.6.
- (3) \mathcal{M}_0 is φ -nilpotent (respectively, ψ -nilpotent) in the sense of Definition 3.2.6.

Proof. We clearly have (1) \Leftrightarrow (3) and (2) \Rightarrow (3). The implication (3) \Rightarrow (2) follows because $\varphi^n(I_0) \subset pS$ for $n \gg 1$ where $I_0 = \ker(S \twoheadrightarrow R_0)$. \square

Lemma 6.2. *Assume that R satisfies the p -basis condition (§2.2.1). If $p > 2$ then the functors $\mathbf{Mod}_{\mathfrak{S}}(\varphi) \rightarrow \mathbf{MF}_S(\varphi)$ and $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla) \rightarrow \mathbf{MF}_S(\varphi, \nabla)$, defined by $S \otimes_{\varphi, \mathfrak{S}} (\cdot)$, is fully faithful. If $p = 2$ then the full faithfulness holds up to isogeny.*

The similar statement holds for $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ and $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$.

Proof. It suffices to show that $\mathbf{Mod}_{\mathfrak{S}}(\varphi) \rightarrow \mathbf{MF}_S(\varphi)$ is fully faithful when $p > 2$, and $\mathbf{Mod}_{\mathfrak{S}}(\varphi)[\frac{1}{p}] \rightarrow \mathbf{MF}_S(\varphi)[\frac{1}{p}]$ is fully faithful when $p = 2$. When R is a discrete valuation ring with perfect residue field, this lemma is can be obtained from the classification of p -divisible groups (combining Theorem 2.2.7 and Proposition A.6 in [Kis06]).

To handle the general case, let R'_0 and R' be the p -adic completions of $\varinjlim_{\varphi} R_{0(p)}$ and $\varinjlim_{\varphi} R_{(\varpi)}$, respectively. We accordingly define \mathfrak{S}' and S' , etc. Then there is a natural φ -compatible map $\mathfrak{S} \rightarrow \mathfrak{S}'$ lifting the natural map $R \rightarrow R'$; cf. §3.6 (Ex.6). Inside $S'[\frac{1}{p}]$ we have

$$(6.2.1) \quad S \cap \mathfrak{S}' = \mathfrak{S} \text{ and } S[1/p] \cap \mathfrak{S}'[1/p] = \mathfrak{S}[1/p].$$

Now, let $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$, and write $\mathcal{M}_i := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i \in \mathbf{MF}_S(\varphi)$, $\mathfrak{M}'_i := \mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M}_i \in \mathbf{Mod}_{\mathfrak{S}'}(\varphi)$, and $\mathcal{M}'_i := S' \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}_i \in \mathbf{MF}_{S'}(\varphi)$ for $i = 1, 2$. We naturally view \mathfrak{M}_i as a submodule in \mathcal{M}_i , \mathfrak{M}'_i , and \mathcal{M}'_i .

Assume that $p > 2$ and we have a morphism $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ in $\mathbf{MF}_S(\varphi)$. We want to show that f maps \mathfrak{M}_1 into \mathfrak{M}_2 . Since R' is a discrete valuation ring with perfect residue field, it follows that $S' \otimes_S f : \mathcal{M}'_1 \rightarrow \mathcal{M}'_2$ maps \mathfrak{M}'_1 into \mathfrak{M}'_2 . Now, from (6.2.1) we have $\mathfrak{M}_i = \mathcal{M}_i \cap \mathfrak{M}'_i$ for $i = 1, 2$, which proves the lemma when $p > 2$. The same argument works when $p = 2$ by inverting p everywhere. \square

When $p = 2$ Lemma 6.2 can be strengthened for φ - and ψ - nilpotent objects (cf. Definition 6.1.6) as follows:

Lemma 6.2.2. *Assume that $p = 2$ and R satisfies the p -basis condition (§2.2.1). Then the functors $\mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\varphi \text{ nilp}} \rightarrow \mathbf{MF}_S(\varphi)^{\varphi \text{ nilp}}$ and $\mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\psi \text{ nilp}} \rightarrow \mathbf{MF}_S(\varphi)^{\psi \text{ nilp}}$, defined by $S \otimes_{\varphi, \mathfrak{S}} (\cdot)$, are fully faithful. The same statement holds for the full subcategories of φ - and ψ - nilpotent objects in $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ and $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$.*

Proof. Note that $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$ has a dualith $\mathfrak{M} \rightsquigarrow \mathfrak{M}^\vee$, where the underlying \mathfrak{S} -module of \mathfrak{M}^\vee is a \mathfrak{S} -linear dual of \mathfrak{M} , and $\varphi_{\mathfrak{M}^\vee}$ is induced from $\psi_{\mathfrak{M}}$. Under this duality, \mathfrak{M} is φ -nilpotent if and only if \mathfrak{M}^\vee is ψ -nilpotent. Therefore, it suffices to show that $\mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\psi \text{ nilp}} \rightarrow \mathbf{MF}_S(\varphi)^{\psi \text{ nilp}}$ is fully faithful.

The same proof of Lemma 6.2 shows that it suffices to prove the lemma when R is a p -adic discrete valuation ring with perfect residue field, which follows from combining Proposition 1.1.9 and Theorem 1.2.8 in [Kis09]. \square

Proposition 6.3. *Assume that R satisfies the p -basis condition (§2.2.1). Then the functors $\mathbf{Mod}_{\mathfrak{S}}(\varphi) \rightarrow \mathbf{MF}_S(\varphi)$ and $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla) \rightarrow \mathbf{MF}_S(\varphi, \nabla)$, defined by $S \otimes_{\varphi, \mathfrak{S}} (\cdot)$, are essentially surjective. In particular, they are equivalences of categories if $p > 2$ or if they are restricted to φ - and ψ - nilpotent objects, and are equivalence of categories up to isogeny if $p = 2$.*

We prove the proposition later in §6.4. Let us record some interesting corollaries. The following is immediate from

Corollary 6.3.1. *Assume that R satisfies the p -basis condition (§2.2.1). If $p > 2$ then there exists an exact contravariant functor*

$$\mathfrak{M}^* : \{p\text{-divisible groups over } R\} \rightarrow \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla) \xrightarrow{\sim} \mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$$

such that for any p -divisible group G over R there exists a natural isomorphism $\mathbb{D}^(G)(S) \cong S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^*(G)$ in $\mathbf{MF}_S(\varphi, \nabla)$. Furthermore, \mathfrak{M}^* is fully faithful if R satisfies the normality assumption (§2.2.4), and an anti-equivalence of categories if R satisfies the formally finite-type assumption (§2.2.2).*

If $p = 2$ then we have an exact contravariant functor $\mathfrak{M}^[\frac{1}{p}] : G \mapsto \mathfrak{M}^*(G)[\frac{1}{p}]$ on the isogeny categories*

$$\{p\text{-divisible groups over } R\}[1/p] \rightarrow \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)[1/p],$$

which is fully faithful if R satisfies the formally finite-type assumption (§2.2.2).

When $p = 2$, we have the following strengthening of Corollary 6.3.1 for φ - and ψ - nilpotent objects:

Corollary 6.3.2. *Let $p = 2$. Then the functor \mathfrak{M}^* is defined on the category of formal and unipotent p -divisible groups (without passing to the isogeny categories), and is fully faithful if R satisfies the normality assumption (§2.2.4). If R satisfies the formally finite-type assumption (§2.2.2), then \mathfrak{M}^* induces an anti-equivalence of categories from the category of formal (respectively, unipotent) p -divisible groups to $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)^{\varphi \text{ nilp}}$ (respectively, $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)^{\psi \text{ nilp}}$).*

Proof. This follows from Proposition 6.3, Theorem 3.5.4, and Lemmas 3.1.6 and 6.1.7. \square

There will be another strengthening of Corollary 6.3.1 when $p = 2$ and R satisfies the formally finite-type assumption (§2.2.2); see Corollary 8.6.1.

Remark 6.3.3. If R is a complete regular local ring with perfect residue field, then Eike Lau [Lau10c] proved a stronger result than Corollary 6.3.1; namely, the classification theorem without connection, which also holds when $p = 2$. Although we

cannot replace $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$ in Corollary 6.3.1 with $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$ in general, there are cases we can “forget the connection”; cf. Corollary 10.3.1. The role of connections will be studied further in §10.

Remark 6.3.4 (lci case). Assume that $R/(\varpi)$ is excellent and locally complete intersection in addition to the p -basis assumption (§2.2.1). Then, as mentioned in Remark 3.5.6 we can still show that the functor

$$G \rightsquigarrow \mathcal{M}^*(G) := \mathbb{D}^*(G)(S) \in \mathbf{MF}_S(\varphi, \nabla)$$

is fully faithful if $p > 2$ (respectively, fully faithful up to isogeny if $p = 2$). Therefore, one can obtain the above corollaries for \mathfrak{M}^* from Proposition 6.3.

Remark 6.3.5. The functors in Proposition 6.3 and Corollary 6.3.1 commute with the base change which satisfy the condition stated at the beginning of §3.6. In particular, we can define base change for “étale morphisms” and “completions” as in §3.6 (Ex1) and (Ex5).

6.4. Proof of Proposition 6.3. We want to show that the functor $\mathbf{Mod}_{\mathfrak{S}}(\varphi) \rightarrow \mathbf{MF}_S(\varphi)$ is essentially surjective. We do this by modifying the proof of [CL09, Theorem 2.2.1]. The proof also works when $p = 2$ with little modification.

Let us begin the proof of Proposition 6.3 with a few preliminary lemmas. From now on, we consider $(\mathcal{M}, \text{Fil}^1, \varphi_1) \in \mathbf{MF}_S(\varphi)$, and set $\mathcal{M}_0 := R_0 \otimes_S \mathcal{M}$. The following lemma is not trivial when \mathcal{M} is not necessarily free over S .

Lemma 6.4.1. *For \mathcal{M} as above, there exists a projective \mathfrak{S} -module \mathfrak{N} equipped with an S -isomorphism $S \otimes_{\mathfrak{S}} \mathfrak{N} = \mathcal{M}$. Any two such projective \mathfrak{S} -modules \mathfrak{N} and \mathfrak{N}' are (non-canonically) isomorphic.*

Proof. Since $\mathcal{M}/(\text{Fil}^1 S)\mathcal{M}$ is a projective R -module, we can lift it to a projective \mathfrak{S} -module $\tilde{\mathfrak{N}}$. Then by Nakayama lemma, there is an isomorphism $S \otimes_{\mathfrak{S}} \tilde{\mathfrak{N}} \cong \mathcal{M}$. If there are two such \mathfrak{S} -modules $\tilde{\mathfrak{N}}$ and $\tilde{\mathfrak{N}}'$, then one can find an isomorphism $\tilde{\mathfrak{N}} \cong \tilde{\mathfrak{N}}'$ which lifts an isomorphism $\tilde{\mathfrak{N}}/E\tilde{\mathfrak{N}} \cong \mathcal{M}/(\text{Fil}^1 S)\mathcal{M} \cong \tilde{\mathfrak{N}}'/E\tilde{\mathfrak{N}}'$. \square

Lemma 6.4.2. *There exists an S -linear injective map $B : \mathcal{M} \rightarrow \text{Fil}^1 \mathcal{M}$ such that $E\mathcal{M} \subseteq B(\mathcal{M})$, $\text{Fil}^1 \mathcal{M} = B(\mathcal{M}) + (\text{Fil}^p S)\mathcal{M}$, and $\varphi_1(B(\mathcal{M}))$ generates \mathcal{M} .*

Proof. Consider the “Hodge filtration”

$$0 \rightarrow \text{Fil}^1 \mathcal{M}/(\text{Fil}^1 S)\mathcal{M} \rightarrow \mathcal{M}/(\text{Fil}^1 S)\mathcal{M} \rightarrow \mathcal{M}/\text{Fil}^1 \mathcal{M} \rightarrow 0,$$

which splits as modules over $R = S/\text{Fil}^1 S$. We lift this filtration to an S -direct factor $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N} \subset \text{Fil}^1 \mathcal{M}$, and choose a splitting $\mathcal{M} \cong \mathcal{N} \oplus \mathcal{M}/\mathcal{N}$. We define B to be identity on \mathcal{N} and multiplication by E on \mathcal{M}/\mathcal{N} .

By construction, we have $\text{Fil}^1 \mathcal{M} = B(\mathcal{M}) + (\text{Fil}^1 S)\mathcal{M}$ and $B(\mathcal{M})$ contains $E\mathcal{M}$. Since $\text{Fil}^1 S = ES + \text{Fil}^p S$, it follows that $\text{Fil}^1 \mathcal{M} = B(\mathcal{M}) + (\text{Fil}^p S)\mathcal{M}$. Now $\varphi_1(B(\mathcal{M}))$ has to generate \mathcal{M} since $\frac{\varphi}{p}(\text{Fil}^p S) \subseteq pS$. \square

The following is the key technical lemma for proving Proposition 6.3:

Lemma 6.4.3. *There exist a projective \mathfrak{S} -module \mathfrak{M} and an injective map $B : \mathcal{M} \rightarrow \text{Fil}^1 \mathcal{M}$ which satisfies the following:*

- *There is an isomorphism $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cong \mathcal{M}$. (From now on, we view, via the natural inclusions, \mathfrak{M} as a $\varphi(\mathfrak{S})$ -submodule of \mathcal{M} , and $\varphi^*\mathfrak{M}$ as a \mathfrak{S} -submodule of \mathcal{M} .)*
- *We have $E\mathcal{M} \subseteq B(\mathcal{M})$, $\text{Fil}^1 \mathcal{M} = B(\mathcal{M}) + (\text{Fil}^p S)\mathcal{M}$, and $\mathfrak{M} = c^{-1}\varphi_1(B(\varphi^*\mathfrak{M}))$ in \mathcal{M} , where $c = \varphi(E)/p$.*
- *The map $B : \mathcal{M} \rightarrow \text{Fil}^1 \mathcal{M} \subseteq \mathcal{M}$ takes $\varphi^*\mathfrak{M}$ into itself.*

Proof. (Cf. [CL09, Lemma 2.2.2]) We let $\text{Fil}^1(\varphi^*\mathcal{M}) \subseteq \varphi^*\mathcal{M}$ denote the image of $\varphi^*(\text{Fil}^1\mathcal{M})$ in $\varphi^*\mathcal{M}$. As observed in the proof of Lemma 3.5.2, we have an isomorphism $1 \otimes \varphi_1 : \text{Fil}^1(\varphi^*\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$.

For any $n \geq 0$ we recursively construct a projective \mathfrak{S} -module $\mathfrak{M}^{(n)}$, and S -linear maps $B^{(n)}, C^{(n)}, D^{(n)} : \mathcal{M} \rightarrow \mathcal{M}$ such that

- (1) $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{(n)} \cong \mathcal{M}$ (so we view $\varphi^*\mathfrak{M}^{(n)}$ as a \mathfrak{S} -submodule of \mathcal{M} , and $\mathfrak{M}^{(n)}$ as a $\varphi(\mathfrak{S})$ -submodule of \mathcal{M});
- (2) $B^{(n)}(\mathcal{M}), C^{(n)}(\mathcal{M}) \subseteq \text{Fil}^1\mathcal{M}$, and $(C^{(n)} - B^{(n)})(\mathcal{M}) \subseteq (p^n \text{Fil}^{n+p} S)\mathcal{M}$.
- (3) we have $E\mathcal{M} \subseteq B^{(n)}(\mathcal{M})$, $\text{Fil}^1\mathcal{M} = B^{(n)}(\mathcal{M}) + (\text{Fil}^p S)\mathcal{M}$, and $\mathfrak{M}^{(n)} = c^{-1}\varphi_1(B^{(n)}(\varphi^*\mathfrak{M}^{(n)}))$ where the equalities take place inside \mathcal{M} ;
- (4) $C^{(n)}$ takes $\varphi^*\mathfrak{M}^{(n)}$ into itself (hence, $C^{(n)}$ is automatically injective);
- (5) We have a recursion formula $\mathfrak{M}^{(n+1)} := c^{-1}\varphi_1(C^{(n)}(\varphi^*\mathfrak{M}^{(n)}))$.
- (6) $1 + D^{(n)}$ is an automorphism of \mathcal{M} which takes $\mathfrak{M}^{(n)}$ onto $\mathfrak{M}^{(n+1)}$ and such that $D^{(n)}(\mathcal{M}) \subseteq p^{\mu_n}\mathcal{M}$ for some strictly increasing sequence $\mu_n \in \mathbb{Z}_{>0}$.

Let us first construct $\mathfrak{M}^{(0)}$, $B^{(0)}$, and $C^{(0)}$. Choose a projective \mathfrak{S} -module \mathfrak{N} with $S \otimes_{\mathfrak{S}} \mathfrak{N} \cong \mathfrak{M}$, which exists by Lemma 6.4.1. Choose $B' : \mathcal{M} \rightarrow \text{Fil}^1\mathcal{M}$ as in Lemma 6.4.2. Set $\mathfrak{M}^{(0)} := c^{-1}\varphi_1(B'(\mathfrak{N}))$, which is a $\varphi(\mathfrak{S})$ -submodule of \mathcal{M} . We view $\mathfrak{M}^{(0)}$ as a \mathfrak{S} -module by letting $s \in \mathfrak{S}$ act via the multiplication by $\varphi(s) \in S$. This \mathfrak{S} -module structure makes $(c^{-1}\varphi_1) \circ B' : \mathfrak{N} \rightarrow \mathfrak{M}^{(0)}$ into a \mathfrak{S} -linear surjective map. But since $\mathfrak{M}^{(0)}$ spans \mathcal{M} by assumption on B' (Lemma 6.4.2), the \mathfrak{S} -linear surjective map $(c^{-1}\varphi_1) \circ B' : \mathfrak{N} \rightarrow \mathfrak{M}^{(0)}$ should be an isomorphism and $\mathfrak{M}^{(0)}$ is projective over \mathfrak{S} (as the source of the map is so). Now $\mathfrak{M}^{(0)}$ satisfies (1).

By the uniqueness assertion in Lemma 6.4.1, we can find an automorphism of \mathcal{M} which takes $\varphi^*\mathfrak{M}^{(0)}$ onto \mathfrak{N} . Let $B^{(0)}$ denote the composition of this automorphism with B' , which clearly satisfies (3). To find $C^{(0)}$ satisfying (4), we first observe that $S = \mathfrak{S} + \text{Fil}^p S$, so we have $\mathcal{M} = \varphi^*\mathfrak{M}^{(0)} + (\text{Fil}^p S)\mathcal{M}$. Now, choose a map $C^{(0)} : \varphi^*\mathfrak{M}^{(0)} \rightarrow \varphi^*\mathfrak{M}^{(0)}$ which lifts

$$\varphi^*\mathfrak{M}^{(0)} \xrightarrow{B^{(0)}} \mathcal{M} \twoheadrightarrow \mathcal{M}/(\text{Fil}^p S)\mathcal{M} \cong \varphi^*\mathfrak{M}^{(0)}/E^p(\varphi^*\mathfrak{M}^{(0)}).$$

Then by construction the image of $B^{(0)} - C^{(0)}$ is inside $(\text{Fil}^p S)\mathcal{M}$.

From now on, let us assume that we have $\mathfrak{M}^{(n)}$, $B^{(n)}$, and $C^{(n)}$, which satisfy (1)–(4). We now define $\mathfrak{M}^{(n+1)}$ using the formula given in (5) as a $\varphi(\mathfrak{S})$ -submodule of \mathcal{M} , and we view $\mathfrak{M}^{(n+1)}$ as a \mathfrak{S} -module by letting $s \in \mathfrak{S}$ act via the multiplication by $\varphi(s) \in S$. The map $c^{-1}\varphi_1 : C^{(n)}(\varphi^*\mathfrak{M}^{(n)}) \rightarrow \mathfrak{M}^{(n+1)}$ is a \mathfrak{S} -linear surjection, so $\mathfrak{M}^{(n+1)}$ would be a projective \mathfrak{S} -module satisfying (1) provided that its S -linear span is \mathcal{M} (i.e., $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^{(n+1)} = \mathcal{M}$).

Now let us construct an automorphism $1 + D^{(n)}$ of \mathcal{M} which satisfies (6). From this we will deduce that $\mathfrak{M}^{(n+1)}$ is projective over \mathfrak{S} and satisfies (1). By induction hypothesis (3) on $B^{(n)}$, the \mathfrak{S} -linear map $(c^{-1}\varphi_1) \circ B^{(n)} : \varphi^*\mathfrak{M}^{(n)} \rightarrow \mathfrak{M}^{(n)}$ is an isomorphism. We set

(6.4.4)

$$D^{(n)} : \mathcal{M} \xrightarrow{((c^{-1}\varphi_1) \circ (\varphi^*B^{(n)}))^{-1}} \varphi^*\mathcal{M} \xrightarrow{\varphi^*(C^{(n)} - B^{(n)})} \text{Fil}^1(\varphi^*\mathcal{M}) \xrightarrow{c^{-1}\varphi_1} \mathcal{M},$$

where $\text{Fil}^1(\varphi^*\mathcal{M})$ is the image of $\varphi^*(\text{Fil}^1\mathcal{M})$ in $\varphi^*\mathcal{M}$. By (5), the endomorphism $1 + D^{(n)}$ of \mathcal{M} takes $\mathfrak{M}^{(n)}$ onto $\mathfrak{M}^{(n+1)}$.

Claim 6.4.5. *We set $\lambda_n := n + p - \lceil \frac{n+p}{p-1} \rceil$ if $p > 2$, where $\lceil \alpha \rceil = \inf\{x \in \mathbb{Z} \mid x \geq \alpha\}$; and $\lambda_n := 1$ if $p = 2$. Then we have $D^{(n)}(\mathcal{M}) \subseteq p^{\lambda_n + n}\mathcal{M}$.*

Granting this claim, it follows that $1 + D^{(n)}$ is an automorphism of \mathcal{M} , and $\mathfrak{M}^{(n+1)}$ is projective over \mathfrak{S} and satisfies (1). (Note that $\lambda_n \geq 1$ for any n .)

Let us prove Claim 6.4.5. For any $s \in \text{Fil}^1 S$ and $m \in \mathcal{M}$ we have $\varphi_1(sm) = c^{-1}\varphi_1(s)\varphi_1(Em)$. Since $(C^{(n)} - B^{(n)})(\mathcal{M}) \subseteq (p^n \text{Fil}^{n+p} S)\mathcal{M}$ by assumption, it suffices to show that $\varphi_1(s) \in p^{\lambda_n} S$ for any $s \in \text{Fil}^{n+p} S$. By writing $s = \sum_{i \geq n+p} a_i E^i / i!$ with $a_i \in \mathfrak{S}$, this assertion is reduced to showing the inequality $i - \text{ord}_p(i!) \geq \lambda_n$ for any $i \geq n+p$. When $p > 2$, Claim 6.4.5 follows from $\text{ord}_p(i!) < \frac{i}{p-1}$. When $p = 2$, we can directly check $i - \text{ord}_2(i!) \geq 1 = \lambda_n$ for any $i \geq p = 2$. (Indeed, $2^n - \text{ord}_2(2^n!) = 1$ for any $n \geq 1$, and this is exactly when the lower bound is achieved.) This proves Claim 6.4.5.

Now we set

$$(6.4.6) \quad B^{(n+1)} := C^{(n)}(1 + D^{(n)})^{-1} = C^{(n)} \sum_{i=0}^{\infty} (-D^{(n)})^i.$$

From the hypotheses on $B^{(n)}$, $C^{(n)}$, and $D^{(n)}$, it follows that $B^{(n+1)}$ satisfies (3). It remains to construct $C^{(n+1)}$ which satisfies (4). We first observe that $B^{(n+1)}(\mathfrak{M}^{(n+1)}) \subseteq \mathfrak{M}^{(n)} = (1 + D^{(n)})^{-1}(\mathfrak{M}^{(n+1)})$, so to construct $C^{(n+1)}$ it suffices to find a decomposition $D^{(n)} = D_1^{(n)} + D_2^{(n)}$ where $D_1^{(n)}(\mathfrak{M}^{(n)}) \subseteq \mathfrak{M}^{(n+1)}$ and $D_2^{(n)}(\mathfrak{M}^{(n)}) \subseteq (p^{n+1} \text{Fil}^{n+1+p} S)\mathcal{M}$. Indeed, we will show that

$$(6.4.7) \quad D^{(n)}(\mathcal{M}) \subseteq p^{\lambda_n+n} \mathcal{M} \subseteq \mathfrak{M}^{(n+1)} + (p^{n+1} \text{Fil}^{n+1+p} S)\mathcal{M}.$$

(Once this is done, one can repeat the argument for $n = 0$ and obtain the desired expression $D^{(n)} = D_1^{(n)} + D_2^{(n)}$.)

Now to see the last inclusion in (6.4.7) it suffices to show that $p^{\lambda_n+n} S \subseteq \mathfrak{S} + p^{n+1} \text{Fil}^{n+1+p} S$; i.e., $p^{\lambda_n+n} \frac{E^i}{i!} \in \mathfrak{S}$ for any $i \leq n+p$. Indeed, when $p > 2$ we have $\lambda_n + n - \text{ord}_p(i!) \geq \lambda_n + n - \frac{i}{p-1} \geq 0$, and when $p = 2$ we have remarked that $\text{ord}_2(i!) \leq i - 1$. Hence, we obtain $C^{(n+1)}$ as in (4). This concludes the ‘‘induction step’’.

Now, let \mathfrak{M} denote the direct limit of $\mathfrak{M}^{(n)}$; in other words,

$$\mathfrak{M} := \left(\prod_{n=0}^{\infty} (1 + D^{(n)}) \right) (\mathfrak{M}^{(0)}),$$

which makes sense thanks to (6). Since $\prod_{n=0}^{\infty} (1 + D^{(n)})$ is an automorphism of \mathcal{M} , it follows that \mathfrak{M} is projective over \mathfrak{S} and $\mathcal{M} \cong S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$.

We now define $B : \mathcal{M} \rightarrow \text{Fil}^1 \mathcal{M}$ by the following limit

$$B := \varprojlim_{n \rightarrow \infty} B^{(n)} = B^{(0)} + \sum_{n=0}^{\infty} (B^{(n+1)} - B^{(n)}),$$

which converges since $(B^{(n+1)} - B^{(n)})(\mathcal{M}) \subset p^n \mathcal{M}$ by construction (6.4.6). Note that $C^{(n)}$ also converges to B by induction hypothesis (2). Now all the desired properties of B can be deduced from the properties of $B^{(n)}$ and $C^{(n)}$; especially from induction hypotheses (3) and (4). \square

Proof of Proposition 6.3. Let \mathfrak{M} and B as in Lemma 6.4.3. Recall that $\varphi_{\mathcal{M}}(m) = c^{-1}\varphi_1(Em)$ for any $m \in \mathcal{M}$. Since B takes $\varphi^*\mathfrak{M}$ into itself and $EM \subseteq B(\mathcal{M})$, we have $E(\varphi^*\mathfrak{M}) \subseteq B(\varphi^*\mathfrak{M})$. So from $c^{-1}\varphi_1(B(\varphi^*\mathfrak{M})) = \mathfrak{M}$, we see that $\mathfrak{M} \subset \mathcal{M}$ is stable under $\varphi_{\mathcal{M}}$. We set $\varphi_{\mathfrak{M}} := \varphi_{\mathcal{M}}|_{\mathfrak{M}}$.

We now show that $E\mathfrak{M} \subseteq (1 \otimes \varphi_{\mathfrak{M}})(\varphi^*\mathfrak{M})$. Note that $(1 \otimes \varphi_{\mathfrak{M}})(m') = c^{-1}\varphi_1(Em')$ for any $m' \in \varphi^*\mathfrak{M}$. For any $m \in \mathfrak{M}$ let $m' \in \varphi^*\mathfrak{M}$ be the (unique) element such that $c^{-1}\varphi_1(B(m')) = m$. Clearly, we have $c^{-1}\varphi_1(B(Em')) = \varphi(E)m$ as elements of \mathcal{M} ; i.e., $(1 \otimes \varphi_{\mathfrak{M}})(m') = E \cdot m$ as elements of \mathfrak{M} since we defined \mathfrak{S} -action on $\mathfrak{M} \subseteq \mathcal{M}$ via $\varphi : \mathfrak{S} \rightarrow S$. This shows that \mathfrak{M} is a quasi-Kisin \mathfrak{S} -module.

It remains to show $\mathrm{Fil}^1(S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) = \mathrm{Fil}^1 \mathcal{M} = B(\mathcal{M}) + (\mathrm{Fil}^p S)\mathcal{M}$. To show $\mathrm{Fil}^1(S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \subseteq \mathrm{Fil}^1 \mathcal{M}$, it suffices to observe that for any $m' \in \varphi^* \mathfrak{M}$, we have $(1 \otimes \varphi)(m') \in E\mathfrak{M}$ if and only if $m' \in B(\varphi^* \mathfrak{M})$; indeed, we have already seen the “if” direction, and the “only if” direction easily follows from $c^{-1}\varphi_1(B(\varphi^* \mathfrak{M})) = \mathfrak{M}$. To show $\mathrm{Fil}^1(S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \supseteq \mathrm{Fil}^1 \mathcal{M}$, it suffices to show that $\mathrm{Fil}^1(S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \supseteq B(\varphi^* \mathfrak{M})$. Note that the image of the natural inclusion $S \otimes_{\mathfrak{S}} \mathfrak{M} \hookrightarrow S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \mathcal{M}$ is the $\varphi(S)$ -span of \mathfrak{M} in \mathcal{M} , and for any $s \in S$ the natural multiplication by s on $S \otimes_{\mathfrak{S}} \mathfrak{M}$ is translated as multiplication by $\varphi(s)$ on $\varphi(S) \cdot \langle \mathfrak{M} \rangle$. Then we can identify $(\mathrm{Fil}^1 S) \otimes_{\mathfrak{S}} \mathfrak{M}$ with $\varphi(\mathrm{Fil}^1 S)\mathcal{M}$, and $(1 \otimes \varphi_{\mathfrak{M}})(B(m'))$ with $c^{-1}\varphi_1(EB(m'))$ for any $m' \in \varphi^* \mathfrak{M}$. Now the claim follows from a direct computation. \square

Remark 6.4.8. With the suitable generalisation of the notion of relative Breuil S -module for the “semi-stable” case with weights $\leq r$ for $r < p-1$ (cf., Remark 3.3.3), Lemma 6.2 and Proposition 6.3 hold in this generalisation with the same proofs.

7. ÉTALE φ -MODULES AND GALOIS REPRESENTATIONS

We generalise the theory of étale φ -modules to our relative setting. All the main ingredients are in Scholze’s work on perfectoid spaces [Sch11]. We refer to [Sch11] for the basic definitions on perfectoid algebras.

Throughout this section, we suppose that R is a *domain* which satisfies the formally finite-type assumption (§2.2.2). Furthermore, we *assume* that there exists a Cohen subring $W \subset R_0$ such that $E(u) \in W[u]$; or equivalently, there exists a complete discrete valuation subring $\mathcal{O}_K \subset R$ which contains ϖ as a uniformiser. (If this is the case then we have $\mathcal{O}_K := W[u]/E(u)$ and $R = \mathcal{O}_K \otimes_W R_0$.) We set $K := \mathcal{O}_K[1/\varpi]$. See Remark 7.1.4 for the reason for this additional assumption.

7.1. Review: Perfectoid algebras. Let L be a field complete with respect to a non-discrete valuation of rank 1 (i.e., $|\bullet| : L^\times \rightarrow \mathbb{R}$ which satisfies the axioms for non-archimedean absolute value). Recall that L is called a *perfectoid field* if the p th power map $L^\circ/(p) \rightarrow L^\circ/(p)$ is surjective, where L° is the valuation ring (cf. [Sch11, Definitions 3.1, Proposition 5.9]). For example, $\widehat{\mathbb{Q}}_p$ is a perfectoid field.

A Banach L -algebra A is called *perfectoid* if the subring of powerbounded elements $A^\circ \subset A$ is open and bounded and the p th power map $A^\circ/(p) \rightarrow A^\circ/(p)$ is surjective (cf. [Sch11, Definitions 5.1, Proposition 5.9]). By a *perfectoid affinoid L -algebra*, we mean a pair (A, A^+) where A is a perfectoid L -algebra and $A^+ \subset A^\circ$ is an open and integrally closed L° -subalgebra. As a trivial example, (L, L°) is a perfectoid affinoid L -algebra.

For an affinoid L -algebra (A, A^+) , we define (A^b, A^{b+}) as follows:

$$A^{b+} := \varprojlim_{x \mapsto x^p} A^+/(p); \quad A^b := L^b \otimes_{L^{b^\circ}} A^{b+},$$

where $L^{b^\circ} := \varprojlim_{x \mapsto x^p} L^\circ/(p)$ and $L^b = \mathrm{Frac} L^{b^\circ}$. (Cf. [Sch11, Proposition 5.17, Lemma 6.2].) Note that L^b is a perfectoid field of characteristic p , and A^b and A^{b+} are respectively L^b - and L^{b° -algebras. We call (A^b, A^{b+}) the *tilt* of (A, A^+) .

Theorem 7.1.1 (Scholze, [Sch11, Theorem 5.2, Lemma 6.2, Theorem 7.9]). *The “tilting” $(A, A^+) \rightsquigarrow (A^b, A^{b+})$ induces an equivalence of categories from the category of perfectoid affinoid L -algebras to the category of perfectoid affinoid L^b -algebras. Furthermore, a morphism $(A, A^+) \rightarrow (B, B^+)$ of perfectoid affinoid L -algebras is finite étale (i.e., $A \rightarrow B$ is finite étale and B^+ is the normalisation of A^+ in B) if and only if $(A^b, A^{b+}) \rightarrow (B^b, B^{b+})$ is finite étale.*

Let us now introduce the perfectoid field which will be the base field of all perfectoid algebras in characteristic 0. Here, we assume that $E(u) \in W[u]$ for some

Cohen subring $W \subset R_0$, and let $\mathcal{O}_K = W[u]/E(u)$ and $K := \text{Frac } \mathcal{O}_K$. We fix \bar{R} as in §4.1, and let $\widehat{\bar{R}}$ be its p -adic completion. Then we define $L \subset \widehat{\bar{R}}[\frac{1}{p}]$ to be the smallest p -adically closed subfield with perfect residue field which contains K and $\varpi^{(n)}$ for all n , where $\{\varpi^{(n)}\}$ are compatible p^n th root of ϖ chosen in §5.1. In fact, L is isomorphic to the p -adic completion of $\bigcup_{n \geq 0} (\text{Frac } W(k^{\text{perf}}))(\varpi^{(n)})$, where $k^{\text{perf}} := \varinjlim_{\varphi} k$. Clearly, L is a perfectoid field, and $(\widehat{\bar{R}}[\frac{1}{p}], \widehat{\bar{R}})$ is a perfectoid affinoid L -algebra

Set $\tilde{\varpi} := (\varpi^{(n)}) \in L^\flat$ (as in §5.1). Then we have $L^{\flat\circ} = k^{\text{perf}}[[\tilde{\varpi}^{1/p^\infty}]]$; i.e., the $\tilde{\varpi}$ -adic completion of $\varinjlim_{\varphi} k^{\text{perf}}[[\tilde{\varpi}]]$. Then $(\bar{R}^\flat[1/\tilde{\varpi}], \bar{R}^\flat)$ is the tilt of $(\widehat{\bar{R}}[1/p], \widehat{\bar{R}})$, so it is a perfectoid affinoid L^\flat -algebra (in the obvious way).

We set $\mathbf{E}_{R_\infty}^+ := \mathfrak{S}/(p) = R/(\varpi)[[u]]$, and let $\tilde{\mathbf{E}}_{R_\infty}^+$ be the u -adic completion of the perfect closure $\varinjlim_{\varphi} \mathbf{E}_{R_\infty}^+$. Set $\mathbf{E}_{R_\infty} := \mathbf{E}_{R_\infty}^+[1/u]$ and $\tilde{\mathbf{E}}_{R_\infty} := \tilde{\mathbf{E}}_{R_\infty}^+[1/u]$. Note that $\tilde{\mathbf{E}}_{R_\infty}^+ \subset \mathbf{E}_{R_\infty}^+$ is open and bounded for the u -adic topology, and it coincides with the subring of power-bounded elements. So by [Sch11, Proposition 5.9], $(\tilde{\mathbf{E}}_{R_\infty}, \tilde{\mathbf{E}}_{R_\infty}^+)$ is a perfectoid affinoid L^\flat -algebra. By sending u to $\tilde{\varpi}$, we obtain a map $(\tilde{\mathbf{E}}_{R_\infty}, \tilde{\mathbf{E}}_{R_\infty}^+) \hookrightarrow (\bar{R}^\flat[1/\tilde{\varpi}], \bar{R}^\flat)$ of perfectoid affinoid L^\flat -algebras. Using this, we view all of these rings as subrings of $\bar{R}^\flat[1/\tilde{\varpi}]$.

Let $(\tilde{R}_\infty[\frac{1}{p}], \tilde{R}_\infty)$ is a perfectoid affinoid L -algebra whose tilt is $(\tilde{\mathbf{E}}_{R_\infty}, \tilde{\mathbf{E}}_{R_\infty}^+)$. (Here, we give the p -adic topology on $(\tilde{R}_\infty[\frac{1}{p}], \tilde{R}_\infty)$.) If, for example, $R = \mathcal{O}_K \langle T_i \rangle_{i=1, \dots, d}$, then we have $\tilde{R}_\infty = L^\circ \langle T_i^{p^{-\infty}} \rangle = \varprojlim_n L^\circ [T_i^{p^{-n}}] / (p^n)$ by [Sch11, Proposition 5.20]. In general, we can construct \tilde{R}_∞ explicitly as follows (cf., [Sch11, Remark 5.19]):

$$\tilde{R}_\infty \cong W(\tilde{\mathbf{E}}_{R_\infty}^+) \otimes_{W(L^{\flat\circ}, \theta)} L^\circ,$$

where θ is defined as in (4.1.2). Note also that \tilde{R}_∞ has a natural structure of p -adic \widehat{R}_∞ -algebra using the Cartier morphism $R_0 \rightarrow W(\tilde{\mathbf{E}}_{R_\infty}^+)$ (i.e., the unique φ -equivariant morphism which lifts the natural map $R_0/(p) \rightarrow \tilde{\mathbf{E}}_{R_\infty}^+$).

Let us show that \tilde{R}_∞ is the p -adic completion of an integral extension of R which becomes ind-étale after inverting p . Since the assertion is Zariski local on $\text{Spf}(R, (\varpi))$, we may assume that $R/(\varpi)$ has (globally) a finite p -basis. Then for any lift $\{T_i\} \subset R_0$ of a p -basis of $R/(\varpi)$ we have

$$(7.1.2) \quad \tilde{R}_\infty \cong \widehat{R}_\infty \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p \langle T_i^{p^{-\infty}} \rangle,$$

where the completed tensor product is taken with respect to the p -adic topology. Indeed, we have $\widehat{R}_\infty/(\varpi) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \langle T_i^{p^{-\infty}} \rangle \cong \tilde{\mathbf{E}}_{R_\infty}^+/(u)$ by mapping T_i to the corresponding p -basis of $R/(\varpi)$ contained in $\tilde{\mathbf{E}}_{R_\infty}^+/(u)$, so by the characterisation of the tilting (cf., Theorem 7.1.1) we have the isomorphism (7.1.2). Now, the right hand side of (7.1.2) is the p -adic completion of $R_\infty \otimes_{\mathbb{Z}_p} \mathbb{Z}_p [T_i^{p^{-\infty}}]$, which is an integral extension of R that becomes ind-étale after inverting p , as desired.

Let $(\tilde{R}_\infty[1/p], \tilde{R}_\infty) \hookrightarrow (\widehat{\bar{R}}[1/p], \widehat{\bar{R}})$ be the injective morphism whose tilt is $(\tilde{\mathbf{E}}_{R_\infty}, \tilde{\mathbf{E}}_{R_\infty}^+) \hookrightarrow (\bar{R}^\flat[1/\tilde{\varpi}], \bar{R}^\flat)$ (defined by $u \mapsto \tilde{\varpi}$). So we have a continuous map $\mathcal{G}_{\tilde{R}_\infty} \rightarrow \mathcal{G}_{\bar{R}}$, where $\mathcal{G}_{\tilde{R}_\infty}$ is the étale fundamental group of $\text{Spec } \tilde{R}_\infty[1/p]$ (with respect to the common geometric generic point as a base point). It follows from [GR03, Proposition 5.4.54] that this map is a closed embedding as \tilde{R}_∞ is the ϖ -adic completion of an integral R -subalgebra of \bar{R} which is henselian along (ϖ) . (Note that a ϖ -adic ring is henselian along (ϖ) , and the property of being henselian along (ϖ) is preserved under taking direct limits.)

The following is a direct consequence of Scholze’s “almost purity theorem” (stated in Theorem 7.1.1):

Theorem 7.1.3. *For any fixed \overline{R} , $\overline{R}^\flat[1/\varpi]$ can be canonically identified with the ϖ -adic completion of the affine ring of a pro-universal covering of $\mathrm{Spec} \widetilde{\mathbf{E}}_{R_\infty}$. Defining $\mathcal{G}_{\widetilde{\mathbf{E}}_{R_\infty}}$ using this pro-universal covering, there is a canonical isomorphism $\mathcal{G}_{\widetilde{\mathbf{E}}_{R_\infty}} \cong \mathcal{G}_{\widetilde{R}_\infty}$.*

Remark 7.1.4. Let us elaborate more on the assumption that $\varpi \in R$ is a uniformiser of some discrete valuation subring $\mathcal{O}_K \subset R$. We use this assumption to obtain the perfectoid base field L such that $\varpi \in L^\flat$. Without this assumption, it is unclear whether there exists a perfectoid subfield $L \subset \widehat{\overline{R}}[\frac{1}{p}]$ so that the “untilt” \widetilde{R}_∞ of $\widetilde{\mathbf{E}}_{R_\infty}$ contains $\varpi^{(n)}$ for each $n \geq 0$ and $u \in \widetilde{\mathbf{E}}_{R_\infty}$ corresponds to ϖ .

7.2. Étale φ -modules and Galois representations. Let $\mathcal{O}_\mathcal{E}$ be the p -adic completion of $\mathfrak{S}[\frac{1}{u}]$, and set $\mathcal{E} := \mathcal{O}_\mathcal{E}[\frac{1}{p}]$. As R satisfies the formally finite-type assumption (§2.2.2), the rings \mathfrak{S} , $\mathcal{O}_\mathcal{E}$, and \mathcal{E} are regular (in particular, normal). The endomorphism $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ extends to $\mathcal{O}_\mathcal{E}$ and \mathcal{E} , which we also denote by φ . Note that $\mathcal{O}_\mathcal{E}$ is a Cohen ring with residue field \mathbf{E}_{R_∞} , and φ lifts the p th power map on the residue field.

Definition 7.2.1. An étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module is a finitely generated $\mathcal{O}_\mathcal{E}$ -module M equipped with a φ -linear endomorphism $\varphi_M : M \rightarrow M$ such that the linearisation $1 \otimes \varphi_M : \varphi^* M \rightarrow M$ is an isomorphism. We say that an étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module M is *projective* (respectively, *torsion*) if the underlying $\mathcal{O}_\mathcal{E}$ -module M is projective (respectively p -power torsion).

Let $\mathrm{Mod}_{\mathcal{O}_\mathcal{E}}^{\mathrm{ét}}(\varphi)$ denote the category of étale $(\varphi, \mathcal{O}_\mathcal{E})$ -modules. Let $\mathrm{Mod}_{\mathcal{O}_\mathcal{E}}^{\mathrm{ét}, \mathrm{pr}}(\varphi)$ and $\mathrm{Mod}_{\mathcal{O}_\mathcal{E}}^{\mathrm{ét}, \mathrm{tor}}(\varphi)$ respectively denote the full subcategories of projective and torsion objects.

Any étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module annihilated by p is automatically projective over $\mathbf{E}_{R_\infty} := \mathcal{O}_\mathcal{E}/(p)$, which follows from the same proof as [And06, Lemma 7.10].

For any quasi-Kisin \mathfrak{S} -module \mathfrak{M} , the scalar extension $M := \mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$ together with $\varphi_M := \varphi_{\mathcal{O}_\mathcal{E}} \otimes \varphi_{\mathfrak{M}}$ is a projective étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module, since $E(u)$ is a unit in $\mathcal{O}_\mathcal{E}$.

There exists a natural notion of subquotient, direct sum, \otimes -product for étale φ -modules. Duality is only defined for projective and torsion objects. For a projective étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module M , we define a dual étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module M^* to be the $\mathcal{O}_\mathcal{E}$ -linear dual of M where φ_{M^*} is defined so that $1 \otimes \varphi_{M^*} = ((1 \otimes \varphi_M)^{-1})^*$. We can similarly define duality for torsion étale $(\varphi, \mathcal{O}_\mathcal{E})$ -modules using Pontragin duality $M \rightsquigarrow \mathrm{Hom}_{\mathcal{O}_\mathcal{E}}(M, \mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$.

The natural inclusion $\mathbf{E}_{R_\infty} \hookrightarrow \widetilde{\mathbf{E}}_{R_\infty}$ has a unique lift $\mathcal{O}_\mathcal{E} \hookrightarrow W(\widetilde{\mathbf{E}}_{R_\infty})$ with the property that the Witt vector Frobenius restricts to φ on $\mathcal{O}_\mathcal{E}$. Indeed, the p -adic completion of $\varinjlim_{\varphi} \mathcal{O}_\mathcal{E}$ can be naturally identified with $W(\mathbf{E}_{R_\infty}^{\mathrm{perf}})$, so the desired morphism is obtained as follows:

$$(7.2.2) \quad \mathcal{O}_\mathcal{E} \hookrightarrow \varinjlim_{\varphi} \mathcal{O}_\mathcal{E} \hookrightarrow W(\mathbf{E}_{R_\infty}^{\mathrm{perf}}) \hookrightarrow W(\widetilde{\mathbf{E}}_{R_\infty}).$$

We will view $W(\overline{R}^\flat[1/\varpi])$ as an $\mathcal{O}_\mathcal{E}$ -algebra via $u \mapsto [\varpi]$.

Let $\mathcal{O}_\mathcal{E}^{\mathrm{ur}}$ be the integral closure of $\mathcal{O}_\mathcal{E}$ in $W(\overline{R}^\flat[\frac{1}{u}])$ and $\widehat{\mathcal{O}}_\mathcal{E}^{\mathrm{ur}}$ the p -adic closure of $\mathcal{O}_\mathcal{E}^{\mathrm{ur}}$. Let $\mathfrak{S}^{\mathrm{ur}}$ be the integral closure of \mathfrak{S} in $\widehat{\mathcal{O}}_\mathcal{E}^{\mathrm{ur}}$, and $\widehat{\mathfrak{S}}^{\mathrm{ur}}$ be its p -adic closure. Note that $\widehat{\mathfrak{S}}^{\mathrm{ur}}$ is indeed contained in $W(\overline{R}^\flat)$, not just in $W(\overline{R}^\flat[1/\varpi])$.

We now define the $\mathcal{G}_{\tilde{R}_\infty}$ -action on these rings. As $\mathcal{O}_\mathcal{E}$ is normal, it follows from Theorem 7.1.3 that $\mathcal{O}_\mathcal{E}^{\text{ur}}$ is the union of finite étale $\mathcal{O}_\mathcal{E}$ -subalgebras in $W(\overline{R}^\flat[1/\tilde{\omega}])$, and $\text{Aut}_{\mathcal{O}_\mathcal{E}}(\mathcal{O}_\mathcal{E}^{\text{ur}})$ is isomorphic to the fundamental group of $\text{Spec } \mathcal{O}_\mathcal{E}$ (with suitable base point). Now, not that we have the following natural equivalences of categories induced by base change

$$(7.2.3) \quad \{\text{finite étale covers of } \text{Spec } \mathcal{O}_\mathcal{E}\} \xrightarrow{\sim} \{\text{finite étale covers of } \text{Spec } \mathbf{E}_{R_\infty}\} \\ \xrightarrow{\sim} \{\text{finite étale covers of } \text{Spec } \tilde{\mathbf{E}}_{R_\infty}\}.$$

Indeed, the first arrow is an equivalence because $\mathcal{O}_\mathcal{E}$ is p -adic and finite étale morphisms uniquely lift under infinitesimal thickenings. The second arrow is an equivalence because $\tilde{\mathbf{E}}_{R_\infty}$ is the u -adic completion of a radicial extension of \mathbf{E}_{R_∞} (which is henselian along (u)), so we may apply [GR03, Proposition 5.4.54]. Now, the isomorphism $\mathcal{G}_{\tilde{R}_\infty} \cong \mathcal{G}_{\tilde{\mathbf{E}}_{R_\infty}}$ in Theorem 7.1.3 and the above equivalences of categories produce the following isomorphisms:

$$(7.2.4) \quad \text{Aut}_{\mathcal{O}_\mathcal{E}}(\mathcal{O}_\mathcal{E}^{\text{ur}}) \cong \mathcal{G}_{\mathbf{E}_{R_\infty}} \cong \mathcal{G}_{\tilde{\mathbf{E}}_{R_\infty}} \cong \mathcal{G}_{\tilde{R}_\infty}.$$

Since $\mathcal{G}_{\tilde{R}_\infty}$ -action on $\mathcal{O}_\mathcal{E}^{\text{ur}}$ fixes \mathfrak{S} , it stabilises the subring $\mathfrak{S}^{\text{ur}} \subset \mathcal{O}_\mathcal{E}^{\text{ur}}$. Since the $\mathcal{G}_{\tilde{R}_\infty}$ -action on $\mathcal{O}_\mathcal{E}^{\text{ur}}$ is p -adically continuous, it extends to a continuous $\mathcal{G}_{\tilde{R}_\infty}$ -action on $\widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}}$ and $\widehat{\mathfrak{S}}^{\text{ur}}$, respectively. In particular, we obtain a $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant embedding

$$(7.2.5) \quad \widehat{\mathfrak{S}}^{\text{ur}} \hookrightarrow W(\overline{R}^\flat) \subseteq A_{\text{cris}}^\nabla(R).$$

Lemma 7.2.6. *We have $\mathcal{O}_\mathcal{E} = (\widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_\infty}}$ and $\mathfrak{S} = (\widehat{\mathfrak{S}}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_\infty}}$. In particular, the same statement holds modulo p^n as well.*

Proof. It suffices to prove $\mathcal{O}_\mathcal{E} \supseteq (\widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_\infty}}$. Since $(\widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}})^{\mathcal{G}_{\tilde{R}_\infty}} \subset \widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}}$ is a closed subspace under the p -adic topology (by continuity of $\mathcal{G}_{\tilde{R}_\infty}$ -action), this claim follows from $\mathbf{E}_{R_\infty} = \mathcal{O}_\mathcal{E}/(p) \cong (\mathcal{O}_\mathcal{E}^{\text{ur}}/(p))^{\mathcal{G}_{\tilde{R}_\infty}}$. \square

Lemma 7.2.7. *There exists a unique $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant ring endomorphism $\varphi : \widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}} \rightarrow \widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}}$ which lifts the p th power map $\varphi : \mathcal{O}_\mathcal{E}^{\text{ur}}/(p) \rightarrow \mathcal{O}_\mathcal{E}^{\text{ur}}/(p)$ and extends $\varphi : \mathcal{O}_\mathcal{E} \rightarrow \mathcal{O}_\mathcal{E}$. Furthermore, this map restricts to $\varphi : \widehat{\mathfrak{S}}^{\text{ur}} \rightarrow \widehat{\mathfrak{S}}^{\text{ur}}$, and the natural inclusion $\widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}} \hookrightarrow W(\overline{R}^\flat[1/\tilde{\omega}])$ is φ -equivariant.*

Proof. Let us first show that for any finite étale $\mathcal{O}_\mathcal{E}$ -algebra A , there exists a unique $\mathcal{O}_\mathcal{E}$ -algebra isomorphism $\varphi^* A \rightarrow A$ which lifts the relative Frobenius (iso)morphism $\varphi^* A/(p) \rightarrow A/(p)$ over $\mathcal{O}_\mathcal{E}/(p)$. This follows from the same argument as the proof of Lemma 2.3.1 (the existence of $\varphi : R_0 \rightarrow R_0$). The uniqueness follows from $\mathcal{O}_\mathcal{E}$ -étaleness of A , and clearly φ sends any element integral over \mathfrak{S} to an element integral over \mathfrak{S} .

Using the unique lift of Frobenius $\varphi : A \rightarrow A$, we can now construct a φ -equivariant embedding

$$A \hookrightarrow W(A \otimes_{\mathcal{O}_\mathcal{E}} \tilde{\mathbf{E}}_{R_\infty})$$

as in (7.2.2). By choosing an $\tilde{\mathbf{E}}_{R_\infty}$ -embedding $A \otimes_{\mathcal{O}_\mathcal{E}} \tilde{\mathbf{E}}_{R_\infty} \hookrightarrow \overline{R}^\flat[1/\tilde{\omega}]$, we now obtain the unique φ -equivariant lift $A \hookrightarrow W(\overline{R}^\flat[1/\tilde{\omega}])$. Note that $\mathcal{O}_\mathcal{E}^{\text{ur}}$ is the union of the image of such A . This gives a unique lift of Frobenius $\varphi : \mathcal{O}_\mathcal{E}^{\text{ur}} \rightarrow \mathcal{O}_\mathcal{E}^{\text{ur}}$ over $\mathcal{O}_\mathcal{E}$ with respect to which the natural inclusion $\mathcal{O}_\mathcal{E}^{\text{ur}} \hookrightarrow W(\overline{R}^\flat[1/\tilde{\omega}])$ is φ -equivariant. The desired lift of Frobenius $\varphi : \widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}} \rightarrow \widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}}$ is obtained by p -adically extending this map. The uniqueness follows from the uniqueness of φ on $\mathcal{O}_\mathcal{E}^{\text{ur}}$ and the density of $\mathcal{O}_\mathcal{E}^{\text{ur}} \subset \widehat{\mathcal{O}}_\mathcal{E}^{\text{ur}}$.

Finally, the $\mathcal{G}_{\tilde{R}_\infty}$ -equivariance claim follows from the uniqueness of φ . Indeed, for any $g \in \mathcal{G}_{\tilde{R}_\infty}$ the uniqueness implies $g\varphi g^{-1} = \varphi$. \square

For any $M \in \mathbf{Mod}_{\mathcal{O}_\varepsilon}^{\text{ét}}(\varphi)$ and $T \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$, we define:

$$(7.2.8) \quad \mathcal{T}(M) := (\widehat{\mathcal{O}}_\varepsilon^{\text{ur}} \otimes_{\mathcal{O}_\varepsilon} M)^{\varphi=1}$$

$$(7.2.9) \quad \mathcal{D}(T) := (\widehat{\mathcal{O}}_\varepsilon^{\text{ur}} \otimes_{\mathbb{Z}_p} T)^{\mathcal{G}_{\tilde{R}_\infty}}.$$

Note that $\mathcal{G}_{\tilde{R}_\infty}$ continuously acts on $\mathcal{T}(M)$ induced by its natural action on $\widehat{\mathcal{O}}_\varepsilon^{\text{ur}}$, and there is a natural φ -linear endomorphism on $\mathcal{D}(T)$ induced by φ on $\widehat{\mathcal{O}}_\varepsilon^{\text{ur}}$.

For any profinite group \mathcal{G} , let $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ denote the category of finitely generated \mathbb{Z}_p -modules equipped with continuous \mathcal{G} -action. Let $\text{Rep}_{\mathbb{Z}_p}^{\text{free}}(\mathcal{G})$ and $\text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(\mathcal{G})$ respectively denote the full subcategories of free and torsion objects.

Proposition 7.3 ([Kat73, Lemma 4.1.1]). *The constructions \mathcal{T} and \mathcal{D} are exact quasi-inverse equivalences of \otimes -categories between $\mathbf{Mod}_{\mathcal{O}_\varepsilon}^{\text{ét}}(\varphi)$ and $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\tilde{R}_\infty})$. Furthermore, \mathcal{T} and \mathcal{D} restrict to rank-preserving equivalences of categories between $\mathbf{Mod}_{\mathcal{O}_\varepsilon}^{\text{ét,pr}}(\varphi)$ and $\text{Rep}_{\mathbb{Z}_p}^{\text{free}}(\mathcal{G}_{\tilde{R}_\infty})$, and length-preserving equivalences of categories between $\mathbf{Mod}_{\mathcal{O}_\varepsilon}^{\text{ét,tor}}(\varphi)$ and $\text{Rep}_{\mathbb{Z}_p}^{\text{tor}}(\mathcal{G}_{\tilde{R}_\infty})$. In both cases, \mathcal{T} and \mathcal{D} commute with duality.*

Proof. (Cf. [And06, Theorem 7.11]) By dévissage, it suffices to prove the proposition for the objects killed by p , which is done in [Kat73, Lemma 4.1.1]; note that \mathbf{E}_{R_∞} is a normal domain (which is required for [Kat73, Lemma 4.1.1]), and we have identified $\mathcal{G}_{\tilde{R}_\infty}$ with a fundamental group of $\text{Spec } \mathbf{E}_{R_\infty}$ in (7.2.4). \square

When M is a projective étale $(\varphi, \mathcal{O}_\varepsilon)$ -module, we can define the following contravariant functor:

$$(7.3.1) \quad \mathcal{T}^*(M) := \mathcal{T}(M^*) = \text{Hom}_{\mathcal{O}_\varepsilon, \varphi}(M, \widehat{\mathcal{O}}_\varepsilon^{\text{ur}}).$$

By Proposition 7.3, we have a natural isomorphism $\mathcal{T}^*(M) \cong (\mathcal{T}(M))^*$. One can define \mathcal{T}^* for torsion étale $(\varphi, \mathcal{O}_\varepsilon)$ -modules using Pontragin duality:

$$(7.3.2) \quad \mathcal{T}^*(M) := \text{Hom}_{\mathcal{O}_\varepsilon, \varphi}(M, \widehat{\mathcal{O}}_\varepsilon^{\text{ur}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p).$$

Clearly, \mathcal{T}^* preserves exact sequences of projective or torsion objects. Furthermore, if we have a short exact sequence $0 \rightarrow \widetilde{M}' \rightarrow \widetilde{M} \rightarrow M \rightarrow 0$ of étale $(\varphi, \mathcal{O}_\varepsilon)$ -module, where \widetilde{M} and \widetilde{M}' are \mathcal{O}_ε -projective, and M is p -power torsion, then we have a natural $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant short exact sequence

$$(7.3.3) \quad 0 \rightarrow \mathcal{T}^*(\widetilde{M}) \rightarrow \mathcal{T}^*(\widetilde{M}') \rightarrow \mathcal{T}^*(M) \rightarrow 0,$$

where the second map is defined by viewing $f : \widetilde{M}' \rightarrow \widetilde{M}$ as $f : \widetilde{M}' \rightarrow \widehat{\mathcal{O}}_\varepsilon^{\text{ur}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ using the isomorphism $\widetilde{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \widetilde{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

7.4. Base change. Let $R, W \subset R_0$, and $E(u)$ be as in the beginning of §7, and consider a topological adic R_0 -algebra R'_0 such that $R'_0/(p)$ locally admits a finite p -basis and is formally finitely generated over some field k' . Set $R' := R'_0[u]/E(u)$, and let $f : R \rightarrow R'$ denote the structure morphism. (Cf. §2.2.2, §3.6.) Choose a map $\bar{f} : \bar{R} \rightarrow \bar{R}'$ that extends $f : R \rightarrow R'$ (as in §5.6). We use the superscript \prime to denote the construction for R' (such as \mathfrak{S}' and $\mathcal{O}_{\varepsilon'}$). Note that \bar{f} induces a continuous group homomorphism $\mathcal{G}_{R'} \rightarrow \mathcal{G}_R$, and a continuous ring homomorphism $\bar{f}^b : \bar{R}^b \rightarrow \bar{R}'^b$ which respects the Galois action in a suitable sense.

As in (7.2.5), we identify $\widehat{\mathfrak{S}}^{\text{ur}}$ and $\widehat{\mathfrak{S}}'^{\text{ur}}$ as subrings of $W(\bar{R}^b)$ and $W(\bar{R}'^b)$, respectively. Since the map $W(\bar{f}^b) : W(\bar{R}^b) \rightarrow W(\bar{R}'^b)$ takes \mathfrak{S} into \mathfrak{S}' , $W(\bar{f}^b)$ restricts to a map $\widehat{\mathfrak{S}}^{\text{ur}} \rightarrow \widehat{\mathfrak{S}}'^{\text{ur}}$ which commutes with φ 's and $\mathcal{G}_{R'_\infty}$ -action if we let $\mathcal{G}_{R'_\infty}$ act on

$\widehat{\mathfrak{S}}^{\text{ur}}$ via the map $\mathcal{G}_{R'_\infty} \rightarrow \mathcal{G}_{R_\infty}$. We also obtain a map $\widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}} \rightarrow \widehat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}}$, which respects φ and Galois actions.

Now for any étale $(\varphi, \mathcal{O}_{\mathcal{E}})$ -module M , we obtain a natural $\mathcal{G}_{R'_\infty}$ -equivariant map $\mathcal{T}(M) \rightarrow \mathcal{T}(\mathcal{O}_{\mathcal{E}'} \otimes_{\mathcal{O}_{\mathcal{E}}} M)$, where $\mathcal{G}_{R'_\infty}$ acts on $\mathcal{T}(M)$ via the natural map $\mathcal{G}_{R'_\infty} \rightarrow \mathcal{G}_{R_\infty}$. This map $\mathcal{T}(M) \rightarrow \mathcal{T}(\mathcal{O}_{\mathcal{E}'} \otimes_{\mathcal{O}_{\mathcal{E}}} M)$ is indeed an isomorphism, as it is an injective map of finite free \mathbb{Z}_p -modules of same rank with saturated image. (To see the image is saturated, note that $\mathcal{E} \cap \mathcal{O}_{\mathcal{E}'} = \mathcal{O}_{\mathcal{E}}$ inside \mathcal{E}' .) Similarly, we obtain $\mathcal{T}^*(M) \xrightarrow{\sim} \mathcal{T}^*(\mathcal{O}_{\mathcal{E}'} \otimes_{\mathcal{O}_{\mathcal{E}}} M)$.

Lemma 7.5. *For any quasi-Kisin \mathfrak{S} -module \mathfrak{M} , the natural morphism*

$$\text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \widehat{\mathfrak{S}}^{\text{ur}}) \rightarrow \mathcal{T}^*(\mathfrak{M}) = \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}),$$

induced by the natural inclusion $\widehat{\mathfrak{S}}^{\text{ur}} \hookrightarrow \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$, is an isomorphism.

Proof. When $R = \mathcal{O}_K$ with perfect residue field the proposition is proved in [Fon90, §B, Proposition 1.8.3]. For the general case, let $k' := \varinjlim_{\varphi} \text{Frac}(R/\varpi)$ be the perfect closure of $\text{Frac}(R/\varpi)$. As discussed in §3.6 (Ex6), there is an \mathcal{O}_K -algebra homomorphism $R \rightarrow \mathcal{O}_{K'} = W(k') \otimes_W \mathcal{O}_K$ which satisfies the assumption in §3.6. We apply the discussion in §7.4 to this setting.

Let us write \mathfrak{S}' , $\mathcal{O}_{\mathcal{E}'}$, $\widehat{\mathfrak{S}}'^{\text{ur}}$, and $\widehat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}}$ for the rings constructed from $\mathcal{O}_{K'}$. Now, note that for any quasi-Kisin \mathfrak{S} -module \mathfrak{M} , the scalar extension $\mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M}$ is a quasi-Kisin \mathfrak{S}' -module. In particular, we obtain a natural $\mathcal{G}_{K'_\infty}$ -equivariant isomorphism $\mathcal{T}^*(\mathfrak{M}) \xrightarrow{\sim} \mathcal{T}^*(\mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M})$ (cf., §7.4). Now, applying the result for $\mathcal{O}_{K'}$ (which is known by [Fon90, §B, Proposition 1.8.3]), it follows that any φ -compatible map $x : \mathfrak{M} \rightarrow \widehat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}}$ has image in $\widehat{\mathfrak{S}}'^{\text{ur}} \cap \widehat{\mathcal{O}}_{\mathcal{E}'}^{\text{ur}}$. To conclude it suffices to show that $\widehat{\mathfrak{S}}^{\text{ur}} = \widehat{\mathfrak{S}}'^{\text{ur}} \cap \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$. In turn, it suffices to show that for any $s \in \widehat{\mathfrak{S}}'^{\text{ur}} \cap \widehat{\mathcal{O}}_{\mathcal{E}}^{\text{ur}}$ the subalgebra $\mathfrak{S}[s] \subseteq \mathcal{O}_{\mathcal{E}'}^{\text{ur}}$ is finite over \mathfrak{S} . But $\{\text{Spec } \mathfrak{S}', \text{Spec } \mathcal{O}_{\mathcal{E}}\}$ is an fpqc covering of $\text{Spec } \mathfrak{S}$, and the assumption on s implies that $\mathfrak{S}'[s]$ is finite over \mathfrak{S}' and $\mathcal{O}_{\mathcal{E}}[s]$ is finite over $\mathcal{O}_{\mathcal{E}}$. Now the claim follows from fpqc descent theory. \square

8. COMPARISON BETWEEN GALOIS-STABLE LATTICES

Assume that R satisfies the formally finite-type assumption (§2.2.2). If $p > 2$ then we have the following exact functor

$$\mathfrak{M}^* : \{p\text{-divisible groups over } R\} \xrightarrow{\mathcal{M}^*} \mathbf{MF}_S(\varphi, \nabla) \xleftarrow{\sim} \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla),$$

which is an anti-equivalence of categories if $p > 2$; cf. Corollary 6.3.1. We also associate, to a p -divisible group G over R , a $\mathcal{G}_{\widetilde{R}_\infty}$ -representation $\mathcal{T}^*(\mathfrak{M}^*(G))$ under some additional assumption on R (as stated in §7); cf., (7.3.1). The main goal of this section is to prove the following proposition:

Proposition 8.1. *Assume that R satisfies the “refined almost étaleness” assumption (§2.2.3), and for some Cohen subring $W \subset R_0$ we have $E(u) \in W[u]$ (cf. §7). Let $\mathcal{M} \in \mathbf{MF}_S(\varphi, \nabla)$, and suppose that $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ is a quasi-Kisin \mathfrak{S} -module such that $\mathcal{M} \cong S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ as quasi-Breuil S -modules (which exists by Proposition 6.3). Then there is a natural $\mathcal{G}_{\widetilde{R}_\infty}$ -equivariant injective map*

$$\mathcal{T}^*(\mathfrak{M}) \hookrightarrow T_{\text{cris}}^*(\mathcal{M}),$$

which is an isomorphism if $p > 2$. Furthermore, if $\mathcal{M} = \mathbb{D}^(G)(S)$ for some p -divisible group G , then (for any p) we have a natural $\mathcal{G}_{\widetilde{R}_\infty}$ -equivariant injective map $\mathcal{T}^*(\mathfrak{M}) \hookrightarrow V_p(G)$, whose image is $T_p(G)$.*

If $p > 2$ then the second assertion follows from the first due to Theorem 5.3, while it requires more work if $p = 2$. When $p = 2$, we can use Proposition 8.1 to

define the functor $\mathfrak{M}^* : G \rightsquigarrow \mathfrak{M}$ (without passing to the isogeny categories), and obtain some full faithfulness result; cf., Corollary 8.6.1.

Before we begin the proof, let us record an immediate corollary of Proposition 8.1:

Corollary 8.2. *Assume that R satisfies the formally finite-type assumption (§2.2.2), and for some Cohen subring $W \subset R_0$ we have $E(u) \in W[u]$ (cf. §7). Let G be a p -divisible group over R , and let $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ be such that $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cong \mathbb{D}^*(G)(S)$. Then (for any p) there is a natural $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant isomorphism $\mathcal{T}^*(\mathfrak{M}) \cong T_p(G)$, which recovers the isomorphism in Proposition 8.1 when R additionally satisfies the “refined almost étaleness” assumption (§2.2.3).*

Proof. By Lemma 2.3.5 and the discussion in §7.4, for any open immersion of formal schemes $\mathrm{Spf}(R', J_{R'}) \hookrightarrow \mathrm{Spf}(R, J_R)$ we have a natural \mathbb{Z}_p -isomorphism

$$\mathcal{T}^*(\mathfrak{M}) \xrightarrow{\sim} \mathcal{T}^*(\mathfrak{S}_{R'} \otimes_{\mathfrak{S}} \mathfrak{M}).$$

For any (R, J_R) as in the statement, there exists a Zariski open cover $\{\mathrm{Spf}(R_\alpha, J_{R_\alpha})\}$ of $\mathrm{Spf}(R, J_R)$ such that each R_α satisfies the “refined almost étaleness” assumption (§2.2.3). (Note that $R[\frac{1}{p}] = K \otimes_W R_0$ is étale over $R_0[\frac{1}{p}]$.) To prove the corollary, it suffices to prove it for each R_α , for which we can apply Proposition 8.1. \square

Recall that T_{cris}^* is defined via a \mathcal{G}_{R_∞} -invariant embedding $S \hookrightarrow A_{\mathrm{cris}}(R)$ which respects the suitable connections on both sides (cf., (5.4.1)). On the other hand, \mathcal{T}^* is defined using the period ring $\widehat{\mathfrak{S}}^{\mathrm{ur}}$, which admits a $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant embedding into $W(\overline{R}^\flat)$, hence into $A_{\mathrm{cris}}^\nabla(R)$. To “interpolate” these two constructions, we introduce auxiliary semilinear objects over \tilde{S} as below.

8.3. More divided power algebras. As in Proposition 8.1, we assume that R satisfies the “refined almost étaleness” assumption (§2.2.3), and for some Cohen subring $W \subset R_0$ we have $E(u) \in W[u]$ (cf. §7), although some constructions work in more generality. Consider $R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S} \twoheadrightarrow R$ defined by R_0 -linearly extending the natural projection $\mathfrak{S} \twoheadrightarrow \mathfrak{S}/(E(u)) \cong R$. (Here, $\widehat{\otimes}$ is the completed \otimes -product with respect to the p -adic topology.) Let \tilde{S} denote the p -adically completed divided power envelope of $R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}$ with respect to $\ker(R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S} \twoheadrightarrow R)$, which is generated by $E(u)$ and $a \otimes 1 - 1 \otimes a$ for $a \in R_0$.

Let $\mathrm{Fil}^1 \tilde{S}$ denote the divided power ideal with $\tilde{S}/\mathrm{Fil}^1 \tilde{S} \cong R$. Note that $\varphi \otimes \varphi$ on $R_0 \otimes_{\mathbb{Z}_p} \mathfrak{S}$ extend to φ on \tilde{S} , and we have $\langle \varphi(\mathrm{Fil}^1 \tilde{S}) \rangle = p\tilde{S}$ as $\varphi(E(u)) \in p\tilde{S}^\times$. We set $\varphi_1 := \frac{\varphi}{p} : \mathrm{Fil}^1 \tilde{S} \rightarrow \tilde{S}$.

We define a connection $d_{\tilde{S}} : \tilde{S} \rightarrow \tilde{S} \otimes_{R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}} \widehat{\Omega}_{R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}}$ by extending the universal continuous derivation for $R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}$ in a usual way, and define $d_{\tilde{S}}^u : \tilde{S} \rightarrow \tilde{S} \otimes_{R_0} \widehat{\Omega}_{R_0}$ by composing $d_{\tilde{S}}$ with the natural projection $\tilde{S} \otimes_{R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}} \widehat{\Omega}_{R_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}} \twoheadrightarrow \tilde{S} \otimes_{R_0} \widehat{\Omega}_{R_0}$.

One can embed $\mathfrak{S} = R_0[[u]]$ into \tilde{S} in two different ways, as follows: for any $\sum_{n=0}^\infty a_n u^n \in \mathfrak{S}$ with $a_n \in R_0$

$$(8.3.1) \quad \iota_1 : \sum_{n=0}^\infty a_n u^n \mapsto \sum_{n=0}^\infty a_n \otimes u^n \in \tilde{S}$$

$$(8.3.2) \quad \iota_2 : \sum_{n=0}^\infty a_n u^n \mapsto 1 \otimes \left(\sum_{n=0}^\infty a_n u^n \right) \in \tilde{S}.$$

Both maps extend to $\iota_1, \iota_2 : S \hookrightarrow \tilde{S}$, which respect φ and the divided power structures. Note that ι_1 is horizontal when S is given a connection d_S^u , and $\iota_2(S) =$

$(\tilde{S})^{d_{\tilde{S}}^u=0}$. Let us also consider the map

$$(8.3.3) \quad \delta : \tilde{S} \rightarrow S; \quad a \otimes s \mapsto a \cdot s \text{ for any } a \in R_0, s \in \mathfrak{S},$$

which respects φ , the divided power structures, and the connections when S is endowed with $d_{\tilde{S}}^u$. Also note that $\delta \circ \iota_1 = \delta \circ \iota_2 = \text{id}_S$.

We now define a φ -compatible $\mathcal{G}_{\tilde{R}_\infty}$ -invariant R_0 -algebra map $\tilde{j} : \tilde{S} \hookrightarrow A_{\text{cris}}(R)$ as follows. First, recall that we have a φ -compatible $\mathcal{G}_{\tilde{R}_\infty}$ -invariant map $\mathfrak{S} \hookrightarrow W(\overline{R}^b)$. (Cf. (7.2.5).) We may R_0 -linearly extend this map to $R_0 \otimes_{\mathbb{Z}_p} \mathfrak{S} \hookrightarrow R_0 \otimes_{\mathbb{Z}_p} W(\overline{R}^b)$, which extends to $\tilde{S} \hookrightarrow A_{\text{cris}}(R)$. This map, which we denote by \tilde{j} , is clearly compatible with φ and ∇ , and the image of $\text{Fil}^1 \tilde{S}$ is precisely the intersection of the image of \tilde{S} and $\text{Fil}^1 A_{\text{cris}}(R)$.

We remark that the embedding $S \xrightarrow{\iota_1} \tilde{S} \xrightarrow{\tilde{j}} A_{\text{cris}}(R)$ is precisely the embedding we use to define T_{cris}^* (5.4.1). The embedding $\mathfrak{S} \xrightarrow{\iota_2} \tilde{S} \xrightarrow{\tilde{j}} A_{\text{cris}}(R_0)$ has image in $W(\overline{R}^b) \subset A_{\text{cris}}^\nabla(R)$ and coincides with the map (7.2.5).

8.4. More filtered modules with connection. By applying Proposition 3.2.5 to $\widehat{D} = \tilde{S}$, we obtain an equivalence of categories $\mathcal{E} \rightsquigarrow \mathcal{E}(\tilde{S})$ from the category of Dieudonné crystals over $\text{Spf}(R, (\varpi))$ to $\mathbf{MF}_{\tilde{S}}(\varphi, \nabla)$ (following the notation of Definition 3.2.3). Since $\mathcal{E} \rightsquigarrow \mathcal{E}(\tilde{S})$ also defines an equivalence of categories into $\mathbf{MF}_S(\varphi, \nabla)$, we obtain an equivalence of categories $\mathbf{MF}_S(\varphi, \nabla) \xrightarrow{\sim} \mathbf{MF}_{\tilde{S}}(\varphi, \nabla)$.

We can indeed make the functor $\mathbf{MF}_S(\varphi, \nabla) \rightarrow \mathbf{MF}_{\tilde{S}}(\varphi, \nabla)$ explicit as follows. The maps $\iota_1, \iota_2 : S \rightarrow \tilde{S}$ and $\delta : \tilde{S} \rightarrow S$ are divided power morphisms over R , so they induce a horizontal φ -compatible filtered morphisms $\mathcal{E}(S) \xrightarrow[\delta]{\iota_j} \mathcal{E}(\tilde{S})$ (with $j = 1, 2$) for any Dieudonné crystal \mathcal{E} over R . Since $\delta \circ \iota_1 = \text{id}_S$, we obtain natural isomorphisms

$$(8.4.1) \quad \mathcal{E}(\tilde{S}) \cong \tilde{S} \otimes_{\iota_j, S} \mathcal{E}(S), \quad \mathcal{E}(S) \cong S \otimes_{\delta, \tilde{S}} \mathcal{E}(\tilde{S}), \text{ for } j = 1, 2.$$

Set $\mathcal{M} := \mathcal{E}(S)$ and $\widetilde{\mathcal{M}} := \tilde{S} \otimes_{\iota_1, S} \mathcal{E}(S)$, using ι_1 . Then one can check that the following coincides with the natural extra structure on $\mathcal{E}(\tilde{S})$ as in Proposition 3.2.5:

- (1) $\varphi_{\widetilde{\mathcal{M}}} = \varphi_{\tilde{S}} \otimes \varphi_{\mathcal{M}}$;
- (2) $\nabla_{\widetilde{\mathcal{M}}}$ is the connection over $d_{\tilde{S}}$, such that if for $m \in \mathcal{M}$ we have $\nabla_{\mathcal{M}}(m) = \sum_i m_i \otimes ds_i$ then $\nabla_{\widetilde{\mathcal{M}}}(1 \otimes m) = \sum_i m_i \otimes d(\iota_1(s_i))$;
- (3) $\text{Fil}^1 \widetilde{\mathcal{M}}$ is the preimage of the Hodge Filtration on $\mathcal{E}(R)$ by the natural projection $\mathcal{E}(\tilde{S}) \rightarrow \mathcal{E}(R)$.

Remark 8.4.2. Note that we can use ι_2 instead of ι_1 and repeat the above construction, but it follows from Proposition 3.2.5 that there is a natural isomorphism $\widetilde{\mathcal{M}} := \tilde{S} \otimes_{\iota_1, S} \mathcal{E}(S) \cong \tilde{S} \otimes_{\iota_2, S} \mathcal{E}(S)$ in $\mathbf{MF}_{\tilde{S}}(\varphi, \nabla)$.

For any $\widetilde{\mathcal{M}} \in \mathbf{MF}_{\tilde{S}}(\varphi, \nabla)$, we define a connection $\nabla_{\widetilde{\mathcal{M}}}^u : \widetilde{\mathcal{M}} \rightarrow \widetilde{\mathcal{M}} \otimes_{R_0} \widehat{\Omega}_{R_0}$ compatible with $d_{\tilde{S}}^u$ on \tilde{S} by

$$(8.4.3) \quad \nabla_{\widetilde{\mathcal{M}}}^u : \widetilde{\mathcal{M}} \xrightarrow{\nabla_{\widetilde{\mathcal{M}}}} \widetilde{\mathcal{M}} \otimes_{R_0 \widehat{\otimes} \mathfrak{S}} (\widehat{\Omega}_{R_0 \widehat{\otimes} \mathfrak{S}}) \twoheadrightarrow \widetilde{\mathcal{M}} \otimes_{R_0} \widehat{\Omega}_{R_0},$$

where the second map is the natural projection.

The relation between $\nabla_{\widetilde{\mathcal{M}}}^u$ and $\nabla_{\mathcal{M}}^u$ is as follows:

$$(8.4.4) \quad \nabla_{\widetilde{\mathcal{M}}}^u(\tilde{s} \otimes m) := \tilde{s} \otimes \nabla_{\mathcal{M}}^u(m) + m \otimes d_{\tilde{S}}^u(\tilde{s}),$$

for any $m \in \mathcal{M}$ and $\tilde{s} \in \tilde{S}$. Here we use ι_1 to get $\widetilde{\mathcal{M}} \cong \tilde{S} \otimes_{\iota_1, S} \mathcal{M}$. We will often work with $\nabla_{\widetilde{\mathcal{M}}}^u$ instead of $\nabla_{\widetilde{\mathcal{M}}}$.

Remark 8.4.5. By abuse of notation, let $\iota_1 : \mathcal{M} \rightarrow \tilde{S} \otimes_{\iota_1, S} \mathcal{M} =: \tilde{\mathcal{M}}$ be the natural map, and $\iota_2 : \mathcal{M} \rightarrow \tilde{S} \otimes_{\iota_2, S} \mathcal{M} \cong \tilde{\mathcal{M}}$ the composition of the natural map and the natural isomorphism discussed in Remark 8.4.2. Then while ι_1 is horizontal with respect to $\nabla_{\mathcal{M}}^u$ and $\nabla_{\tilde{\mathcal{M}}}^u$, one can directly see that $\iota_2(\mathcal{M}) = \tilde{\mathcal{M}}^{\nabla^u=0}$. This observation will be used later in §8.5.

Using the embedding $\tilde{j} : \tilde{S} \hookrightarrow A_{\text{cris}}(R)$ constructed in §8.3, we define

$$(8.4.6) \quad \tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}}) := \text{Hom}_{\tilde{S}, \text{Fil}^1, \varphi, \nabla^u}(\tilde{\mathcal{M}}, A_{\text{cris}}(R))$$

for $\tilde{\mathcal{M}} \in \mathbf{MF}_{\tilde{S}}(\varphi, \nabla)$. Here, the morphisms are required to respect the connections $\nabla_{\tilde{\mathcal{M}}}^u$ on $\tilde{\mathcal{M}}$ and ∇ on $A_{\text{cris}}(R)$ defined in §4.1. Note that $\tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}})$ is p -adic, and has a continuous $\mathcal{G}_{\tilde{R}_\infty}$ -action (as $\tilde{S} \hookrightarrow A_{\text{cris}}(R)$ is $\mathcal{G}_{\tilde{R}_\infty}$ -invariant).

For any $\mathcal{M} \in \mathbf{MF}_S(\varphi, \nabla)$, there is a p -divisible group G' over R such that $\mathcal{M}[\frac{1}{p}] \cong \mathcal{M}^*(G')[\frac{1}{p}] := \mathbb{D}^*(G')(S)[\frac{1}{p}]$ by Theorem 3.5. Set $\tilde{\mathcal{M}} := \tilde{S} \otimes_{\iota_1, S} \mathcal{M} \in \mathbf{MF}_{\tilde{S}}(\varphi, \nabla)$. Then we have a natural isomorphism $D^*(G') := \mathbb{D}^*(G')(R_0)[\frac{1}{p}] \cong R_0[\frac{1}{p}] \otimes_{\tilde{S}} \tilde{\mathcal{M}}$ as objects of $\mathbf{MF}_{R/R_0}(\varphi, \nabla)$ (cf., Example 4.6.1), where we view R_0 as an \tilde{S} -algebra via $\tilde{S} \xrightarrow{\delta} S \rightarrow R_0$, and the Hodge Filtration on $R_0[\frac{1}{p}] \otimes_{\tilde{S}} \tilde{\mathcal{M}}$ is given by the image of $\text{Fil}^1 \tilde{\mathcal{M}}$. Note that the following defines a φ -compatible section of the natural projection $\tilde{\mathcal{M}}[\frac{1}{p}] \rightarrow D^*(G')$:

$$(8.4.7) \quad \tilde{s} : D^*(G') \xrightarrow{s} \mathcal{M}[1/p] \xrightarrow{\iota_1} \tilde{\mathcal{M}}[1/p],$$

where s is as in Lemma 3.3.5, and ι_1 is as in Remark 8.4.5. Clearly, \tilde{s} is a unique φ -compatible section.

Lemma 8.4.8. *We use the notation as above. There exists a natural $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant isomorphism $V_p(G') \cong \tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}})[\frac{1}{p}]$, and the image of $\tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}})$ in $V_p(G')$ is same as the image of $T_{\text{cris}}^*(\mathcal{M})$ in $V_p(G')$.*

Proof. By Theorem 5.3 we have a natural \mathcal{G}_R -isomorphism $V_p(G') \cong V_{\text{cris}}^*(D^*(G'))$. Since the section \tilde{s} (8.4.7) induces an isomorphism

$$1 \otimes \tilde{s} : B_{\text{cris}}(R) \otimes_{R_0[1/p]} D^*(G') \xrightarrow{\sim} B_{\text{cris}}(R) \otimes_{\tilde{S}} \tilde{\mathcal{M}}[\frac{1}{p}],$$

one obtains a $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant isomorphism $\tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}})[\frac{1}{p}] \xrightarrow{\sim} V_{\text{cris}}^*(D^*(G')) \cong V_p(G')$.

For the second assertion, $\iota_1 : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ (cf., Remark 8.4.5) induces an $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant map $\tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}}) \rightarrow T_{\text{cris}}^*(\mathcal{M})$, which is surjective (as $\delta : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ defines a section) and respects the embeddings into $V_p(G')$. The lemma now follows. \square

8.5. Proof of Proposition 8.1. Consider $\mathcal{M} \in \mathbf{MF}_S(\varphi, \nabla)$ and $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ such that $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cong \mathcal{M}$. Set $\tilde{\mathcal{M}} := \tilde{S} \otimes_{\iota_1, S} \mathcal{M}$, as before. By Lemma 8.4.8, it suffices to construct a natural $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant injective map $\mathcal{T}^*(\mathfrak{M}) \rightarrow \tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}})$ and show that it is an isomorphism when $p > 2$.

The map $\iota_2 : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ (cf., Remark 8.4.5) induces an \mathcal{G}_{R_∞} -isomorphism

$$(8.5.1) \quad \text{Hom}_{S, \text{Fil}^1, \varphi, \varphi_1}(\mathcal{M}, A_{\text{cris}}^\nabla(R)) \cong \text{Hom}_{\tilde{S}, \text{Fil}^1, \varphi, \nabla^u}(\tilde{\mathcal{M}}, A_{\text{cris}}(R)) = \tilde{T}_{\text{cris}}^*(\tilde{\mathcal{M}}).$$

This is an isomorphism because $\iota_2(\mathcal{M}) = \tilde{\mathcal{M}}^{\nabla^u=0}$. Since we have $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} = \mathcal{M}$, we obtain a $\mathcal{G}_{\tilde{R}_\infty}$ -equivariant map

$$(8.5.2) \quad \mathcal{T}^*(\mathfrak{M}) = \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \widehat{\mathfrak{S}}^{\text{ur}}) \rightarrow \text{Hom}_{S, \text{Fil}^1, \varphi, \varphi_1}(\mathcal{M}, A_{\text{cris}}^\nabla(R)),$$

where the arrow is induced by $\varphi : \widehat{\mathfrak{S}}^{\text{ur}} \rightarrow A_{\text{cris}}^{\nabla}(R)$. This map is clearly injective. Combining (8.5.1) and (8.5.2), we obtain a $\mathcal{G}_{\widetilde{R}_{\infty}}$ -equivariant injective map $\mathcal{T}^*(\mathfrak{M}) \rightarrow \widetilde{T}_{\text{cris}}^*(\mathcal{M})$.

Let us show that the map (8.5.2) is an isomorphism when $p > 2$. We follow the same strategy as in [Kis06, Theorem 2.2.7]. It suffices to show that the following map, obtained by reducing the map (8.5.2) modulo p , is an isomorphism when $p > 2$:

$$(8.5.3) \quad \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}/(p), \widehat{\mathfrak{S}}^{\text{ur}}/(p)) \rightarrow \text{Hom}_{S, \text{Fil}^1, \varphi, \varphi_1}(\mathcal{M}/(p), A_{\text{cris}}^{\nabla}(R)/(p)).$$

Since both sides are finite-dimensional \mathbb{F}_p -vector spaces of same dimension, it suffices to prove injectivity. Note that $A_{\text{cris}}^{\nabla}(R)/(p)$ is naturally isomorphic to the divided power envelope of \overline{R}^{\flat} with respect to $\widetilde{\omega}^e$, so it easily follows that the kernel of $\varphi : \widehat{\mathfrak{S}}^{\text{ur}}/(p) \rightarrow A_{\text{cris}}^{\nabla}(R)$ is principally generated by u^e . Therefore, $x : \mathfrak{M}/(p) \rightarrow \widehat{\mathfrak{S}}^{\text{ur}}/(p)$ is in the kernel of (8.5.3) if and only if the image of x is contained in $u^e \widehat{\mathfrak{S}}^{\text{ur}}$.

Recall that $(1 \otimes \varphi_{\mathfrak{M}})(\mathfrak{M}/(p))$ contains $u^e(\mathfrak{M}/(p)) = E(u)(\mathfrak{M}/(p))$. Since x commutes with φ 's, we have

$$(8.5.4) \quad u^e \cdot x(m) = x(u^e m) = x((1 \otimes \varphi_{\mathfrak{M}})(m')) \in \varphi(u^e) \cdot \widehat{\mathfrak{S}}^{\text{ur}}/(p) = u^{pe} \cdot \widehat{\mathfrak{S}}^{\text{ur}}/(p),$$

where $m \in \mathfrak{M}/(p)$ and $m' \in \varphi^*(\mathfrak{M}/(p))$ such that $(1 \otimes \varphi_{\mathfrak{M}})(m') = u^e m$. So we have $x(\mathfrak{M}/(p)) \subseteq u^{(p-1)e} \cdot \widehat{\mathfrak{S}}^{\text{ur}}/(p)$. If $p > 2$, then we can iterate this process to get $x = 0$, which implies that the map (8.5.3) is injective. This concludes the proof of Proposition 8.1 when $p > 2$.

Remark 8.5.5. The map (8.5.2) is not necessarily an isomorphism when $p = 2$. For example, set $\mathfrak{M} := \mathfrak{M}^*(\widehat{\mathbb{G}}_m)$ and $\mathcal{M} := \mathcal{M}^*(\widehat{\mathbb{G}}_m)$. Note that we can find a \mathfrak{S} -basis $\mathbf{e} \in \mathfrak{M}$ such that $\varphi_{\mathfrak{M}}(\mathbf{e}) = E(u)\mathbf{e}$. Then one can show that the image of $\mathcal{T}^*(\mathfrak{M})$ in $T_{\text{cris}}^*(\mathcal{M})$ is $2 \cdot T_{\text{cris}}^*(\mathcal{M})$ by the same proof as [Kim12, Proposition 5.4(2)]. On the other hand, it is not difficult to show that $T_p(\widehat{\mathbb{G}}_m) = 2 \cdot T_{\text{cris}}^*(\mathcal{M})$, so we nonetheless have $T_p(\widehat{\mathbb{G}}_m) = \mathcal{T}^*(\mathfrak{M})$. We can show the same statement when G' is a multiplicative-type p -divisible group.

It remains to show that the \mathbb{Z}_p -lattices $T_p(G)$ and $\mathcal{T}^*(\mathfrak{M}^*(G))$ in $V_p(G)$ coincide when $p = 2$. Recall that there exists a (unique) φ -equivariant map $R_0 \rightarrow W(\text{Frac}(R/\varpi)^{\text{perf}})$ (cf. §3.6 (Ex6)). Set $\mathcal{O}_{K'} := W(\text{Frac}(R/\varpi)^{\text{perf}}) \otimes_W \mathcal{O}_K$, which is an p -adic discrete valuation ring with perfect residue field. As discussed in §7.4, applying the base change $R \rightarrow \mathcal{O}_{K'}$ does not change the \mathbb{Q}_p -vector space $V_0(G)$ and the \mathbb{Z}_p -lattices $T_p(G)$ and $\mathcal{T}^*(\mathfrak{M}^*(G))$. So in order to show $T_p(G) = \mathcal{T}^*(\mathfrak{M}^*(G))$, we may assume that $R = \mathcal{O}_{K'}$ with perfect residue field. But this is known already, independently in [Lau10a] and in [Kim12]. This concludes the proof of Proposition 8.1 when $p = 2$.

8.6. Full faithfulness in the $p = 2$ case. Let $p = 2$. Then we have a functor

$$\{p\text{-divisible groups}/R\}[1/p] \rightarrow \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)[1/p]; \quad G \rightsquigarrow \mathfrak{M}^*(G)[1/p]$$

by Corollary 6.3.1.

Using Proposition 8.1, we can improve Corollary 6.3.1 in the the following case:

Corollary 8.6.1. *Let $p = 2$, and assume that R satisfies the formally finite-type assumption (§2.2.2), and for some Cohen subring $W \subset R_0$ we have $E(u) \in W[u]$ (cf. §7). Then for a p -divisible group G over R , there exists a unique $\varphi(\mathfrak{S})$ -lattice $\mathfrak{M} \subset \mathbb{D}^*(G)(S)$ with $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$. Furthermore, the assignment $G \rightsquigarrow \mathfrak{M}$ defines a fully faithful functor:*

$$\mathfrak{M}^* : \{p\text{-divisible groups}/R\} \rightarrow \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla).$$

Proof. For a p -divisible group G over R , we write $\mathcal{M} := \mathbb{D}^*(G)(S)$ and let $\mathfrak{M}, \mathfrak{M}' \subset \mathcal{M}$ be $\varphi(\mathfrak{S})$ -lattices which are in $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$, which exist by Proposition 6.3. We want to show that $\mathfrak{M} = \mathfrak{M}'$. Since this assertion can be proved Zariski-locally, we may assume that R satisfies the “refined almost étaleness” assumption (§2.2.3). Note that $\mathfrak{M}[\frac{1}{p}] = \mathfrak{M}'[\frac{1}{p}]$ in $\mathcal{M}[\frac{1}{p}]$ by Lemma 6.2, and we have $T_p(G) = \mathcal{T}^*(\mathfrak{M}) = \mathcal{T}^*(\mathfrak{M}')$ inside $T_{\text{cris}}^*(\mathcal{M})$ by Proposition 8.1. Therefore we have $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}'$ inside $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}[\frac{1}{p}]$ by Proposition 7.3, which forces $\mathfrak{M} = \mathfrak{M}'$.

Now let $\mathfrak{M}^*(G) \in \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ be the unique $\varphi(\mathfrak{S})$ -lattice in $\mathcal{M} = \mathbb{D}^*(G)(S)$, and we show that the assignment $\mathfrak{M} : G \rightsquigarrow \mathfrak{M}^*(G)$ is functorial; in other words, for any morphism $f : G \rightarrow G'$ of p -divisible groups over R , the map $\mathbb{D}^*(f)(S)$ restricts to $\mathfrak{M}^*(G') \rightarrow \mathfrak{M}^*(G)$. Since this assertion can be proved Zariski-locally, we may assume that R satisfies the “refined almost étaleness” assumption (§2.2.3). By Lemma 6.2, there is a map $\alpha : \mathfrak{M}^*(G') \rightarrow \mathfrak{M}^*(G)$ in $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)$ which is the restriction of $p^n \mathbb{D}^*(f)(S)$. Then, the image of $\mathcal{T}^*(\alpha) : \mathcal{T}^*(\mathfrak{M}^*(G')) \rightarrow \mathcal{T}^*(\mathfrak{M}^*(G))$ is divisible by p^n since $\mathcal{T}^*(\alpha) = p^n T_p(f)$ by Proposition 8.1. This shows that $\frac{1}{p^n} \alpha : \mathfrak{M}^*(G') \rightarrow \mathfrak{M}^*(G)$ is well-defined by Proposition 7.3. We set $\mathfrak{M}^*(f) := \frac{1}{p^n} \alpha$, which is the restriction of $\mathbb{D}^*(f)(S)$.

Let us now show the full faithfulness of \mathfrak{M} . For $\mathfrak{M} := \mathfrak{M}^*(G)$ and $\mathfrak{M}' := \mathfrak{M}^*(G')$ where G and G' are p -divisible groups over R , consider a map $\alpha : \mathfrak{M}' \rightarrow \mathfrak{M}$ in $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$. Since we already showed that \mathfrak{M}^* is fully faithful up to isogeny (as \mathcal{M}^* is so by the proof of Theorem 3.5), there is a homomorphism $f' : G \rightarrow G'$ with $\mathfrak{M}^*(f') = p^n \alpha$ for some n . Applying the base change $R \rightarrow \widehat{R}_{(\varpi)}$ (cf. §3.6 (Ex5)), it follows from Proposition 8.1 that f' kills $G[p^n]_{R(\varpi)}$. (We can apply Proposition 8.1 for the base ring $\widehat{R}_{(\varpi)}$ as it is a p -adic discrete valuation ring.) So the kernel of $f' : G[p^n] \rightarrow G'[p^n]$ is some finite (not necessarily flat) subgroup of $G[p^n]$ containing $G[p^n]_{R(\varpi)}$, so it should be equal to $G[p^n]$. It now follows that there is a homomorphism $f : G \rightarrow G'$ such that $p^n f = f'$. Clearly, $\mathfrak{M}^*(f) = \alpha$. Faithfulness of \mathfrak{M}^* is proved very similarly. \square

9. FINITE LOCALLY FREE GROUP SCHEMES

In this section, we state some results on classifications of finite locally free³ group schemes via torsion versions of Breuil S -modules and Kisin \mathfrak{S} -modules; cf. Theorem 9.4. The proof will be given in §10.5.

Throughout this section, we let R be a p -adic ring satisfying the p -basis assumption (§2.2.1) (possibly with stronger assumptions on R , which will be specified), with the choice of (\mathfrak{S}, φ) , $E(u)$, and S as in §3.3.

9.1. Torsion filtered S -modules. The following definition is the torsion version of Breuil S -modules.

Definition 9.1.1. Let $\mathbf{MF}'_S(\varphi)^{\text{tor}}$ denote the category of tuples $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_1)$ where

- (1) \mathcal{M} is a finitely generated S -module which is killed by some power of p , and $\text{Fil}^1 \mathcal{M} \subset \mathcal{M}$ is an S -submodule containing $(\text{Fil}^1 S)\mathcal{M}$;
- (2) $\varphi_1 : \text{Fil}^1 \mathcal{M} \rightarrow \mathcal{M}$ is a φ -linear endomorphism of \mathcal{M} such that $\varphi_1(\mathcal{M})$ generates \mathcal{M} and for any $s \in \text{Fil}^1 S$ and $m \in \mathcal{M}$ we have

$$\varphi_1(sm) = \frac{\varphi_1(s)}{\varphi_1(E)} \varphi_1(Em).$$

³Note that we always assume our finite locally free group schemes are commutative. We use “finite locally free group schemes” instead of more popular terminology “finite flat group schemes” as it is essential to deal with possibly non-noetherian base rings in the proof.

Note that $\varphi_1(E) \in S^\times$. We define $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\varphi_{\mathcal{M}}(m) := \varphi_1(E)^{-1} \varphi_1(Em)$$

for any $m \in \mathcal{M}$. Then we have $\varphi_{\mathcal{M}} = p\varphi_1$ on $\text{Fil}^1 \mathcal{M}$.

Let $\mathbf{MF}'_S(\varphi, \nabla)^{\text{tor}}$ denote the category of tuples $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_1, \nabla_{\mathcal{M}})$ where $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_1) \in \mathbf{MF}'_S(\varphi)^{\text{tor}}$ and $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathfrak{S}} \widehat{\Omega}_{\mathfrak{S}}$ is a quasi-nilpotent integrable connection which commutes with $\varphi_{\mathcal{M}}$.

Let $\mathbf{MF}'_S^{\text{Br}}(\varphi, \nabla^0)^{\text{tor}}$ denote the category of tuples $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_1, \nabla_{\mathcal{M}_0})$ where $(\mathcal{M}, \text{Fil}^1 \mathcal{M}, \varphi_1) \in \mathbf{MF}'_S(\varphi)^{\text{tor}}$ and $\nabla_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_0 \otimes_{R_0} \widehat{\Omega}_{R_0}$ is a quasi-nilpotent integrable connection on $\mathcal{M}_0 := R_0 \otimes_S \mathcal{M}$ which commutes with $\varphi_{\mathcal{M}_0}$.

The proof of Lemma 3.3.4 shows that the natural functor $\mathbf{MF}'_S(\varphi, \nabla)^{\text{tor}} \rightarrow \mathbf{MF}'_S^{\text{Br}}(\varphi, \nabla^0)^{\text{tor}}$ (defined in the same way as in Remark 3.3.3) is fully faithful.

Not all objects $\mathcal{M} \in \mathbf{MF}'_S(\varphi, \nabla)^{\text{tor}}$ are expected to come from a finite locally free group schemes (by the recipe as in the proof of Proposition 9.3), as the definition of $\mathbf{MF}'_S(\varphi, \nabla)^{\text{tor}}$ does not prevent \mathcal{M} from being a R_0 -module. For the purpose of studying the p -power order finite locally free group schemes, it is much more convenient to use torsion Kisin modules, which we introduce in Definition 9.2.1.

9.2. Classification via torsion Kisin modules. The following definition is the torsion version of Kisin \mathfrak{S} -modules.

Definition 9.2.1. Let $\mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$ denote the category of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where

- (1) \mathfrak{M} is a finitely presented \mathfrak{S} -module killed by some power of p , and of \mathfrak{S} -projective dimension ≤ 1 ;
- (2) $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a φ -linear map such that $\text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by E .

Let $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}}$ denote the category of pairs $(\mathfrak{M}, \nabla_{\mathcal{M}})$ where $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$ and $\nabla_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathfrak{S}} \widehat{\Omega}_{\mathfrak{S}}$ on $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ is a quasi-nilpotent integrable connection which commutes with $\varphi_{\mathcal{M}} := \varphi_S \otimes \varphi_{\mathfrak{M}}$.

Let $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)^{\text{tor}}$ denote the category of pairs $(\mathfrak{M}, \nabla_{\mathcal{M}_0})$ where $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$ and $\nabla_{\mathcal{M}_0} : \mathcal{M}_0 \rightarrow \mathcal{M}_0 \otimes_{R_0} \widehat{\Omega}_{R_0}$ is a quasi-nilpotent integrable connection on $\mathcal{M}_0 := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ which commutes with $\varphi_{\mathcal{M}_0}$. We call an object in $\mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)^{\text{tor}}$ a *torsion Kisin \mathfrak{S} -module*.

We let $(\mathbf{Mod FI})_{\mathfrak{S}}(\varphi, \nabla)$, $(\mathbf{Mod FI})_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$, and $(\mathbf{Mod FI})_{\mathfrak{S}}(\varphi)$ denote the full subcategories of \mathfrak{M} such that $\mathfrak{M} \cong \bigoplus \mathfrak{M}_i$ as a \mathfrak{S} -module where \mathfrak{M}_i are projective over $\mathfrak{S}/(p^{n_i})$.

Remark 9.2.2. As before, one can define a natural functor $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}} \rightarrow \mathbf{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)^{\text{tor}}$ by the same way as in Remark 3.3.3, and the same proof of Lemma 3.3.4 shows that this is fully faithful. We will see later that this functor is an equivalence of categories if R satisfies the formally finite-type assumption (§2.2.2) for any p (including $p = 2$) – indeed, we show that both categories are naturally anti-equivalent to the category of p -power order finite locally free group schemes over R ; cf., Theorem 9.4.

Let \mathfrak{M} be an object of either $\mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$ or $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}}$, and set $\mathcal{M} := S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. One can define $\text{Fil}^1 \mathcal{M} \subset \mathcal{M}$ and $\varphi_1 : \text{Fil}^1 \mathcal{M} \rightarrow \mathcal{M}$ using the same formula for (quasi-)Kisin modules (cf., (6.1.3), (6.1.4)). Clearly, \mathcal{M} with this structure is an object in $\mathbf{MF}'_S(\varphi)^{\text{tor}}$ (or $\mathbf{MF}'_S(\varphi, \nabla)^{\text{tor}}$ if $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}}$).

Remark 9.2.3. For $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$ there exists a unique (injective) \mathfrak{S} -linear map $\psi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \varphi^* \mathfrak{M}$ such that $\psi_{\mathfrak{M}} \circ (1 \otimes \varphi_{\mathfrak{M}}) = E(u) \text{id}_{\varphi^* \mathfrak{M}}$ and $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi_{\mathfrak{M}} = E(u) \text{id}_{\mathfrak{M}}$. Indeed, the multiplications by $E(u)$ on \mathfrak{M} and $\varphi^* \mathfrak{M}$ are injective, and the image of $1 \otimes \varphi_{\mathfrak{M}}$ contains $E(u) \mathfrak{M}$.

Similarly, if $\mathcal{M} = S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ for some $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$, then we set

$$\psi_{\mathcal{M}} = \varphi_1(E(u))^{-1}(1 \otimes \psi_{\mathfrak{M}}) : \mathcal{M} \rightarrow \varphi^* \mathcal{M}.$$

Then we have $\psi_{\mathcal{M}} \circ (1 \otimes \varphi_{\mathcal{M}}) = p \text{id}_{\varphi^* \mathcal{M}}$ and $(1 \otimes \varphi_{\mathcal{M}}) \circ \psi_{\mathcal{M}} = p \text{id}_{\mathcal{M}}$.

Proposition 9.3. *Assume that the functor \mathfrak{M}^* on p -divisible groups over R is fully faithful. (cf., Corollaries 6.3.1 and 8.6.1). Then there exists an exact fully faithful functor \mathfrak{M}^* from the category of p -power order finite locally free group schemes over R to $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}}$ with the following properties:*

- (1) *For any p -power order finite locally free group scheme H over $\text{Spf}(R, (\varpi))$, there exists a natural horizontal isomorphism $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^*(H) \cong \mathbb{D}^*(H)(S)$ which matches the quotient $e_H^* \Omega_{H/R}$ of $\mathbb{D}^*(H)(S)$ with*

$$\text{coker}(1 \otimes \varphi_{\mathfrak{M}^*(H)}) \cong (S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^*(H)) / \text{Fil}^1(S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}^*(H)),$$

and (F, V) on $\mathbb{D}^(H)(S)$ with $(1 \otimes \varphi, \psi)$ on $\mathcal{M}^*(H)$ (cf., Definition 9.1.1, Remark 9.2.3).*

- (2) *If $H = \ker[d : G^0 \rightarrow G^1]$ for some isogeny d , then there exists a natural isomorphism $\mathfrak{M}^*(H) \cong \text{coker}[\mathfrak{M}^*(d)]$.*
- (3) *The formation of \mathfrak{M}^* commutes with a base change by $R \rightarrow R'$ that lifts to some φ -equivariant map $\mathfrak{S} \rightarrow \mathfrak{S}'$, where $\mathfrak{S}' = R'_0[[u]]$ and φ are chosen as in §2.2.1 and §3.3; cf. §3.6.*

As a preparation for the proof, we recall the following theorem of Raynaud [BBM82, Théorème 3.1.1]:

Theorem 9.3.1. *Let H be a commutative finite locally free group scheme over some scheme X . Then there exists a Zariski covering $\{U_\alpha\}_\alpha$ of X and an abelian scheme \mathcal{A}_α over each U_α such that H_{U_α} can be embedded in \mathcal{A}_α as a closed subgroup.*

Now assume that R is p -adic, and let H be a finite locally free group scheme over R of p -power order. (Note that there is no difference in viewing H either over $\text{Spf}(R, (p))$ or over $\text{Spec } R$.) We apply Raynaud's theorem for H over $\text{Spec } R$ to obtain an open affine cover $\{U_\alpha = \text{Spec } R_\alpha\}$ and an abelian scheme \mathcal{A}_α over each U_α . So we obtain a p -divisible group $G_\alpha := \mathcal{A}_\alpha[p^\infty]$ over the p -adic completion \widehat{R}_α of R_α into which $H_{\widehat{R}_\alpha}$ can be embedded. (Note that $\{\text{Spf}(\widehat{R}_\alpha, (p))\}$ is a Zariski open covering of $\text{Spf}(R, (p))$.) The same discussion holds if we work with J_R -adic topology if R is J_R -adically separated and complete.

We need another lemma for the proof of Proposition 9.3. Let us consider a triangulated full subcategory Isog_R^\bullet of the derived category of fppf sheaves of abelian groups over R , generated by two-term complexes $G^\bullet := [G^0 \xrightarrow{d} G^1]$ at degree $[0, 1]$ such that G^i are p -divisible groups and d is an isogeny. Note that the only non-zero cohomology of $G^\bullet \in \text{Isog}_R^\bullet$ is $H^0(G^\bullet)$, which is a finite locally free group scheme over R with p -power order.

Lemma 9.3.2. *The functor $H^0 : [G^0 \xrightarrow{d} G^1] \rightsquigarrow \ker(d)$ from Isog_R^\bullet to the category of p -power order finite locally free group schemes over R is fully faithful.*

Proof. Let $G^\bullet := [G^0 \xrightarrow{d} G^1]$ and $G'^\bullet := [G'^0 \xrightarrow{d'} G'^1]$ be objects in Isog_R^\bullet , and set $H := H^0(G^\bullet)$ and $H' := H^0(G'^\bullet)$.

Let $f : G^\bullet \rightarrow G'^\bullet$ be a morphism such that $H^0(f)$ is a zero map, and we seek to show that f is a zero map. This can be checked after composing with any isomorphism, so we may assume, without loss of generality, that f is given by morphisms $f^i : G^i \rightarrow G'^i$ of p -divisible groups. If $H^0(f) : H \rightarrow H'$ is a zero map, then f^0 factors through G^1 giving a null homotopy for f . Now it easily follows that H^0 is faithful.

If $f : H \rightarrow H'$ be any homomorphism, then we can factor f into $H \xrightarrow{(1, f)} H \times H' \rightarrow H'$. Since $H \times H' := \ker[(d, d') : G^0 \times G'^0 \rightarrow G^1 \times G'^1]$, we may assume that f is a closed immersion by replacing f by $(1, f)$ if necessary. Set $G''^\bullet := [G'^0 \rightarrow G'^0/f(H)]$. Then we have $H = H^0[G''^\bullet]$ and f comes from $g : G''^\bullet \rightarrow G'^\bullet$ where $g^0 = \text{id}$ and g^1 is the natural projection. \square

Proof of Proposition 9.3/Construction. Let us first construct \mathfrak{M}^* . For any isogeny $d : G^0 \rightarrow G^1$ of p -divisible groups over R , $\mathfrak{M} := \text{coker}[\mathfrak{M}^*(d) : \mathfrak{M}^*(G^1) \rightarrow \mathfrak{M}^*(G^0)]$ is clearly a torsion Kisin \mathfrak{S} -module. By Corollary 6.3.1 (Corollary 8.6.1 when $p = 2$) and Lemma 9.3.2, $H^0 \circ \mathfrak{M}^*$ induces a fully faithful functor $\text{Isog}_R^\bullet \rightarrow \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}}$ which commutes with étale base change $\text{Spf}(R', (\varpi)) \rightarrow \text{Spf}(R, (\varpi))$. If $H := \ker(d)$, then we set $\mathfrak{M}^*(H) := \text{coker}[\mathfrak{M}^*(d)]$.

Let H be a finite locally free group scheme over R of p -power order. Then by Raynaud's theorem (Theorem 9.3.1 and the subsequent discussion) there exists a Zariski covering $\{\text{Spf}(R_\alpha, (p))\}_\alpha$ of $\text{Spf}(R, (p))$ and p -divisible groups G_α over R_α such that $H_{R_\alpha} \subset G_\alpha$.

Recall that for each R_α there is a natural \mathfrak{S} -algebra \mathfrak{S}_α together with a unique choice of $\varphi_{\mathfrak{S}_\alpha}$ over $\varphi_{\mathfrak{S}}$, such that $\mathfrak{S}_\alpha/(E) = R_\alpha$. (Cf. Lemma 2.3.5, Remark 6.3.5.) Note also that $\{\text{Spf}(\mathfrak{S}_\alpha, (p, E))\}$ is a Zariski covering of $\text{Spf}(\mathfrak{S}, (p, E))$. Now, $\mathfrak{M}^*(H_{R_\alpha}) \in \mathbf{Mod}_{\mathfrak{S}_\alpha}(\varphi, \nabla)^{\text{tor}}$ makes sense, and the natural glueing datum on $\{H_{R_\alpha}\}$ induces a glueing datum on $\{\mathfrak{M}^*(H_{R_\alpha})\}$. We define $\mathfrak{M}^*(H) \in \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}}$ by glueing $\{\mathfrak{M}^*(H_{R_\alpha})\}$. \square

For $\mathfrak{M} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$, we say \mathfrak{M} is φ -nilpotent if $\varphi_{\mathfrak{M}}^n = 0$ for some n , and similarly define ψ -nilpotent objects. (Cf. Definition 3.2.6.) We can improve Proposition 9.3 in the following case:

Theorem 9.4. *Assume that R satisfies the formally finite-type assumption (§2.2.2). If $p > 2$ then the functor $(\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}(\varphi, \nabla) \rightarrow (\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$, defined in Remark 9.2.2, is an equivalence of categories, and the following functors is an anti-equivalence of categories:*

$$\left\{ \begin{array}{l} p\text{-power order finite locally} \\ \text{free group schemes } H \text{ over } R \\ \text{such that } H[p^n] \text{ is locally free } \forall n \end{array} \right\} \xrightarrow{\mathfrak{M}^*} (\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}(\varphi, \nabla) \xrightarrow{\sim} (\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0).$$

Let $R_0 = W(k)[[T_1, \dots, T_d]]$ with perfect field k of characteristic $p > 2$, and define $\varphi : R_0 \rightarrow R_0$ by extending the Witt vector Frobenius by $\varphi(T_i) = T_i^p$. Set $R = R_0[[u]]/E(u)$. Then \mathfrak{M}^* composed with the forgetful functor $\mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\text{tor}} \rightarrow \mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$ is an anti-equivalence of categories.

Allowing any prime p , each of the following functors is an anti-equivalence of categories:

$$\left\{ \begin{array}{l} \text{infinitesimal finite locally} \\ \text{free group schemes over } R \end{array} \right\} \xrightarrow{\mathfrak{M}^*} \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\varphi \text{ nilp}} \xrightarrow{\sim} \mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\varphi \text{ nilp}};$$

$$\left\{ \begin{array}{l} \text{unipotent finite locally} \\ \text{free group schemes over } R \end{array} \right\} \xrightarrow{\mathfrak{M}^*} \mathbf{Mod}_{\mathfrak{S}}(\varphi, \nabla)^{\psi \text{ nilp}} \xrightarrow{\sim} \mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\psi \text{ nilp}},$$

where \mathfrak{M}^* is as in Proposition 9.3, the other arrows are suitable forgetful functors, and the superscripts $\varphi \text{ nilp}$ and $\psi \text{ nilp}$ respectively denote the full subcategory of φ -nilpotent and ψ -nilpotent objects.

We prove this theorem later in §10.5.

Remark 9.4.1. Let us review previous results on Theorem 9.4. When R is a p -adic discrete valuation ring with perfect residue field and $p > 2$, then Theorem 9.4 is

a theorem of Kisin [Kis06, Theorem 2.3.5]. (Note that in this case, $\widehat{\Omega}_{R_0} = 0$ so ∇^0 is forced to be zero.) The strategy of Kisin to deduce Theorem 9.4 from the classification of p -divisible groups can be applied in more general setting. Most general result of this type is the theorem of Eike Lau [Lau10c, Corollary 7.7, Proposition 8.1] in the case when R is a complete regular local ring with perfect residue field. (Note that his result does not require $R/(\varpi)$ to be a formal power series.)

Unfortunately, there are difficulties in using the strategy of Kisin to deduce Theorem 9.4 from Corollary 6.3.1. The most serious problem is the presence of connections ∇^0 , which cannot be removed in general (unless it is asserted in Theorem 9.4). In order to handle the connections, we use the theory of moduli of connections [Vas12, §3]. We review (and slightly generalise) the theory in §10.

9.5. Galois representations. We continue to assume that R satisfies the formally finite-type assumption (§2.2.2). Let H be a p -power order finite locally free group scheme over R . Then $H(\overline{R})$ is a finite torsion \mathbb{Z}_p -module equipped with continuous (i.e., discrete) action of \mathcal{G}_R . On the other hand, under the additional assumption (stated in Proposition 9.5.1) we obtain a torsion $\mathcal{G}_{\overline{R}_\infty}$ -representation $\mathcal{T}^*(\mathfrak{M}^*(H))$ since $\mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}^*(H)$ is an étale $(\varphi, \mathcal{O}_\mathcal{E})$ -module.

Proposition 9.5.1. *Assume that R satisfies the formally finite-type assumption (§2.2.2), and for some Cohen subring $W \subset R_0$ we have $E(u) \in W[u]$ (cf. §7). Then there exists a natural $\mathcal{G}_{\overline{R}_\infty}$ -equivariant isomorphism $H(\overline{R}) \cong \mathcal{T}^*(\mathfrak{M}^*(H))$ with the following properties:*

- (1) *If $H = \ker[d : G^0 \rightarrow G^1]$ for some isogeny d , then the isomorphism $H(\overline{R}) \cong \mathcal{T}^*(\mathfrak{M}^*(H))$ is compatible with the isomorphism $T_p(G^i) \cong \mathcal{T}^*(\mathfrak{M}^*(G^i))$ as in Corollary 8.2.*
- (2) *The formation of the isomorphism $H(\overline{R}) \cong \mathcal{T}^*(\mathfrak{M}^*(H))$ commutes (in the obvious sense) with a base change by $R \rightarrow R'$ that lifts to some φ -equivariant map $\mathfrak{S} \rightarrow \mathfrak{S}'$, where $\mathfrak{S}' = R'_0[[u]]$ and φ are chosen as in §2.2.1 and §3.3; cf. §3.6.*

Proof. If $H = \ker[d : G^0 \rightarrow G^1]$, then the $\mathcal{G}_{\overline{R}_\infty}$ -isomorphism $T_p(G^i) \cong \mathcal{T}^*(\mathfrak{M}^*(G^i))$ as in Corollary 8.2 induces a $\mathcal{G}_{\overline{R}_\infty}$ -isomorphism $H(\overline{R}) \cong \mathcal{T}^*(\mathfrak{M}^*(H))$ with all the desired properties. (Here we use the exact sequence (7.3.3).) But since the formation of the isomorphism commutes with Zariski localisation, it suffices to define the isomorphism on some Zariski covering where H can be embedded in some p -divisible group. \square

10. MODULI OF CONNECTIONS

In this section we study the role of connections using Vasu's construction of "moduli of connections" [Vas12, §3]. This construction will play an important role in the classification of finite locally free group schemes in §9. The main results in this section are Corollaries 10.3.1 and 10.4.1.

Throughout this section, we assume that R satisfies the formally finite-type assumption (§2.2.2) and choose (R_0, φ) as usual. Set $R_{0,k} := R_0/(p) (\cong R/(\varpi))$. Let J_R and $J_0 (= J_{R_0})$ respectively denote the Jacobson radicals of R and R_0 . Note that R and R_0 are respectively J -adic and J_0 -adic.

10.1. Preliminaries. Let $I \subset R$ be a J_R -adically closed ideal such that $(\varpi) \subseteq I \subseteq J_R$, and set $I_0 := \ker(R_0 \twoheadrightarrow R/I)$. (In practice, I is either (ϖ) or J_R .) Let A be a I -adic formally étale R -algebra such that A/IA is finitely generated over R/I (i.e., $\text{Spec } A/I^n A \rightarrow \text{Spec } R/I^n$ is étale for any n). Then applying §3.6 (Ex1), there exists a unique I_0 -adic formally étale lift $R_0 \rightarrow A_0$ of $R/I \rightarrow A/IA$, and A_0 can naturally be viewed as a R_0 -subalgebra of A . We set $\mathfrak{S}_A := A_0[[u]]$, and define S_A , etc.,

correspondingly. More generally, if A is “ind-étale” (i.e., A is a I -adic completion of $\varinjlim_i A^{(i)}$ for some I -adic formally étale R -algebras $A^{(i)}$) then we define A_0 by I_0 -adically completing $\varinjlim A_0^{(i)}$, where $A_0^{(i)}$ is the R_0 -lift of $A^{(i)}/IA^{(i)}$, and define \mathfrak{S}_A, S_A , etc., accordingly. Note that A_0 is flat over W and $A/(\varpi)$ locally admits a finite p -basis, so it satisfies the requirement for R_0 as in §2.2.1 and A satisfies the p -basis assumption (§2.2.1) using this A_0 and $E(u) \in R_0[u] \subset A_0[u]$.

10.2. Review of the construction of moduli of connections. For any $\mathcal{M} \in \mathbf{MF}_S(\varphi)$ (respectively, $\mathcal{M}_0 \in \mathbf{MF}_{R_0}(\varphi)$), we have $S_A \otimes_S \mathcal{M} \in \mathbf{MF}_{S_A}(\varphi)$ (respectively, $A_0 \otimes_{R_0} \mathcal{M} \in \mathbf{MF}_{A_0}(\varphi)$).

Let us recall (and slightly extend) Vasiu’s construction of moduli of connections [Vas12, Theorem 3.2]. Let $\mathcal{M}_0, \mathcal{M}'_0 \in \mathbf{MF}_{R_0}(\varphi)$ and a morphism $f : \mathcal{M}_0 \rightarrow \mathcal{M}'_0$ be given. Fix an R_0 -direct factor $(\mathcal{M}_0)^1 \subseteq \mathcal{M}_0$ which lifts $\mathrm{Fil}^1 \mathcal{M}_0/p\mathcal{M}_0 \subseteq \mathcal{M}_0/p\mathcal{M}_0$, and similarly choose $(\mathcal{M}'_0)^1 \subseteq \mathcal{M}'_0$. Set

$$\widetilde{\mathcal{M}}_0 := \varphi^* \left(\mathcal{M}_0 + \frac{1}{p}(\mathcal{M}_0)^1 \right) \subset \mathcal{M}_0[\frac{1}{p}].$$

Note that $1 \otimes \varphi_{\mathcal{M}_0}$ induces an R_0 -linear isomorphism $\widetilde{\mathcal{M}}_0 \xrightarrow{\sim} \mathcal{M}_0$, and one can recover $\mathcal{M}_0 \in \mathbf{MF}_{R_0}(\varphi)$ from $(\mathcal{M}_0, (\mathcal{M}_0)^1, \widetilde{\mathcal{M}}_0 \xrightarrow[1 \otimes \varphi_{\mathcal{M}_0}]{\sim} \mathcal{M}_0)$. We similarly define $\widetilde{\mathcal{M}}'_0$ so that f induces $\widetilde{\mathcal{M}}_0 \rightarrow \widetilde{\mathcal{M}}'_0$.

By passing to a Zariski covering of $\mathrm{Spf}(R_0, J_0)$, we may assume that $(\mathcal{M}_0)^1, \mathcal{M}_0/(\mathcal{M}_0)^1, (\mathcal{M}'_0)^1, \mathcal{M}'_0/(\mathcal{M}'_0)^1$, and $\widehat{\Omega}_{R_0}$ are all free over R_0 . Let us fix a R_0 -basis of $\widehat{\Omega}_{R_0}$ and an R_0 -basis of \mathcal{M}_0 adapted to the direct factor $(\mathcal{M}_0)^1$, and same for \mathcal{M}'_0 .

Consider a functor \mathcal{Q}_n which associates to any étale map $\mathrm{Spf}(A_0, (p)) \rightarrow \mathrm{Spf}(R_0, (p))$ the set of (naturally defined) equivalence classes of additive morphisms

$$\nabla_{A_0, n} : A_0/(p^n) \otimes_{R_0} \mathcal{M}_0 \rightarrow A_0/(p^n) \otimes_{R_0} \mathcal{M}'_0 \otimes_{R_0} \widehat{\Omega}_{R_0}$$

which satisfies the Leibnitz rule (i.e., $\nabla_{A_0, n}(ax) = a\nabla_{A_0, n}(x) + x \otimes da$ for any $a \in A_0/(p^n)$ and $x \in \mathcal{M}_0$, where we identify $\widehat{\Omega}_{A_0} \cong A_0 \widehat{\otimes}_{R_0} \widehat{\Omega}_{R_0}$ by étaleness) and makes the following diagram commute:

$$(10.2.1) \quad \begin{array}{ccc} A_0/(p^n) \otimes_{R_0} \widetilde{\mathcal{M}}_0 & \xrightarrow{\varphi^*(\nabla_{A_0, n})} & A_0/(p^n) \otimes_{R_0} \widetilde{\mathcal{M}}'_0 \otimes_{R_0} \widehat{\Omega}_{R_0} \\ \downarrow 1 \otimes \varphi_{\mathcal{M}_0} & & \downarrow (1 \otimes \varphi_{\mathcal{M}'_0}) \otimes \mathrm{id}_{\widehat{\Omega}_{R_0}} \\ A_0/(p^n) \otimes_{R_0} \mathcal{M}_0 & \xrightarrow{\nabla_{A_0, n}} & A_0/(p^n) \otimes_{R_0} \mathcal{M}'_0 \otimes_{R_0} \widehat{\Omega}_{R_0} \end{array}$$

Here we define $\varphi^*(\nabla_{A_0, n})$ by choosing an arbitrary lift of $\nabla_{A_0, n}$ over $A_0/(p^{n+1})$, and one can check without difficulty that $\varphi^*(\nabla_{A_0, n})$ does not depend on the choice of lift. (Cf. equation (9) in [Vas12, §3.1.1].)

If $\mathcal{M}_0 = \mathcal{M}'_0$ then a connection $\nabla_{A_0, n}$ on $A_0/(p^n) \otimes_{R_0} \mathcal{M}_0$ commutes with $\varphi_{A_0} \otimes \varphi_{\mathcal{M}_0}$ if and only it satisfies (10.2.1) modulo any p^n . More generally, if $\mathcal{M}_0, \mathcal{M}'_0 \in \mathbf{MF}_{R_0}(\varphi, \nabla)$ and $f : \mathcal{M}_0 \rightarrow \mathcal{M}'_0$ respects φ and takes $(\mathcal{M}_0)^1$ into $(\mathcal{M}'_0)^1$, then both $\nabla_{\mathcal{M}'_0} \circ f$ and $(f \otimes \mathrm{id}) \circ \nabla_{\mathcal{M}_0}$ satisfy (10.2.1) modulo any p^n .

We claim that the functor \mathcal{Q}_n is representable by a p -adic formally étale R_0 -algebra (again denoted by \mathcal{Q}_n). Fixing R_0 -bases as above, the condition for $\nabla_{A_0, 1}$ to satisfy (10.2.1) for any A_0 we consider is given by a system of equations of the form $\underline{x} = B\varphi(\underline{x}) + C_0$ for some matrices B, C_0 with entries in $R_{0, k} = R_0/(p)$, where the variables x are the “matrix entries” of $\nabla_{A_0, 1}$ with respect to the fixed bases (Cf. the proof of [Vas12, Theorem 3.2].) Such a system of equations is an example of *Artin-Schreier system of equation* (defined in [Vas12, §2.4]), and defines a faithfully

flat étale algebra $\mathcal{Q}_{1,k}$ over $R_{0,k}$ by [Vas12, Theorem 2.4.1], and we take \mathcal{Q}_1 to be the unique p -adic formally étale R_0 -algebra lifting $\mathcal{Q}_{1,k}$.

Now, assume that \mathcal{Q}_n is defined, and let $\nabla_{\mathcal{Q}_n,n}$ denote the universal object. Pick an arbitrary lift $\tilde{\nabla}_{\mathcal{Q}_n,n+1} : \mathcal{Q}_n/(p^{n+1}) \otimes_{R_0} \mathcal{M}_0 \rightarrow \mathcal{Q}_n/(p^{n+1}) \otimes_{R_0} \mathcal{M}_0 \otimes_{R_0} \hat{\Omega}_{R_0}$ of $\nabla_{\mathcal{Q}_n,n}$. For any étale map $j : \mathrm{Spf}(A_0, (p)) \rightarrow \mathrm{Spf}(\mathcal{Q}_n, (p))$, the condition for some $\nabla_{A_0,n+1}$ to satisfy (10.2.1) is given an system of equations of the form $\underline{x} = B\varphi(\underline{x}) + C_n$ over $\mathcal{Q}_n/(p)$, where the variables are the matrix entries of $\delta_{A_0,n+1}$ where $\nabla_{A_0,n+1} = j \otimes \tilde{\nabla}_{\mathcal{Q}_n,n+1} + p^n \delta_{A_0,n+1}$. (See the proof of [Vas12, Theorem 3.2] for the details.) We take \mathcal{Q}_{n+1} to be the unique p -adic formally étale \mathcal{Q}_n -algebra lifting the faithfully flat étale $\mathcal{Q}_{n,k}$ -algebra $\mathcal{Q}_{n+1,k}$ defined by these equations. The system of étale algebras $\{\mathcal{Q}_{n,k}\}$ is an example of *Artin-Schreier tower* defined in [Vas12, §2.4].

Definition 10.2.2. The system $\{\mathcal{Q}_n\}$ constructed above is called the *moduli of connections* for \mathcal{M}_0 and \mathcal{M}'_0 .

The following lemma states that any “connected component” of $\mathrm{Spec} \mathcal{Q}_\infty/(p)$ enjoys nice properties, where $\mathcal{Q}_\infty := \varinjlim_n \mathcal{Q}_n$.

Lemma 10.2.3. *Let $\{\mathcal{Q}_n\}$ be the p -adic R_0 -formally étale lift of some “Artin-Schreier tower” over $R_{0,k} := R_0/(p)$ in the sense of [Vas12, §2.4]; for example, $\{\mathcal{Q}_n\}$ can be a “moduli of connections”. Let $\hat{\mathcal{Q}}_\infty$ denote the p -adic completion of $\mathcal{Q}_\infty := \varinjlim_n \mathcal{Q}_n$, and choose a direct summand A_0 of $\hat{\mathcal{Q}}_\infty$ such that $\mathrm{Spec} A_0$ has finitely many connected components. Set $A_{0,k} := A_0/(p)$. Then the following properties hold:*

- (1) *Let $\mathfrak{p} \subset A_0$ be a prime ideal containing p , and set $\bar{\mathfrak{p}} := \mathfrak{p}/(p) \subset A_{0,k}$. Then the localisation $(A_{0,k})_{\bar{\mathfrak{p}}}$ is regular and excellent,⁴ and $A_{0,k}$ locally admit a finite p -basis.*
- (2) *The image of any open subset of $\mathrm{Spec} A_{0,k}$ in $\mathrm{Spec} R_{0,k}$ is open.*

Proof. Note that $R_0/(p)$ is regular and excellent by the theory of excellence; cf, [Val75, Theorem 4], [dJ95, Lemma 1.3.3], and [Mat80, §33.I, Theorem 79]. Set $\bar{\mathfrak{q}} := \bar{\mathfrak{p}} \cap R_{0,k}$. Since $\mathrm{Spec}(A_{0,k})_{\bar{\mathfrak{p}}}$ is an projective limit of étale neighbourhoods of $\bar{\mathfrak{q}} \in \mathrm{Spec}(R_{0,k})_{\bar{\mathfrak{q}}}$, we obtain a (faithfully) flat local morphism $(A_{0,k})_{\bar{\mathfrak{p}}} \rightarrow (R_{0,k})_{\bar{\mathfrak{q}}}^{\mathrm{sh}}$, where $(R_{0,k})_{\bar{\mathfrak{q}}}^{\mathrm{sh}}$ is a strict henselisation at $\bar{\mathfrak{q}}$. Now it follows that $(A_{0,k})_{\bar{\mathfrak{p}}}$ satisfies ascending chain condition. (Recall that the strict henselisation of a noetherian local ring is noetherian [EGA, IV₄, Proposition 18.8.8(iv)].)

Since the strict henselisation of $(A_{0,k})_{\bar{\mathfrak{p}}}$ is naturally isomorphic to $(R_{0,k})_{\bar{\mathfrak{q}}}^{\mathrm{sh}}$, $A_{0,k}$ is regular by [EGA, IV₄, Corollaire 18.8.13]. Excellence of $A_{0,k}$ follows from [Gre76, Corollary 5.6(iv)] since $R_{0,k}$ is excellent and normal.

Recall that $R_{0,k}$ locally admits a finite p -basis by [dJ95, Lemmas 1.1.3, 1.3.3]. So $A_{0,k}$ locally admits a finite p -basis since it is a direct limit of étale $R_{0,k}$ -algebras. This proves (1).

The assertion (2) is a consequence of [Vas12, Theorem 2.4.1(c)], which asserts that there exists a “stratification” of $\mathrm{Spec} R_{0,k}$ such that over each stratum $\mathrm{Spec} \mathcal{Q}_{n,k}$ is finite étale cover. Therefore, the map $\mathrm{Spec} A_{0,k} \rightarrow \mathrm{Spec} R_{0,k}$ is open and closed over each stratum (unless the fibre at the stratum is empty) by [Mat86, Theorem 9.6], from which (2) follows. \square

Let us specialise to the moduli of connections $\{\mathcal{Q}_n\}$ for $\mathcal{M}_0 = \mathcal{M}'_0$. Under the notation as in Lemma 10.2.3, let $\hat{\mathcal{Q}}_\infty$ denote the p -adic completion of $\mathcal{Q}_\infty := \varinjlim_n \mathcal{Q}_n$, and choose a quotient A_0 of $\hat{\mathcal{Q}}_\infty$ which is p -adic and formally étale over R_0 such that $A_{0,k} := A_0/(p)$ is constructed as in Lemma 10.2.3.

⁴Note that this does not necessarily imply that $A_{0,k}$ is noetherian. The author does not know whether or when to expect that $A_{0,k}$ is noetherian.

Proposition 10.2.4. *In the above setting, $A_0 \otimes_{R_0} \mathcal{M}_0$ together with the universal connection is an object in $\mathbf{MF}_{A_0}(\varphi, \nabla)$.*

Proof. The universal connection is integrable and topologically quasi-nilpotent, which is shown in the last three paragraphs in the proof of [Vas12, Theorem 3.2]. \square

10.3. Special cases. There are some special cases where the moduli of connection $\{\mathcal{Q}_\infty\}$ turns out to be very close to R_0 . We begin with some definitions:

Let $\{\mathcal{Q}_n\}$ be the moduli of connections for $\mathcal{M}_0, \mathcal{M}'_0 \in \mathbf{MF}_{R_0}(\varphi)$, and consider one of the following cases.

Formal/Unipotent case: \mathcal{M}_0 and \mathcal{M}'_0 are either both φ -nilpotent or both ψ -nilpotent (in the sense of Definition 3.2.6).

Local case: Assume that k is perfect, $R_0 = W(k)[[T_1, \dots, T_d]]$, and φ satisfies $d\varphi(T_i)/p \in \mathfrak{m}_{R_0} \widehat{\Omega}_{R_0}$ where \mathfrak{m}_{R_0} is the maximal ideal (e.g., $\varphi(T_i) = T_i^p$). We allow \mathcal{M}_0 and \mathcal{M}'_0 to be arbitrary.

The proof of [Vas12, Theorem 3.3.1] shows that in the above cases there exists a *unique* geometric point of $\mathrm{Spec} \mathcal{Q}_n/(p)$ over any geometric closed point of $\mathrm{Spec} R_{0,k}$. Set $J_0 := \ker(R_0 \twoheadrightarrow R_{\mathrm{red}})$ as before. From this, one can check that there exists a unique connected component $\mathrm{Spf}(A_0^{(n)}, (p)) \subset \mathrm{Spf}(\mathcal{Q}_n, (p))$ for each n which intersects with the fibre over some closed point in $\mathrm{Spec} R_{0,k}$, and furthermore we have $R_0 \xrightarrow{\sim} A_0^{(n)}$ for all n . (Indeed, all closed points lie in $\mathrm{Spec} R/J_R$, so if we set $J_0 := \ker(R_0 \twoheadrightarrow R_{\mathrm{red}})$ then $R_0/J_0 \rightarrow \mathcal{Q}_n/J_0 \mathcal{Q}_n$ is an isomorphism for all n because it is étale and radicial covering. Now exploit the J_0 -adic completeness of R_0 .) In particular, it follows that for any $\mathcal{M}_0, \mathcal{M}'_0 \in \mathbf{MF}_{R_0}(\varphi)$ in one of the above cases, one can uniquely define a connection on each of \mathcal{M}_0 and \mathcal{M}'_0 so that they become an object of $\mathbf{MF}_{R_0}(\varphi, \nabla)$ by Proposition 10.2.4, and any φ -equivariant map $\mathcal{M}_0 \rightarrow \mathcal{M}'_0$ is horizontal. Combining this with Theorems 3.5 and 3.5.4, we obtain:

Corollary 10.3.1. *In the Local case above with $p > 2$, the functors \mathfrak{M}^* and \mathcal{M}^* below*

$$\{p\text{-divisible groups}/R\} \xrightarrow{\mathcal{M}^*} \mathbf{MF}_S(\varphi) \xleftarrow[\cong]{S \otimes_{\varphi, \varepsilon} (-)} \mathbf{Mod}_{\mathfrak{S}}(\varphi),$$

\mathfrak{M}^*

defined by forgetting the connections, are anti-equivalences of categories.

Let p be any prime. Then \mathfrak{M}^ and \mathcal{M}^* induce anti-equivalences of categories from the category of formal p -divisible groups over R to $\mathbf{Mod}_{\mathfrak{S}}(\varphi)^{\varphi \text{ nilp}}$ and $\mathbf{MF}_S(\varphi)^{\varphi \text{ nilp}}$, and similarly for unipotent p -divisible groups.*

Remark 10.3.2. Corollary 10.3.1 can be alternatively obtained using display theory for formal p -divisible groups and Dieudonné display theory (even in the **Local case** when $p = 2$); cf. [Zin02, Lau08] and [Lau10c]. Indeed, the “display” corresponding to a p -divisible group can be recovered from $\mathbf{MF}_S(\varphi)$ (without connection).

Let us now state the main result of this section:

Theorem 10.4. *Let A_0 be a finite product of p -adic \mathbb{Z}_p -flat domains which satisfies the conclusions of Lemma 10.2.3(1); namely, $A_{0,k} := A_0/(p)$ locally admits a finite p -basis, and for any prime ideal $\mathfrak{p} \subset A_0$ containing p , $(A_0)_{\mathfrak{p}}/(p)$ is regular and excellent. We also fix a lift of Frobenius $\varphi = \varphi_{A_0} : A_0 \rightarrow A_0$ over φ_W , which exists by Lemma 2.3.1.*

Then, the contravariant functor

$$\mathcal{M}_0^* : \{p\text{-divisible groups over } A_{0,k} := A_0/(p)\} \rightarrow \mathbf{MF}_{A_0}(\varphi, \nabla),$$

given by $G_{A_{0,k}} \rightsquigarrow \mathbb{D}^(G_{A_{0,k}})(A_0)$ is an anti-equivalence of categories.*

Let $\{\mathcal{Q}_n\}$ denote a moduli of connections for $\mathcal{M}_0 = \mathcal{M}'_0 \in \mathbf{MF}_{R_0}(\varphi, \nabla)$, and let $\widehat{\mathcal{Q}}_\infty$ denote the p -adic completion of $\mathcal{Q}_\infty := \varinjlim \mathcal{Q}_n$, as usual. By Lemma 10.2.3(1), Theorem 10.4 can be applied to any direct factor A_0 of $\widehat{\mathcal{Q}}_\infty$ which is a finite product of domains.

Let us first record a consequence. Let A_0 be as in Theorem 10.4, and set $A := A_0 \otimes_{R_0} R \cong A_0[u]/E(u)$, which is a ϖ -adic formally étale R -algebra. Set $\mathfrak{S}_A := A_0[[u]]$ and construct S_A , etc., accordingly.

Corollary 10.4.1. *In the above setting, the contravariant functor*

$$\mathcal{M}^* : \{p\text{-divisible groups over } A\} \rightarrow \mathbf{MF}_{S_A}^{\text{Bf}}(\varphi, \nabla^0),$$

given by $G \rightsquigarrow \mathbb{D}^(G)(S_A)$ is an exact anti-equivalence of categories.*

Proof. The full assertion for $p > 2$ and the essential surjectivity claim for $p = 2$ follows from Theorems 3.5 and 10.4. When $p = 2$, full faithfulness of \mathcal{M}^* follows from Corollary 8.6.1, which can be applied thanks to Lemma 10.2.3(1). \square

We first begin with the following standard commutative algebra lemma, which will be used in the proof of Theorem 10.4:

Lemma 10.4.2. *Let B be a (not necessarily noetherian) ring, and $I \subset B$ a finitely generated ideal. Let $\widehat{B} := \varprojlim B/I^n$ denote the I -adic completion, equipped with the inverse limit topology. Then the topological closure of I^n in \widehat{B} is $I^n \widehat{B}$, and the natural topology of \widehat{B} coincides with the $I\widehat{B}$ -adic topology. Furthermore, if B/I is noetherian, then so is \widehat{B} .*

We will apply this lemma when $I = (p)$.

Proof. The first assertion (on the completion of I^n and the topology of \widehat{B}) is exactly Corollaire 2 in [Bou06, Ch. III, §2, no. 12, page 228]. Since I is finitely generated, the associated graded ring $\bigoplus_{n \geq 0} I^n/I^{n+1}$ for \widehat{B} is finitely generated over B/I , so it is noetherian if B/I is noetherian. Now the noetherian-ness claim follows from Corollaire 1 in [Bou06, Ch. III, §2, no. 9, page 217]. \square

Proof of Theorem 10.4. Full faithfulness of \mathcal{M}_0^* follows from [BM90, Théorème 4.1.1] and Proposition 3.2.5. We prove essential surjectivity by generalising the proof of [Vas12, Theorem 3.4.1]: the main idea is fpqc descent from the completion at each point of $\text{Spec } A_{0,k}$. (The strategy of descending from completion using the theory of excellent rings appeared in the proof of [Zin02, Theorem 103] as well.)

Let us set up the notation. Let $\mathfrak{p} \subset A_0$ be a prime ideal containing p , and set $\bar{\mathfrak{p}} := \mathfrak{p}/(p)$ and $k(\mathfrak{p}) := \text{Frac}(A_0/\mathfrak{p}) = \text{Frac}(A_{0,k}/\bar{\mathfrak{p}})$. Let $B_{0,\mathfrak{p}}$ be the p -adic completion of $(A_0)_{\mathfrak{p}}$. Note that $B_{0,\mathfrak{p}}/(p) = (A_{0,k})_{\bar{\mathfrak{p}}}$, which is noetherian by assumption. Applying Lemma 10.4.2, it follows that $B_{0,\mathfrak{p}}$ is p -adic and noetherian.

Let $(\widehat{A_0})_{\mathfrak{p}}$ be the \mathfrak{p} -adic completion of $(A_0)_{\mathfrak{p}}$. One can easily check that $(\widehat{A_0})_{\mathfrak{p}}$ is faithfully flat over $B_{0,\mathfrak{p}}$, and $(\widehat{A_0})_{\mathfrak{p}}/(p)$ is naturally isomorphic to the $\bar{\mathfrak{p}}$ -adic completion of $(A_{0,k})_{\bar{\mathfrak{p}}}$. Since $(A_{0,k})_{\bar{\mathfrak{p}}}$ is regular, $(\widehat{A_0})_{\mathfrak{p}}/(p)$ is isomorphic to a formal power series ring over $k(\mathfrak{p})$ with finitely many variables, and $(\widehat{A_0})_{\mathfrak{p}}$ is isomorphic to a formal power series ring over some Cohen ring of $k(\mathfrak{p})$ with finitely many variables.

Note that $(A_{0,k})_{\bar{\mathfrak{p}}}$ has a finite p -basis because the relative Frobenius morphism $(A_{0,k})_{\bar{\mathfrak{p}}} \rightarrow A_{0,k} \otimes_{\varphi, A_{0,k}} (A_{0,k})_{\bar{\mathfrak{p}}}$ is an isomorphism. (Indeed, $(A_{0,k})_{\bar{\mathfrak{p}}} = \varinjlim_{f \notin \bar{\mathfrak{p}}} A_{0,k}[\frac{1}{f}]$, and the relative Frobenius morphism is clearly an isomorphism for the $A_{0,k}$ -algebra $A_{0,k}[\frac{1}{f}]$.) Hence, $(\widehat{A_0})_{\mathfrak{p}}/(p)$, being the $\bar{\mathfrak{p}}$ -adic completion of $(A_{0,k})_{\bar{\mathfrak{p}}}$, has a finite p -basis by [dJ95, Lemma 1.1.3].

As observed in §3.6 (Ex5), φ_{A_0} uniquely extends to lifts of Frobenius on $B_{0,p}$ and $(\widehat{A_0})_p$. Therefore, for any $\mathcal{M}_0 \in \mathbf{MF}_{A_0}(\varphi, \nabla)$, the scalar extensions $(\widehat{A_0})_p \otimes_{A_0} \mathcal{M}_0$ and $B_{0,p} \otimes_{A_0} \mathcal{M}_0$ can respectively be viewed as objects in $\mathbf{MF}_{(\widehat{A_0})_p}(\varphi, \nabla)$ and $\mathbf{MF}_{B_{0,p}}(\varphi, \nabla)$.

Now, by [dJ95, Main Theorem 1], there exists a p -divisible group $\widehat{G}_{\overline{p}}$ over $(\widehat{A_0})_p/(p)$ equipped with an isomorphism $\mathbb{D}^*(\widehat{G}_{\overline{p}})((\widehat{A_0})_p) \cong (\widehat{A_0})_p \otimes_{A_0} \mathcal{M}_0$ as objects in $\mathbf{MF}_{(\widehat{A_0})_p}(\varphi, \nabla)$.

We will proceed by descending $\widehat{G}_{\overline{p}}$ to a p -divisible group over $(A_{0,k})_{\overline{p}}$, and then to a p -divisible group over $A_{0,k}$. To produce descent data for p -divisible groups, we will first produce descent data for “Dieudonné crystals”, and show full faithfulness of crystalline Dieudonné functors over relevant base schemes.

Viewing $B_{0,p} \otimes_{A_0} \mathcal{M}_0$ and $(\widehat{A_0})_p \otimes_{A_0} \mathcal{M}_0$, respectively, as vector bundles on $\mathrm{Spf}(B_{0,p}, (p))$ and $\mathrm{Spf}((\widehat{A_0})_p, (p))$, the “vector bundle” $(\widehat{A_0})_p \otimes_{A_0} \mathcal{M}_0$ carries a descent datum for the fpqc covering $\mathrm{Spf}((\widehat{A_0})_p, (p)) \rightarrow \mathrm{Spf}(B_{0,p}, (p))$, as it is a pull back of the “vector bundle” $B_{0,p} \otimes_{A_0} \mathcal{M}_0$. To make it more precise, we obtain a projective system of descent data on $\{(\widehat{A_0})_p/(p^n) \otimes_{A_0} \mathcal{M}_0\}_n$ for faithfully flat maps $\{B_{0,p}/(p^n) \rightarrow (\widehat{A_0})_p/(p^n)\}_n$. It is now clear that the standard results on full faithfulness and effectivity of descent can be extended to our setting. In particular, we can consider “descent data with Frobenius structure and connections”, and the natural Frobenius structure and connection on $B_{0,p} \otimes_{A_0} \mathcal{M}_0$ can be uniquely recovered from the descent datum on $(\widehat{A_0})_p \otimes_{A_0} \mathcal{M}_0$.

For the rest of the proof, we let $\widehat{\otimes}$ denote the p -adic completion of the \otimes -product. We respectively define $i_1, i_2 : (\widehat{A_0})_p \rightarrow (\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p$ by $i_1 : a \mapsto a \widehat{\otimes} 1$ and $i_2 : a \mapsto 1 \widehat{\otimes} a$ for any $a \in (\widehat{A_0})_p$. Then the descent datum on $(\widehat{A_0})_p \otimes_{A_0} \mathcal{M}_0$ can be interpreted as an $(\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p$ -linear isomorphism

$$(10.4.3) \quad \left((\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p \right) \otimes_{i_1, A_0} \mathcal{M}_0 \xrightarrow{\sim} \left((\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p \right) \otimes_{i_2, A_0} \mathcal{M}_0$$

which satisfies the “standard” cocycle condition that involves scalar extensions by different choices of maps $(\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p \rightarrow (\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p$; actually, the cocycle condition coincides with the usual cocycle condition for the descent datum for modules, except that \otimes is replaced by $\widehat{\otimes}$. Since the Frobenius structure and connection on $(\widehat{A_0})_p \otimes_{A_0} \mathcal{M}_0$ are induced from $B_{0,p} \otimes_{A_0} \mathcal{M}_0$, the isomorphism (10.4.3) is an isomorphism in $\mathbf{MF}_{(\widehat{A_0})_p \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_p}(\varphi, \nabla)$. Note also that the descent datum (10.4.3) can be interpreted as an isomorphism

$$\mathbb{D}^*(i_1^* \widehat{G}_{\overline{p}}) \xrightarrow{\sim} \mathbb{D}^*(i_2^* \widehat{G}_{\overline{p}})$$

of Dieudonné crystals over $\mathrm{Spec}((\widehat{A_{0,k}})_{\overline{p}} \otimes_{(A_{0,k})_{\overline{p}}} (\widehat{A_{0,k}})_{\overline{p}})$, which satisfies some natural cocycle condition (cf., Proposition 3.2.5).

In order to show that this descent datum induces a descent datum on $\widehat{G}_{\overline{p}}$ for the fpqc covering $\mathrm{Spec}(\widehat{A_{0,k}})_{\overline{p}} \rightarrow \mathrm{Spec}(A_{0,k})_{\overline{p}}$, we need the following claim:

Claim 10.4.4. *The rings $(\widehat{A_{0,k}})_{\overline{p}} \otimes_{(A_{0,k})_{\overline{p}}} (\widehat{A_{0,k}})_{\overline{p}}$ and $(\widehat{A_{0,k}})_{\overline{p}} \otimes_{(A_{0,k})_{\overline{p}}} (\widehat{A_{0,k}})_{\overline{p}} \otimes_{(A_{0,k})_{\overline{p}}} (\widehat{A_{0,k}})_{\overline{p}}$ are normal and admit a finite p -basis.*

The p -basis assertion is clear. To show normality, let us write $(\widehat{A_{0,k}})_{\overline{p}} = \varinjlim_i B^{(i)}$ where $B^{(i)}$'s are finitely generated normal $(A_{0,k})_{\overline{p}}$ -subalgebras in $(\widehat{A_{0,k}})_{\overline{p}}$; indeed,

this is possible as the normalisation of any finitely generated $(A_{0,k})_{\bar{p}}$ -algebra is finite by excellence of $(A_{0,k})_{\bar{p}}$.

Recall that the natural map $(A_{0,k})_{\bar{p}} \rightarrow \widehat{(A_{0,k})_{\bar{p}}}$ is flat with geometrically regular fibres by excellence of $(A_{0,k})_{\bar{p}}$. In particular, $B^{(i)} \rightarrow \widehat{(A_{0,k})_{\bar{p}}} \otimes_{(A_{0,k})_{\bar{p}}} B^{(i)}$ is flat with geometrically regular fibres. Since the source of the map is normal, it follows from (Corollary of) [Mat86, Theorem 23.9] that $\widehat{(A_{0,k})_{\bar{p}}} \otimes_{(A_{0,k})_{\bar{p}}} B^{(i)}$ is normal. Normality of $\widehat{(A_{0,k})_{\bar{p}}} \otimes_{(A_{0,k})_{\bar{p}}} \widehat{(A_{0,k})_{\bar{p}}} = \varinjlim_i \left(\widehat{(A_{0,k})_{\bar{p}}} \otimes_{(A_{0,k})_{\bar{p}}} B^{(i)} \right)$ is now clear. We similarly show that $\widehat{(A_{0,k})_{\bar{p}}} \otimes_{(A_{0,k})_{\bar{p}}} \widehat{(A_{0,k})_{\bar{p}}} \otimes_{(A_{0,k})_{\bar{p}}} \widehat{(A_{0,k})_{\bar{p}}}$ is normal. This proves Claim 10.4.4.

It now follows from Claim 10.4.4 that the functors $\mathbb{D}^*(-)((\widehat{A_0})_{\bar{p}} \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_{\bar{p}})$ and $\mathbb{D}^*(-)((\widehat{A_0})_{\bar{p}} \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_{\bar{p}} \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_{\bar{p}})$ on p -divisible groups (over suitable bases) are fully faithful. (Cf., [BM90, Théorème 4.1.1], Proposition 3.2.5. Recall that $((\widehat{A_0})_{\bar{p}} \widehat{\otimes}_{B_{0,p}} (\widehat{A_0})_{\bar{p}})/(p) = (\widehat{A_{0,k}})_{\bar{p}} \otimes_{(A_{0,k})_{\bar{p}}} (\widehat{A_{0,k}})_{\bar{p}}$, and similarly for the triple tensor product.) Therefore, the descent datum on $(\widehat{A_0})_{\bar{p}} \otimes_{A_0} \mathcal{M}_0$ induces a descent datum on $\widehat{G_{\bar{p}}}$. By descending the finite flat group scheme $\widehat{G_{\bar{p}}}[p^n]$ for each n , we obtain a p -divisible group $G_{\bar{p}}$ over $(A_{0,k})_{\bar{p}}$ such that $\mathbb{D}^*(G_{\bar{p}})(B_{0,p}) \cong B_{0,p} \otimes_{A_0} \mathcal{M}_0$.

Now, a similar (but much easier) consideration produces a descent datum on $\{G_{\bar{p}}\}_{\bar{p} \in \text{Spec } A_{0,k}}$ with respect to the fpqc covering $\{\text{Spec}(A_{0,k})_{\bar{p}}\}$ of $\text{Spec } A_{0,k}$. (Note that Claim 10.4.4 becomes rather straightforward for $(A_{0,k})_{\bar{p}} \otimes_{A_{0,k}} (A_{0,k})_{\bar{p}'}$ and $(A_{0,k})_{\bar{p}} \otimes_{A_{0,k}} (A_{0,k})_{\bar{p}'} \otimes_{A_{0,k}} (A_{0,k})_{\bar{p}''}$, where $\bar{p}, \bar{p}', \bar{p}'' \in \text{Spec } A_{0,k}$.) The resulting p -divisible group $G_{A_{0,k}}$ comes equipped with a natural isomorphism $\mathbb{D}^*(G_{A_{0,k}})(A_0) \cong \mathcal{M}_0$, as desired. This concludes the proof of Theorem 10.4. \square

10.5. Proof of Theorem 9.4. Assume that R satisfies the formally finite-type assumption (§2.2.2), and let $I \subset R$ be an ideal with $(\varpi) \subset I \subset J_R$. We set $I_0 := \ker(R_0 \twoheadrightarrow R/I)$. Recall from §10.1 that for any I -adic formally étale R -algebra A such that A/IA is an inductive limit of étale R/I -algebras, we have a canonical choice of R_0 -subalgebra $A_0 \subset A$ and a lift of Frobenius φ_{A_0} over φ_{R_0} . We set $\mathfrak{S}_A := A_0[[u]]$ (with $\varphi_{\mathfrak{S}_A}$ extended by $u \mapsto u^p$), and $I_{\mathfrak{S}} := \ker(\mathfrak{S} \twoheadrightarrow R/I)$. Clearly \mathfrak{S} is $I_{\mathfrak{S}}$ -adically separated and complete. If $I = J_R$, then we write $J_{\mathfrak{S}} := I_{\mathfrak{S}}$. (If $I = (\varpi)$ for example, then $I_{\mathfrak{S}} = (p, E(u))$.)

If $\{\text{Spf}(A_{\alpha}, IA_{\alpha})\}$ is an ind-étale fpqc covering of $\text{Spf}(R, I)$ in the formal scheme sense (i.e., for each n , $\text{Spec } A_{\alpha}/I^n A_{\alpha}$ are ind-étale over $\text{Spec } R/I^n$ and form a fpqc covering), then the corresponding $\{\text{Spf}(\mathfrak{S}_{A_{\alpha}}, I_{\mathfrak{S}} \mathfrak{S}_{A_{\alpha}})\}$ is an ind-étale fpqc covering of $\text{Spf}(\mathfrak{S}, I_{\mathfrak{S}})$. (Here, ind-étale fpqc coverings are defined in the formal scheme sense.) For any \mathfrak{M} either in $\text{Mod}_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)^{\text{tor}}$ or $\text{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$, we let $\mathfrak{M}_{A_{\alpha}} := \mathfrak{S}_{A_{\alpha}} \otimes_{\mathfrak{S}} \mathfrak{M}$ denote the scalar extension. Note that we will actually work with ind-étale fpqc coverings that may not be étale coverings; cf., Proposition 10.5.2.

Lemma 10.5.1. *For any $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$, there exists a Zariski open covering $\{\text{Spf}(R_{\alpha}, J_R R_{\alpha})\}$ of $\text{Spf}(R, J_R)$ and $\mathfrak{N}_{\alpha}, \mathfrak{N}'_{\alpha} \in \text{Mod}_{\mathfrak{S}_{R_{\alpha}}}(\varphi)$ such that for each α there is a φ -equivariant exact sequence*

$$0 \rightarrow \mathfrak{N}'_{\alpha} \rightarrow \mathfrak{N}_{\alpha} \rightarrow \mathfrak{M}_{\alpha} := \mathfrak{S}_{R_{\alpha}} \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow 0.$$

Proof. We prove this lemma by adapting the proof of [Kis06, Lemma 2.3.4] as follows. For $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}(\varphi)^{\text{tor}}$, set $L := \text{coker}(1 \otimes \varphi_{\mathfrak{M}})$ and $\mathfrak{F} := \text{im}(1 \otimes \varphi_{\mathfrak{M}})$. Choose a finite free R -module \tilde{L} which surjects onto L and a finite-rank free \mathfrak{S} -module \mathfrak{N} which surjects onto both \mathfrak{M} and \tilde{L} . Set $\tilde{\mathfrak{F}} := \ker(\mathfrak{N} \twoheadrightarrow \tilde{L})$. We may assume that $\tilde{\mathfrak{F}}$ surjects onto \mathfrak{F} under the natural projection $\mathfrak{N} \twoheadrightarrow \mathfrak{M}$ by replacing, if necessary, \mathfrak{N} by $\mathfrak{N} \oplus \mathfrak{S}^r$ where \mathfrak{S}^r surjects onto \mathfrak{F} and maps to zero in \tilde{L} . To summarise, we have

the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\mathfrak{F}} & \longrightarrow & \mathfrak{N} & \longrightarrow & \tilde{L} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{F} & \longrightarrow & \mathfrak{M} & \longrightarrow & L \longrightarrow 0 \end{array}$$

Since \tilde{L} is free over R and R is of \mathfrak{S} -projective dimension 1, it follows that $\tilde{\mathfrak{F}}$ is projective over \mathfrak{S} . We choose $\{R_\alpha\}$ so that $\tilde{\mathfrak{F}}_\alpha := \mathfrak{S}_{R_\alpha} \otimes_{\mathfrak{S}} \tilde{\mathfrak{F}}$ is free over \mathfrak{S}_{R_α} for each α . (Note that since \mathfrak{S}_{R_α} is $J_{\mathfrak{S}}$ -adic, a finitely generated projective \mathfrak{S}_{R_α} -module $\tilde{\mathfrak{F}}_\alpha$ is free if and only if $\tilde{\mathfrak{F}}_\alpha/J_{\mathfrak{S}}\tilde{\mathfrak{F}}_\alpha$ is free over $\mathfrak{S}_{R_\alpha}/J_{\mathfrak{S}}\mathfrak{S}_{R_\alpha} = R_\alpha/J_R R_\alpha$.) So we take R_α so that $\tilde{\mathfrak{F}}_\alpha/J_{\mathfrak{S}}\tilde{\mathfrak{F}}_\alpha$ is free over $R_\alpha/J_R R_\alpha$.) Since both $\tilde{\mathfrak{F}}_\alpha$ and \mathfrak{N}_α are free of the same rank, we may choose an isomorphism $\varphi^*\mathfrak{N}_\alpha \cong \tilde{\mathfrak{F}}_\alpha$, and define $\varphi_{\mathfrak{N}_\alpha}$ so that we have:

$$1 \otimes \varphi_{\mathfrak{N}_\alpha} : \varphi^*\mathfrak{N}_\alpha \cong \tilde{\mathfrak{F}}_\alpha \hookrightarrow \mathfrak{N}_\alpha.$$

It is clear that $(\tilde{\mathfrak{M}}_\alpha, \varphi_{\tilde{\mathfrak{M}}_\alpha}) \in \mathbf{Mod}_{\mathfrak{S}_{R_\alpha}}(\varphi)^{\text{tor}}$ and the natural projection $\tilde{\mathfrak{M}}_\alpha \rightarrow \mathfrak{M}_\alpha$ is φ -equivariant.

Set $\tilde{\mathfrak{M}}'_\alpha := \ker(\tilde{\mathfrak{M}}_\alpha \rightarrow \mathfrak{M}_\alpha)$, which is a φ -stable submodule of $\tilde{\mathfrak{M}}_\alpha$. We claim that $\tilde{\mathfrak{M}}'_\alpha \in \mathbf{Mod}_{\mathfrak{S}_{R_\alpha}}(\varphi)^{\text{tor}}$. Set $\tilde{L}'_\alpha := \text{coker}(1 \otimes \varphi_{\tilde{\mathfrak{M}}'_\alpha})$. Then \tilde{L}'_α naturally embeds into $\tilde{L}_\alpha := R_\alpha \otimes_R \tilde{L}$ since $\tilde{L}'_\alpha[\frac{1}{p}] = \tilde{L}_\alpha[\frac{1}{p}]$ and R is \mathbb{Z}_p -flat. It follows that \tilde{L}'_α is annihilated by E . \square

Proposition 10.5.2. *For any $\mathfrak{M} \in (\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\text{Ki}}(\varphi, \nabla^0)$, there exists a Zariski open covering $\{\text{Spf}(R_\alpha, J_R R_\alpha)\}$ of $\text{Spf}(R, J_R)$, such that for each α there exist a faithfully flat ind-étale map $\text{Spf}(A_\alpha, (\varpi)) \rightarrow \text{Spf}(R_\alpha, (\varpi))$ and $\mathfrak{N}_{A_\alpha} \in \mathbf{Mod}_{\mathfrak{S}_{A_\alpha}}^{\text{Ki}}(\varphi, \nabla^0)$ with the following properties:*

- (1) for each α , $A_{\alpha,0}$ satisfies the assumption for A_0 over $R_{\alpha,0}$ in Theorem 10.4;
- (2) for each α , there is a φ -equivariant surjective map $\mathfrak{N}_{A_\alpha} \rightarrow \mathfrak{M}_{A_\alpha} := \mathfrak{S}_{A_\alpha} \otimes_{\mathfrak{S}} \mathfrak{M}$ such that the induced map $\mathcal{N}_{A_\alpha,0} := A_{\alpha,0} \otimes_{\varphi, \mathfrak{S}_{A_0}} \mathfrak{N}_{A_\alpha} \rightarrow \mathcal{M}_{A_\alpha,0} := A_{\alpha,0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ respects connections.

Proof. In order to prove the proposition, we may replace R by one of R_α 's as in Lemma 10.5.1, so that we have $\mathfrak{N} \in \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ with φ -equivariant surjective map $\mathfrak{N} \rightarrow \mathfrak{M}$. (Cf., Lemma 10.5.1.) The main idea is to find an ind-étale fpqc covering over which $\mathcal{N}_0 := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}$ admits a connection as in the statement. We use some slight variant of Vasiu's construction of "moduli of connections" (cf., [Vas12, Theorem 3.2], and also §10.2 of this paper).

Let us set up the notation. Set $\mathcal{M}_0 := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cong R_0 \otimes_S (S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})$. Since $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \in \mathbf{MF}'_S(\varphi, \nabla)^{\text{tor}}$, one can view $\mathcal{M}_0 \in \mathbf{MF}_{R_0}(\varphi, \nabla)^{\text{tor}}$ where all the extra structure is induced from $S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Similarly, $\mathcal{N}_0 := R_0 \otimes_{\varphi, \mathfrak{S}} \mathfrak{N}$ can naturally be viewed as an object in $\mathbf{MF}_{R_0}(\varphi)$. We also fix a R_0 -direct factor $(\mathcal{N}_0)^1 \subseteq \mathcal{N}_0$ which lifts $(\text{Fil}^1 \mathcal{N}_0)/p\mathcal{N}_0 \subseteq \mathcal{N}_0/p\mathcal{N}_0$.

Assume that $p^n \mathfrak{M} = 0$. Let $\{\text{Spf}(R_\alpha, J_R R_\alpha)\}$ be a Zariski covering of $\text{Spf}(R, J_R)$ with the following properties:

- $(\mathcal{N}_0)^1, \mathcal{N}_0/(\mathcal{N}_0)^1$, and $\widehat{\Omega}_{R_0}$ become free over $R_{\alpha,0}$;
- the underlying R_0 -modules of both \mathcal{M}_0 and the kernel of $\mathcal{N}_0/p^n \mathcal{N}_0 \rightarrow \mathcal{M}_0$ become isomorphic to $\bigoplus (R_{\alpha,0}/p^i)^{d_i}$ after the base change.

We may (and do) replace R by one of R_α , and drop the subscript α from now on.

We now construct an ind-étale fpqc covering $\text{Spf}(A_0, (p)) \rightarrow \text{Spf}(R_0, (p))$ as follows. Recall that for each $n \geq 1$ we have a faithfully flat p -adic formally étale R -algebra \mathcal{Q}_n such that for any p -adic formally étale R_0 -algebra B_0 , $\text{Hom}_{R_0}(\mathcal{Q}_n, B_0)$

is naturally in bijection with the set of connections on $B_0/(p^n) \otimes_{R_0} \mathcal{N}_0$ which satisfy the commutative diagram (10.2.1). One can see without difficulty that there is a universal quotient $\mathcal{Q}'_{1,k}$ of $\mathcal{Q}_1/(p)$ such that the natural projection $\mathcal{Q}'_{1,k} \otimes_{R_0} \mathcal{N}_0 \rightarrow \mathcal{Q}'_{1,k} \otimes_{R_0} \mathcal{M}_0$ is horizontal when we give a universal connection on the source. Furthermore, by writing down this condition in terms of the “matrix entries” of a universal connection, one sees that $\mathcal{Q}'_{1,k}$ is defined by equations of the form $x' = B'\varphi(x') + C'$ where B' and C' are matrices over $R_0/(p)$ and x' is a column vector of variables. One can check (cf., [Vas12, Theorem 2.4.1]) that $\mathcal{Q}'_{1,k}$ is faithfully flat étale over $R_0/(p)$, hence $\mathrm{Spec} \mathcal{Q}'_{1,k}$ is a union of connected components in $\mathrm{Spec} \mathcal{Q}_1/(p)$. Let \mathcal{Q}'_1 denote the unique p -adic formally étale lift of $\mathcal{Q}'_{1,k}$, which can naturally be viewed as a quotient of \mathcal{Q}_1 .

We can repeat this construction to obtain a union of connected components $\mathrm{Spf}(\mathcal{Q}'_n, (p)) \subset \mathrm{Spf}(\mathcal{Q}_n, (p))$ where $p^n \mathfrak{M} = 0$, with the property that $\mathcal{Q}'_n/(p^n) \otimes_{R_0} \mathcal{N}_0 \rightarrow \mathcal{Q}'_n \otimes_{R_0} \mathcal{M}_0$ is horizontal. (Note that both $p^i \mathcal{N}_0/p^{i+1} \mathcal{N}_0$ and $p^i \mathcal{M}_0/p^{i+1} \mathcal{M}_0$ are free over $R_0/(p)$ by assumption.) Pick a formally étale lift \mathcal{Q}'_n of $\mathcal{Q}'_{n,k}$ and view it as a quotient of \mathcal{Q}_n . For any $i \geq n$, we set $\mathcal{Q}'_i := \mathcal{Q}_i \otimes_{\mathcal{Q}_n} \mathcal{Q}'_n$, and let $\widehat{\mathcal{Q}}'_\infty$ denote the p -adic completion of $\mathcal{Q}'_\infty := \varinjlim_i \mathcal{Q}'_i$.

Note that each $\mathrm{Spf}(\mathcal{Q}'_n, (p))$ is faithfully flat and étale over $\mathrm{Spf}(R_0, (p))$, so we can choose a direct summand A_0 of $\widehat{\mathcal{Q}}'_\infty$, such that A_0 is a finite product of domains and $\mathrm{Spec} A_0/(p)$ surjects onto $\mathrm{Spec} R_0/(p)$ (which is possible by Lemma 10.2.3(2)). As $\widehat{\mathcal{Q}}'_\infty$ is a direct factor of $\widehat{\mathcal{Q}}_\infty$, A_0 satisfies (1) in the statement by Lemma 10.2.3(1). The universal connection on $\widehat{\mathcal{Q}}'_\infty$ induces a connection on $A_0 \otimes_{R_0} \mathcal{N}_0$, which makes it an object in $\mathbf{MF}_{A_0}(\varphi, \nabla)$ by Proposition 10.2.4, and satisfies (2) in the statement by construction. Now we set $A := A_0 \otimes_W \mathcal{O}_K$ and $\mathfrak{N}_A := \mathfrak{S}_A \otimes_{\mathfrak{S}} \mathfrak{N}$. \square

Now let us prove Theorem 9.4. By Proposition 9.3, we only need to show essential surjectivity. From Lemma 10.5.1 and Corollary 10.3.1, one easily obtain Theorem 9.4 except the assertions involving $(\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\mathrm{Ki}}(\varphi, \nabla^0)$. We now show the remaining part of Theorem 9.4.

By local flatness criterion (for modules over $\mathfrak{S}/(p^i)$), for a p -power order finite locally free group scheme H over R , $H[p^i]$ is locally free over R for each i if and only if $\mathfrak{M}^*(H) \in (\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\mathrm{Ki}}(\varphi, \nabla^0)$. So by Proposition 9.3 and Remark 9.2.2 it remains to show that for any $\mathfrak{M} \in (\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\mathrm{Ki}}(\varphi, \nabla^0)$ there exists a finite locally free group scheme H over R such that $\mathfrak{M}^*(H) \cong \mathfrak{M}$.

For a given $\mathfrak{M} \in (\mathbf{Mod} \mathbf{FI})_{\mathfrak{S}}^{\mathrm{Ki}}(\varphi, \nabla^0)$ we choose $\{R_\alpha\}$, $\{A_\alpha\}$ and $\mathfrak{N}_{A_\alpha} \in \mathbf{Mod}_{\mathfrak{S}_\beta}^{\mathrm{Ki}}(\varphi, \nabla^0)$ as in Proposition 10.5.2, and set $\mathfrak{N}'_{A_\alpha} := \ker(\mathfrak{N}_{A_\alpha} \rightarrow \mathfrak{M}_{A_\alpha})$, which is an object in $\mathbf{Mod}_{\mathfrak{S}_{A_\alpha}}^{\mathrm{Ki}}(\varphi, \nabla^0)$. By Corollary 10.4.1, we obtain an isogeny of p -divisible groups $G_{A_\alpha} \rightarrow G'_{A_\alpha}$ over $\mathrm{Spf}(A_\alpha, (\varpi))$ corresponding to the natural inclusion $\mathfrak{N}'_{A_\alpha} \hookrightarrow \mathfrak{N}_{A_\alpha}$, and set $H_{A_\alpha} := \ker(G_{A_\alpha} \rightarrow G'_{A_\alpha})$. Clearly, we have $\mathfrak{M}^*(H_{A_\alpha}) \cong \mathfrak{S}_{A_\alpha} \otimes_{\mathfrak{S}} \mathfrak{M}$. We will produce H over R by first descending H_{A_α} over R_α , and glue them together.

We write $\widehat{\otimes}_{R_\alpha}$ for the ϖ -adically completed tensor product, and $\widehat{\otimes}_{\mathfrak{S}_\alpha}$ for the $(p, E(u))$ -adically completed tensor product (or equivalently, (p, u) -adically completed tensor product). Consider maps $i_1, i_2 : A_\alpha \rightarrow A_\alpha \widehat{\otimes}_{R_\alpha} A_\alpha$ defined by $i_1 : a \mapsto a \widehat{\otimes} 1$ and $i_2 : a \mapsto 1 \widehat{\otimes} a$ for $a \in A_\alpha$, and use the same notation to denote the maps $i_1, i_2 : \mathfrak{S}_{A_\alpha} \rightarrow \mathfrak{S}_{A_\alpha} \widehat{\otimes}_{\mathfrak{S}_{R_\alpha}} \mathfrak{S}_{A_\alpha}$ defined similarly.

Since $\mathrm{Spf}(A_\alpha, (\varpi)) \rightarrow \mathrm{Spf}(R_\alpha, (\varpi))$ is faithfully flat map of formal schemes (i.e., $\mathrm{Spec} A_\alpha/(\varpi^n) \rightarrow \mathrm{Spec} R_\alpha/(\varpi^n)$ is faithfully flat for each n), we can apply fpqc descent theory. A descent datum on H_{A_α} is an isomorphism

$$(10.5.3) \quad i_1^* H_{A_\alpha} \xrightarrow{\sim} i_2^* H_{A_\alpha}$$

of finite locally free group schemes over $A_\alpha \widehat{\otimes}_{R_\alpha} A_\alpha$ which satisfies the natural co-cycle conditions. By setting $A_{\alpha,k} := A_\alpha/(\varpi)$ and $R_{\alpha,k} := R_\alpha/(\varpi)$, the tensor products $A_{\alpha,k} \otimes_{R_{\alpha,k}} A_{\alpha,k}$ and $A_{\alpha,k} \otimes_{R_{\alpha,k}} A_{\alpha,k} \otimes_{R_{\alpha,k}} A_{\alpha,k}$ are normal and locally admit finite p -basis. (For normality, note that these rings are filtered direct limits of étale $R_{\alpha,k}$ -algebras, which are normal.) Therefore, by the base change property and full faithfulness of \mathfrak{M}^* as stated in Proposition 9.3, we obtain a descent datum on H_{A_α} as in (10.5.3) from the “ (p, u) -adically continuous descent datum”

$$(10.5.4) \quad (\mathfrak{S}_{A_\alpha} \widehat{\otimes}_{\mathfrak{S}_{R_\alpha}} \mathfrak{S}_{A_\alpha}) \otimes_{i_1, \mathfrak{S}} \mathfrak{M} \xrightarrow{\sim} (\mathfrak{S}_{A_\alpha} \widehat{\otimes}_{\mathfrak{S}_{R_\alpha}} \mathfrak{S}_{A_\alpha}) \otimes_{i_2, \mathfrak{S}} \mathfrak{M}$$

obtained from the fact that $\mathfrak{S}_{A_\alpha} \otimes_{\mathfrak{S}} \mathfrak{M}$ is the scalar extension of $\mathfrak{S}_{R_\alpha} \otimes_{\mathfrak{S}} \mathfrak{M}$. Therefore, we obtain H_{R_α} over $\mathrm{Spf}(R_\alpha, (\varpi))$ such that $\mathfrak{M}^*(H_{R_\alpha}) \cong \mathfrak{S}_{R_\alpha} \otimes_{\mathfrak{S}} \mathfrak{M}$.

A similar consideration produces a glueing datum on $\{H_{R_\alpha}\}_\alpha$, so we obtain a finite locally group scheme H over R with $\mathfrak{M}^*(H) \cong \mathfrak{M}$. This concludes the proof of Theorem 9.4.

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