RAPOPORT-ZINK SPACES OF HODGE TYPE

WANSU KIM

ABSTRACT. When $p > 2$, we construct a Hodge-type analogue of Rapoport-Zink spaces under the unramifiedness assumption, as formal schemes parametrising "deformations" (up to quasi-isogeny) of p-divisible groups with certain crystalline Tate tensors. We also define natural rigid analytic towers with expected extra structure, providing more examples of "local Shimura varieties" conjectured by Rapoport and Viehmann.

CONTENTS

1. INTRODUCTION

Let $(G, \mathfrak{H})$ be a Hodge-type Shimura datum; i.e., $(G, \mathfrak{H})$ can be embedded into the Shimura datum associated to some symplectic similitude group (i.e., Siegel Shimura datum). By choosing such an embedding, the associated complex Shimura variety $\mathrm{Sh}(G, \mathfrak{H})_{\mathbb{C}}$ obtains a family of abelian varieties (coming from the ambient Siegel modular variety) together with Hodge cycles.

In this paper, we construct, in the unramified case, a natural p-adic local analogue of such Shimura varieties; loosely speaking, what we constructed can be regarded as "moduli spaces" of p-divisible groups equipped with certain "crystalline Tate tensors". Since the precise definition is rather technical, let us just indicate the idea. Recall that for a $\mathbb{Q}$-Hodge structure H, a Hodge cycle on H can be understood as a morphism $t : \mathbf{1} \to H$ of $\mathbb{Q}$-Hodge structures, where $\mathbf{1}$ is the trivial $\mathbb{Q}$-Hodge structure of rank 1. Our definition of crystalline Tate tensors is very similar, with $\mathbb{Q}$-Hodge structures replaced by F-crystals[1] equipped with Hodge filtration.[2]

2010 *Mathematics Subject Classification.* 14L05, 14F30.

Key words and phrases. Rapoport-Zink Spaces, local Shimura varieties, crystalline Dieudonné theory.

[1]By F-crystal, we mean Tate twists by *any* integers of crystals equipped with nondegenerate Frobenius action. This is to allow the dual of an F-crystal to be an F-crystal.

[2]We only define the notion of crystalline Tate tensors for p-divisible groups defined over "formally smooth" base rings, to avoid subtleties involving torsions of crystalline Dieudonné theory. See Definition 4.6 for the precise definition over nice enough base rings.

Let G be a connected *unramified*[3] reductive group over $\mathbb{Q}_p$. We fix a reductive $\mathbb{Z}_p$-model of G (which exists by unramifiedness), and also denote it by G. We choose an element $b \in G(\widehat{\mathbb{Q}}_p^{\mathrm{ur}})$ which gives rise to a p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}}_p$ in the following sense: for some finite free $\mathbb{Z}_p$-module Λ with faithful G-action, the F-crystal $\mathbf{M} := (\widehat{\mathbb{Z}}_p^{\mathrm{ur}} \otimes \Lambda^*, b \circ (\sigma \otimes \mathrm{id}))$ gives rise to a p-divisible group $\mathbb{X}$ by the (contravariant) Dieudonné theory. In this case, we can associate to such b an "unramified Hodge-type local Shimura datum" $(G, [b], \{\mu^{-1}\})$. (See §2.5 for the details.)

Let $\mathbf{M}^{\otimes}$ denote the direct sum of the combinations of tensor products, symmetric and alternating products, and duals of $\mathbf{M}$. Then the fact that $b \in G(K_0)$ gets encoded as the existence of certain "Frobenius-invariant tensors" $(\mathbf{t}_\alpha) \in \mathbf{M}^{\otimes}$; *cf.* Lemma 2.5.6, Proposition 2.1.3.

Rapoport and Zink constructed a moduli space $\mathtt{RZ}_{\mathbb{X}}$ parametrising "deformations" of $\mathbb{X}$ up to quasi-isogeny [39, Theorem 2.16]. Here, $\mathtt{RZ}_{\mathbb{X}}$ is a formal scheme which is locally formally of finite type over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$. Roughly speaking, our main result is of the following form:

Theorem (4.9.1). *Let (G, b) and $\mathbb{X}$ be as above, and assume that $p > 2$.*[4] *Then there exists a closed formal subscheme $\mathtt{RZ}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$ which classifies deformations (up to quasi-isogeny) of $\mathbb{X}$ with Tate tensors $(\mathbf{t}_\alpha)$, such that the Hodge filtration of the p-divisible group is étale-locally given by some cocharacter in the conjugacy class $\{\mu\}$. (See Definitions 4.6 and 5.1 for the precise conditions that define $\mathtt{RZ}_{G,b}$.) Furthermore, $\mathtt{RZ}_{G,b}$ is formally smooth, is functorial in (G, b), and only depends on the associated integral Hodge-type local Shimura datum $(G, [b], \{\mu^{-1}\})$ up to isomorphism.*

Rapoport and Viehmann conjectured that to any (not necessarily unramified nor Hodge-type) "local Shimura datum" $(G, [b].\{\mu^{-1}\})$, there exists a rigid analytic tower of "local Shimura varieties" with suitable extra structure [38, §5]. In §7, we construct the rigid analytic tower over the generic fibre of $\mathtt{RZ}_{G,b}$ equipped with suitable extra structure as predicted in [38, §5]; in other words, we construct "local Shimura varieties" associated to any unramified Hodge-type local Shimura data when $p > 2$.[5]

In the case of unramified EL and PEL type, we also show that $\mathtt{RZ}_{G,b}$ recovers the original construction of Rapoport-Zink space in [39, Theorem 3.25]. See Proposition 4.7.1 for the precise statement. On the other hand, the theorem provides Rapoport-Zink spaces for more general class of groups G that do not necessarily arise from any EL or PEL datum. For example, we may allow G to be the spin similitude group associated to a split quadratic space over $\mathbb{Q}_p$, which do not arise from any EL or PEL datum if the rank of the quadratic space is at least 7. Note also that the "functoriality" assertion of the theorem produces some interesting morphisms between EL and PEL Rapoport-Zink spaces, which may not be easily seen from the original construction. See Remark 4.9.7 for such an example involving an "exceptional isomorphism".

Recently, Scholze and Weinstein [42] constructed the "infinite-level" Rapoport-Zink spaces of EL- and PEL- type, which provides a new approach to study Repoport-Zink spaces. As remarked in the introduction of [42], for quite a general "local

[3] I.e., quasi-split and split over $\mathbb{Q}_p^{\mathrm{ur}}$.

[4] The assumption is made in order to use the Grothendieck-Messing deformation theory for the nilpotent ideal generated by p. The main result will be generalised to the case when $p = 2$ in the author's subsequent paper.

[5] Rapoport and Viehmann also conjectured that "local Shimura varieties" could be constructed by a purely group-theoretic means. Note that our construction of "local Shimura varieties" (in the unramified Hodge-type case) is not purely group-theoretic as we make crucial use of p-divisible groups.

Shimura datum" $(G, [b], \{\mu^{-1}\})$ – without requiring G to be unramified, nor $[b]$ to come from a p-divisible group – it should be possible to construct the infinite-level Rapoport-Zink space for $(G, [b], \{\mu\})$ using the technique in [42]. This approach does not require any formal model at the "maximal level" Rapoport-Zink space, nor does it give a natural formal model, while our approach is to start with the formal scheme at the hyperspecial maximal level and build up the rigid analytic tower from there. Perhaps, having a formal scheme at the hyperspecial maximal level could be useful in some applications; for example, p-adic uniformisation of Hodge-type Shimura varieties; see the next paragraph for more details. In §7.6 we give a construction of $\mathtt{RZ}^{\infty}_{G,b}$ using "finite-level" Rapoport-Zink spaces $\mathtt{RZ}^{\mathsf{K}}_{G,b}$ and [42, Theorem D]. Note that there should be more natural "purely infinite-level" construction of $\mathtt{RZ}^{\infty}_{G,b}$, which should work more generally, but we give our "finite-level" construction just to link our work with [42].

In the PEL case, Rapoport and Zink also showed that certain arithmetic quotients of PEL Rapoport-Zink spaces can be related to PEL Shimura varieties, generalising the theorem of Drinfeld and Cerednik on p-adic uniformisation of Shimura curves; *cf.* [39, Ch. VI]. This is a useful tool for studying the mod p geometry of PEL Shimura varieties – especially, the basic (i.e. supersingular) locus – by reducing the question to a purely local problem of studying the corresponding Rapoport-Zink space. In the sequel of this paper [26], we give a Hodge-type generalisation of this result at odd good reduction primes. In particular, the result is applicable to $\mathrm{GSpin}(n, 2)$ Shimura varieties for any n.

Recently, Ben Howard and George Pappas [22] gave another construction of Hodge-type Rapoport-Zink spaces, which recovers our construction, in the case when the Hodge-type local Shimura datum $(G, [b], \{\mu^{-1}\})$ comes from a (global) Hodge-type Shimura datum. Their construction relies on the existence of integral canonical models of Hodge-type Shimura varieties and the Rapoport-Zink uniformisation for Siegel modular varieties. This global construction is simpler than ours and automatically gives the Rapoport-Zink uniformisation of Hodge-type Shimura varieties (recovering [26]), while our construction is purely local (as it should be) and does not require the Hodge-type local Shimura datum $(G, [b], \{\mu^{-1}\})$ to come from a global Shimura datum.

Let us comment on the proof of the main theorem (Theorem 4.9.1). We do not directly extract defining equations of $\mathtt{RZ}_{G,b}$ in $\mathtt{RZ}_{\mathbb{X}}$, but instead we take an indirect approach. As a starting point, observe that the candidate for the set of closed points $\mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p) \subset \mathtt{RZ}_{\mathbb{X}}(\overline{\mathbb{F}}_p)$ is given by the affine Deligne-Lusztig set (*cf.* Proposition 2.5.9). Furthermore, Faltings constructed a formally smooth closed subspace $(\mathtt{RZ}_{G,b})^{\widehat{}}_x$ of the completion $(\mathtt{RZ}_{\mathbb{X}})^{\widehat{}}_x$ at $x \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$, which gives a natural candidate for the completion of $\mathtt{RZ}_{G,b}$ at x.

We then construct a formal algebraic space $\mathtt{RZ}_{G,b}$ by patching together Faltings deformation spaces $(\mathtt{RZ}_{G,b})^{\widehat{}}_x$ via Artin's algebraisation technique (*cf.* §6.1).[6] The key step is to show that Tate tensors, constructed formal-locally over $(\mathtt{RZ}_{G,b})^{\widehat{}}_x$ by Faltings, patch together and smear out to some neighbourhood of x whenever they should (*cf.* Propositions 5.2 and 5.6).

The main reason why we exclude $p = 2$ is that the standard PD structure on pR is not nilpotent unless either $p > 2$ or $pR = 0$. In particular, if $p = 2$ then we cannot apply the Grothendieck-Messing deformation theory [35] for the thickenings $R \twoheadrightarrow R/p$. The main results of this paper will be extended to the case when $p = 2$ in the author's forthcoming paper.

[6]Although Artin's criterion is only for algebraic spaces, not for formal algebraic spaces, we apply Artin's criterion to suitable "closed subspaces" which turn out to be algebraic spaces.

Structure of the paper. In §2 we recall and introduce some basic definitions, such as (iso)crystals with G-structures, filtrations, and cocharacters. In §3 we review Faltings's explicit construction of the "universal deformation" of p-divisible groups with Tate tensors. The theorem on the existence of $\mathtt{RZ}_{G,b}$ is stated in §4, which is proved in §5 and §6.

In §7 we define various extra structures on $\mathtt{RZ}_{G,b}$ as predicted in [38, §5], including rigid analytic tower and the Hecke and quasi-isogeny group actions. In §8, we show the existence and integrality of "étale realisations" of the crystalline Tate tensors, which are needed for constructing the rigid analytic tower in §7.

Acknowledgement. The author would like to thank Chris Blake, James Newton, Tony Scholl, Teruyoshi Yoshida, and Xinwen Zhu for helpful discussions and advice, and especially Brian Conrad and George Pappas for his careful reading of the manuscript and providing helpful suggestions. The author sincerely thanks the anonymous referee who pointed out the gap in the earlier version of the manuscript regarding the torsion of PD hulls.

2. Definitions and Preliminaries

2.1. **Notation.** For any ring R, an R-module M, and an R-algebra R', we write $M_{R'} := R' \otimes_R M$. Similarly, if R is a noetherian adic ring and $\mathfrak{X}$ is a formal scheme over $\mathrm{Spf}\, R$, then for any continuous morphism of adic rings $R \to R'$ we write $\mathfrak{X}_{R'} := \mathfrak{X} \times_{\mathrm{Spf}\, R} \mathrm{Spf}\, R'$.

For any ring $\mathscr{O}$, we let $\mathrm{Alg}_{\mathscr{O}}$ denote the category of $\mathscr{O}$-algebra. If $\mathscr{O}$ is a p-adic discrete valuation ring, then we let $\mathrm{Nilp}_{\mathscr{O}}$ denote the category of $\mathscr{O}$-algebras R where p is nilpotent. Let $\mathfrak{AR}_{\mathscr{O}}$ denote the category of artin local $\mathscr{O}$-algebras with residue field $\mathscr{O}/\mathfrak{m}_{\mathscr{O}}$. Note that any formal scheme $\mathfrak{X}$ over $\mathrm{Spf}\, R$ defines a set-valued functor on Nilp_W by $\mathfrak{X}(R) := \mathrm{Hom}_{\mathscr{O}}(\mathrm{Spec}\, R, \mathfrak{X})$ for $R \in \mathrm{Nilp}_{\mathscr{O}}$.

We will work with formal schemes that satisfy the following finiteness condition:

Definition 2.1.1. Let $\mathscr{O}$ be a p-adic discrete valuation ring. A locally noetherian formal $\mathscr{O}$-scheme $\mathfrak{X}$ is *locally formally of finite type* over $\mathrm{Spf}\, \mathscr{O}$ if $\mathfrak{X}_{\mathrm{red}}$ is locally of finite type over $\mathscr{O}/\mathfrak{m}_{\mathscr{O}}$; or equivalently, if $\mathfrak{X}$ admits an affine open covering $\{\mathrm{Spf}\, R_\xi\}$ where each R_ξ is *formally finitely generated* over $\mathscr{O}$ in the following sense: there exists a surjective map of $\mathscr{O}$-algebras

$$\mathscr{O}[[u_1, \cdots, u_r]]\langle v_1, \cdots, v_s\rangle \twoheadrightarrow R,$$

where we use the p-adic topology on $\mathscr{O}$ to define convergent formal power series. A formal $\mathscr{O}$-scheme $\mathfrak{X}$ is *formally of finite type* if it is quasi-compact and locally formally of finite type over $\mathrm{Spf}\, \mathscr{O}$.

Let $\mathcal{C}$ be a pseudo-abelian[7] symmetric tensor category such that arbitrary (infinite) direct sum exists. (For definitions in category theory, see [46] and references therein.) Let $\mathbf{1}$ denote the unit object for $\otimes$-product in $\mathcal{C}$ (which exists by axiom of tensor categories).

Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ which is stable under direct sums, tensor products, and direct factors. Assume furthermore that $\mathcal{D}$ is rigid; i.e., every object of $\mathcal{D}$ has a dual. (For example, $\mathcal{C}$ can be the category of R-modules filtered by direct factors over R, and $\mathcal{D}$ can be the full subcategory of finitely generated projective R-modules filtered by direct factors.)

[7]Pseudo-abelian categories are defined in the same way as abelian categories, except that we only require the existence of kernel for idempotent morphisms instead of requiring the existence of kernel and cokernel for any morphism. In practice, the pseudo-abelian categories that we will encounter are the category of filtered or graded objects in some abelian category.

Definition 2.1.2. For any object $M \in \mathcal{D}$, we let

$$M^{\otimes} \in \mathcal{C}$$

denote the direct sum of any (finite) combination of tensor products, symmetric products, alternating products, and duals of M. Note that $M^{\otimes} = (M^*)^{\otimes}$.

Let us now recall the following (slight variant of the) standard result:

Proposition 2.1.3. *Let R be either a field of characteristic 0 or a discrete valuation ring of mixed characteristic. Let G be a connected reductive group over R, and Λ a finite free R-module equipped with a closed embedding of algebraic R-groups $G \hookrightarrow \mathrm{GL}(\Lambda)$. (We identify G with its image in $\mathrm{GL}(\Lambda)$.) Then there exist finitely many elements $s_\alpha \in \Lambda^{\otimes}$ such that G coincides with the pointwise stabiliser of (s_α); i.e., for any R-algebra R' we have*

$$G(R') = \{g \in \mathrm{GL}(\Lambda_{R'}) \text{ such that } g(s_\alpha) = s_\alpha \ \forall \alpha\}.$$

Proof. A more general statement is proved in [29, Proposition 1.3.2]. □

Example 2.1.4. For any ring R, let Λ be a finite free R-module, and $(,) : \Lambda \otimes \Lambda \twoheadrightarrow R$ be a perfect alternating form. Then we will construct a tensor $s \in \Lambda^{\otimes}$ from $R^\times \cdot (,)$ so that its pointwise stabiliser is $\mathrm{GSp}(\Lambda, (,))$. Let $c : \mathrm{GSp}(\Lambda, (,)) \to \mathbb{G}_m$ be the similitude character, and let $R(c)$ be R as a R-module equipped with the $\mathrm{GSp}(\Lambda, (,))$-action via c. Then $(,)$ induces a $\mathrm{GSp}(\Lambda, (,))$-equivariant morphism $\Lambda \otimes \Lambda \twoheadrightarrow R(c)$.

Now, we switch the role of Λ and Λ^*, so $(,)$ induces $\Lambda^* \otimes \Lambda^* \twoheadrightarrow R(c)^*$. We then obtain a $\mathrm{GSp}(\Lambda, (,))$-equivariant section $R(c) \hookrightarrow \Lambda \otimes \Lambda$. Post-composing this section, we may view ψ as an $\mathrm{GSp}(\Lambda, (,))$-equivariant endomorphism of $\Lambda \otimes \Lambda$, so it defines a tensor $s \in \Lambda^{\otimes 2} \otimes \Lambda^{*\otimes 2}$. Note that replacing $(,)$ with any $R^\times$-multiple does not modify s, and the pointwise stabiliser of s in $\mathrm{GL}_{\mathbb{Z}_p}(\Lambda)$ is clearly $\mathrm{GSp}(\Lambda, (,))$.

2.2. $\{\mu\}$-filtrations.

Definition 2.2.1. Let R be any ring and M a finitely generated projective R-module. For a cocharacter $\mu : \mathbb{G}_m \to \mathrm{GL}_R(M)$, we say that a grading $\mathrm{gr}^\bullet M$ is *induced by* μ if the $\mathbb{G}_m$-action on M via μ leaves each grading stable, and the resulting $\mathbb{G}_m$-action on $\mathrm{gr}^a(M)$ is given by

$$\mathbb{G}_m \xrightarrow{z \mapsto z^{-a}} \mathbb{G}_m \xrightarrow{z \mapsto z\,\mathrm{id}} \mathrm{GL}(\mathrm{gr}^a(M)).$$

If $\mathrm{gr}^\bullet M$ is induced by μ, then for any map $f : R \to R'$ the scalar extension of the grading on $R' \otimes_R M$ is induced by the pull back $f^*\mu$.

In the defition, we chose the sign so that it is compatible with the standard sign convention in the theory of Shimura varieties (as in [15, 36]).

Let G be a connected reductive group over $\mathbb{Z}_p$. We choose $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ with faithful G-action, and finitely many tensors $(s_\alpha) \subset \Lambda^{\otimes}$ which define G (in the sense of Proposition 2.1.3). Let $\mathfrak{X}$ be a formal scheme over $\mathrm{Spf}\, W$, and $\mathscr{E}$ a finite-rank locally free $\mathcal{O}_{\mathfrak{X}}$-module (i.e., a vector bundle on $\mathfrak{X}$). Let $(t_\alpha) \subset \Gamma(\mathfrak{X}, \mathscr{E}^{\otimes})$ be finitely many global sections.

We will introduce a notion of filtrations on $\mathscr{E}$ which étale-locally admits a splitting given by some cocharacter (but such a splitting may not be defined globally); see Definition 4.6 for the relevant setting. To define such a notion, we need the following formal scheme

$$P_{\mathfrak{X}} := \underline{\mathrm{isom}}_{\mathcal{O}_{\mathfrak{X}}}\big([\mathscr{E}, (t_\alpha)], [\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Z}_p} \Lambda, (1 \otimes s_\alpha)]\big), \tag{2.2.2}$$

classifying isomorphisms of vector bundles on $\mathfrak{X}$ matching t_α and $1 \otimes s_\alpha$. Indeed, it is a closed formal subscheme of $\underline{\mathrm{isom}}_{\mathcal{O}_{\mathfrak{X}}}(\mathscr{E}, \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Z}_p} \Lambda)$, which is a $\mathrm{GL}(\Lambda)$-torsor over $\mathfrak{X}$ (hence, it is a formal scheme). Note that $P_{\mathfrak{X}}$ is a G-pretorsor, but it does not have to be flat over $\mathfrak{X}$.

Definition 2.2.3. Let $\mu : \mathbb{G}_m \to G_W$ be a cocharacter, and let $\{\mu\}$ denote the $G(W)$-conjugacy class of μ. A filtration $\mathrm{Fil}^\bullet \mathscr{E}$ of $\mathscr{E}$ is called a $\{\mu\}$-*filtration* (with respect to (t_α)) if the following conditions are satisfied:

(1) $P_{\mathfrak{X}}$ is a G-torsor; i.e., there exist étale-local sections $\varsigma_\xi : \mathfrak{Y}_\xi \to P_{\mathfrak{X}}$ for some étale covering $\{\mathfrak{Y}_\xi\}$ of $\mathfrak{X}$. Note that ς_ξ can be understood as an isomorphism $\mathscr{E}_{\mathfrak{Y}_\xi} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{Y}_\xi} \otimes_{\mathbb{Z}_p} \Lambda$ matching $(t_\alpha)_{\mathfrak{Y}_\xi}$ and $(1 \otimes s_\alpha)$, and using ς_ξ we can embed $G_{\mathfrak{Y}_\xi}$ into $\mathrm{GL}(\mathscr{E}_{\mathfrak{Y}_\xi})$.
(2) For some étale-local section $\{\varsigma_\xi\}$ as above (which induce an embedding $G_{\mathfrak{Y}_\xi} \hookrightarrow \mathrm{GL}(\mathscr{E}_{\mathfrak{Y}_\xi})$), the filtration $(\mathrm{Fil}^\bullet \mathscr{E})_{\mathfrak{Y}_\xi}$ of $\mathscr{E}_{\mathfrak{Y}_\xi}$ is induced from the conjugate of the cocharacter μ over $\mathfrak{Y}_\xi$ by some element $g \in G(\mathfrak{Y}_\xi)$.

It is clear from the definition that the notion of $\{\mu\}$-filtrations only depend on the conjugacy class $\{\mu\}$ of μ. Note also that the definition of $\{\mu\}$-filtrations is intrinsic to $\mathrm{Fil}^\bullet \mathscr{E}$ and (t_α); i.e., independent of the choice of (†) and $\{\mathfrak{Y}_\xi\}$. To see the independence of the choice of (ς_ξ) (for the same choice of $\{\mathfrak{Y}_\xi\}$), note that different choices of ς_ξ modify the embedding $G_{\mathfrak{Y}_\xi} \xrightarrow{\sim} \mathrm{GL}(\mathscr{E}_{\mathfrak{Y}_\xi})$ by some inner automorphisms of G (i.e., pre-composing the conjugation by some $g' \in G(\mathfrak{Y}_\xi)$), which can be cancelled out by the choice of $g \in G(\mathfrak{Y}_\xi)$ that conjugates μ to define some splitting of $(\mathrm{Fil}^\bullet \mathscr{E})_{\mathfrak{Y}_\xi}$. Now by refining étale coverings $\{\mathfrak{Y}_\xi\}$, we obtain the desired independence claim.

Remark 2.2.4. An equivalent definition of $\{\mu\}$-filtration is given in [48, Definition 3.6]. Let $\mathrm{Fil}^\bullet_\mu$ denote the filtration of $W \otimes_{\mathbb{Z}_p} \Lambda$ induced by some μ in $\{\mu\}$ (*cf.* Definition 2.2.1), and let $P^\mu \subset G_W$ be the stabiliser of $\mathrm{Fil}^\bullet_\mu$. Then the filtration $\mathrm{Fil}^\bullet \mathscr{E}$ is a $\{\mu\}$-filtration with respect to (t_α) if and only if the (étale-locally defined) isomorphisms $\mathscr{E} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}} \otimes \Lambda$ matching $((t_\alpha), \mathrm{Fil}^\bullet \mathscr{E})$ with $((1 \otimes s_\alpha), \mathcal{O}_{\mathfrak{X}} \otimes \mathrm{Fil}^\bullet_\mu)$ form a P^μ-torsor.

Remark 2.2.5. If $G = \mathrm{GL}(\Lambda)$ and $\mathscr{E}$ is a vector bundle over $\mathfrak{X}$ with rank equal to $\mathrm{rank}_{\mathbb{Z}_p}(\Lambda)$, then a filtration $\mathrm{Fil}^\bullet \mathscr{E}$ of $\mathscr{E}$ is a $\{\mu\}$-filtration for some $\{\mu\}$ if and only if each of the graded pieces $\mathrm{gr}^i \mathscr{E}$ is of constant rank. (And one can write down $\{\mu\}$ uniquely up to conjugation from the ranks of $\mathrm{gr}^i \mathscr{E}$.)

In the setting of Definition 2.2.3, let $\mathtt{Fl}^{\mathscr{E},(t_\alpha)}_{G,\{\mu\}}$ denote the functor on formal schemes over $\mathfrak{X}$, which associates to $\mathfrak{Y} \xrightarrow{f} \mathfrak{X}$ the set of $\{\mu\}$-filtration of $f^*\mathscr{E}$ with respect to (f^*t_α). We write $\mathtt{Fl}^{\mathscr{E}}_{\{\mu\}} := \mathtt{Fl}^{\mathscr{E},\emptyset}_{\mathrm{GL}(\Lambda),\{\mu\}}$, and we use the same letter to denote the the formal scheme representing the functor, which is relatively projective and smooth over $\mathfrak{X}$.

Lemma 2.2.6. *Assume that $P_{\mathfrak{X}}$ (2.2.2) is a G-torsor. Then $\mathtt{Fl}^{\mathscr{E},(t_\alpha)}_{G,\{\mu\}}$ can be represented by a closed formal subscheme of $\mathtt{Fl}^{\mathscr{E}}_{\{\mu\}}$ which is smooth over $\mathfrak{X}$ with nonempty geometrically connected fibres.*

Proof. The claim is obvious if the torsor $P_{\mathfrak{X}}$ (2.2.2) is trivial; in this case the representing formal scheme is isomorphic to a certain flag variety for G. The claim is obvious if the torsor $P_{\mathfrak{X}}$ splits Zariski-locally; indeed, formal schemes that Zariski-locally represent the functor $\mathtt{Fl}^{\mathscr{E},(t_\alpha)}_{G,\{\mu\}}$ glue together because the notion of $\{\mu\}$-filtrations is independent of auxiliary choices involved. This shows the lemma when $G = \mathrm{GL}(\Lambda)$.

Now, consider the natural inclusion $\mathtt{Fl}^{\mathscr{E},(t_\alpha)}_{G,\{\mu\}} \hookrightarrow \mathtt{Fl}^{\mathscr{E}}_{\{\mu\}}$. Etale-locally on $\mathfrak{X}$ this morphism of functors is representable by a closed immersion, and it respects the étale descent datum. By effectivity of étale descent for closed immersions, the lemma now follows. □

Let us finish the section with the following trivial but useful lemma:

Lemma 2.2.7. *Let $\mathscr{E}/\mathfrak{X}$ and (t_α) be as in Definition 2.2.3, and let* $\mathrm{Fil}^\bullet\mathscr{E}$ *be a* $\{\mu\}$*-filtration on* $\mathscr{E}$*. Then we have* $t_\alpha \in \Gamma(\mathfrak{X}, \mathrm{Fil}^0\mathscr{E}^\otimes)$ *for each* α*.*

Proof. Since the claim is étale-local on $\mathfrak{X}$, we may assume that the torsor $P_{\mathfrak{X}}$ (2.2.2) is trivial. Fixing a trivialisation of $P_{\mathfrak{X}}$ we may assume that the filtration $\mathrm{Fil}^\bullet\mathscr{E}$ is given by a cocharacter $\mu : \mathbb{G}_m \to G_{\mathfrak{X}}$. This means that $\mathrm{Fil}^\bullet\mathscr{E}$ admits a splitting by the weight decomposition $\mathrm{gr}^\bullet_\mu\mathscr{E}$ using the $\mathbb{G}_m$-action by μ. In particular, $\mathrm{gr}^0_\mu\mathscr{E}$ is precisely the $\mathbb{G}_m$-invariance (with respect to μ). On the other hand, the $\mathbb{G}_m$-action fixes each of t_α as the cocharacter μ factors through the pointwise stabiliser of (t_α). It follows that $t_\alpha \in \mathrm{gr}^0_\mu\mathscr{E}$ for each α. □

2.3. **Review on p-divisible groups and crystalline Dieudonné theory.** Throughout the paper, κ be an algebraically closed field of characteristic $p > 0$ unless stated otherwise. (Most of the time, there is no harm to set $\kappa := \overline{\mathbb{F}}_p$). We will set $W := W(\kappa)$ and $K_0 := W[\frac{1}{p}]$. Let σ denote the Witt vector Frobenius map on W and K_0.

We will only consider *compatible* PD thickenings; i.e., the PD structure is required to be compatible with the standard PD structure on $p\mathbb{Z}_p$.

If B is a $\mathbb{Z}_p$-algebra and $\mathfrak{b} \subset B$ is a PD ideal, then for any $x \in \mathfrak{b}$ we let $x^{[i]}$ denote the ith divided power.

Definition 2.3.1. We define the *isogeny category* of p-divisible groups over a scheme $\mathfrak{X}$ over $\mathrm{Spf}\,\mathbb{Z}_p$ as follows:

- objects are p-divisible groups over $\mathfrak{X}$
- morphisms $\iota : X \dashrightarrow X'$ are global sections of the following Zariski sheaf over $\mathfrak{X}$:
$$\mathrm{QHom}_R(X, X') := \mathrm{Hom}(X, X') \otimes_{\mathbb{Z}} \mathbb{Q}.$$
If $\mathfrak{X}$ is quasi-compact (for example, $\mathfrak{X} = \mathrm{Spec}\,R$ for $R \in \mathrm{Nilp}_W$) then morphisms can be understood as equivalence classes of diagrams of the form
$$X \xleftarrow{[p^n]} X \xrightarrow{\iota'} X'$$
where the equivalence relation is defined by "calculus of fractions". We also say that ι is *defined up to isogeny*, and often write $\iota = \frac{1}{p^n}\iota'$ or $\iota' = p^n\iota$.

We will use dashed arrows $X \dashrightarrow X'$ to denote morphisms defined up to isogeny. By *quasi-isogeny* $\iota : X \dashrightarrow X'$, we mean an isomorphism in the isogeny category; i.e., an invertible global section of $\mathrm{QHom}_R(X, X')$. Let $\mathrm{Qisg}_R(X, X')$ be the set of quasi-isogenies. A quasi-isogeny ι is called an *isogeny* if ι is an actual map of p-divisible groups.

We define the *height* $h(\iota)$ of an isogeny $\iota : X \to X'$ to be a locally constant function on $\mathfrak{X}$ so that over any connected component of $\mathfrak{X}$, the order of $\ker(\iota)$ is $p^{h(\iota)}$. We extend the definition of the *height* to a quasi-isogenies by
$$h(p^{-n}\iota) := h(\iota) - h([p^n])$$
on each component, where ι is an isogeny.

When p is nilpotent on the base, $\mathrm{Qisg}_R(X, X')$ satisfies the "rigidity property" analogous to rigidity of crystals; namely, for any $B \twoheadrightarrow R$ with nilpotent kernel, and p-divisible groups X and X' over B, the natural morphism
$$\mathrm{Hom}_B(X, X')[\tfrac{1}{p}] \to \mathrm{Hom}_R(X_R, X'_R)[\tfrac{1}{p}] \tag{2.3.2}$$

is bijective.(*Cf.* [16].) In particular, for a formal scheme $\mathfrak{X}$ locally formally of finite type over W, the isogeny categories of p-divisible groups over $\mathfrak{X}$ and $\mathfrak{X}_{\mathrm{red}}$ are equivalent via the pull back functor.

Let $\mathfrak{X}$ be a formal scheme over $\operatorname{Spf}\mathbb{Z}_p$, and set $\overline{\mathfrak{X}} := \mathfrak{X}\times_{\operatorname{Spf}\mathbb{Z}_p}\operatorname{Spec}\mathbb{F}_p$. Let $i_{\mathrm{CRIS}} := (i_{\mathrm{CRIS},*}, i^*_{\mathrm{CRIS}}) : (\overline{\mathfrak{X}}/\mathbb{Z}_p)_{\mathrm{CRIS}} \to (\mathfrak{X}/\mathbb{Z}_p)_{\mathrm{CRIS}}$ be the morphism of topoi induced from the closed immersion $\overline{\mathfrak{X}} \hookrightarrow \mathfrak{X}$. Then $i_{\mathrm{CRIS},*}$ and i^*_{CRIS} induce quasi-inverse exact equivalences of categories between the categories of crystals of quasi-coherent (respectively, finite locally free) $\mathcal{O}_{\overline{\mathfrak{X}}/\mathbb{Z}_p}$-modules and $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$-modules. (This follows from [11, Lemma 2.1.4], which can be applied since the natural map $i_{\mathrm{CRIS},*}\mathcal{O}_{\overline{\mathfrak{X}}/\mathbb{Z}_p} \to \mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ is an isomorphism by [5, §5.17.3].) In particular, for any crystal $\mathbb{D}$ of quasi-coherent $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$-modules, we define the pull-back by the absolute Frobenius morphism $\sigma : \overline{\mathfrak{X}} \to \overline{\mathfrak{X}}$ as follows:

$$\sigma^*\mathbb{D} := i_{\mathrm{CRIS},*}(\sigma^*_{\mathrm{CRIS}} i^*_{\mathrm{CRIS}}\mathbb{D}).$$

For a p-divisible group X over $\mathfrak{X}$, we have a contravariant Dieudonné crystal[8] $\mathbb{D}(X)$ equipped with a filtration $(\operatorname{Lie} X)^* \cong \operatorname{Fil}^1_X \subset \mathbb{D}(X)_{\mathfrak{X}}$ by a direct factor as a vector bundle on $\mathfrak{X}$, where $\mathbb{D}(X)_{\mathfrak{X}}$ is the pull-back of $\mathbb{D}(X)$ to the Zariski site of $\mathfrak{X}$. We call Fil^1_X the *Hodge filtration* for X. If $\mathfrak{X} = \operatorname{Spf} R$, then we can regard the Hodge filtration as a filtration on the R-sections $\operatorname{Fil}^1_X \subset \mathbb{D}(X)(R)$. From the relative Frobenius morphism $F : X_{\overline{\mathfrak{X}}} \to \sigma^* X_{\overline{\mathfrak{X}}}$, we obtain the Frobenius morphism $F : \sigma^*\mathbb{D}(X) \to \mathbb{D}(X)$. On tensor products of $\mathbb{D}(X)$'s, we naturally extend the Frobenius structure and filtration.

We set $\mathbf{1} := \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$ and $\mathbf{1}(-1) := \mathbb{D}(\mu_{p^\infty})$. Note that $\mathbf{1} \cong \mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ with the usual Frobenius structure and $\operatorname{Fil}^1 = 0$. We define $\mathbb{D}(X)^*$ to be the $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$-linear dual with the dual filtration. (Note that the Frobenius structure on $\mathbb{D}(X)^*$ is defined "up to isogeny".) We set $\mathbf{1}(0) := \mathbf{1}$ and

$$\mathbf{1}(-c) := \mathbf{1}(-1)^{\otimes c} \ \&\ \mathbf{1}(c) := \mathbf{1}(-c)^* \text{ if } c > 0$$

For any crystal $\mathbb{D}$ with Frobenius structure and Hodge filtration, we set $\mathbb{D}(r) := \mathbb{D} \otimes \mathbf{1}(r)$ for any $r \in \mathbb{Z}$. Note that $\mathbb{D}(X^\vee) \cong \mathbb{D}(X)^*(-1)$ by [4, §5.3].

Definition 2.3.3. We define the *category of isocrystals* over $\mathfrak{X}$ as follows:

- objects are locally free $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$-modules $\mathbb{D}$; we write $\mathbb{D}[\frac{1}{p}]$ if we view $\mathbb{D}$ as an isocrystal;
- morphisms are global sections of the Zariski sheaf $\operatorname{Hom}(\mathbb{D}, \mathbb{D}')[\frac{1}{p}]$ over $\mathfrak{X}$. If $\mathfrak{X}$ is quasi-compact (for example, $\mathfrak{X} = \operatorname{Spec} R$ for $R \in \mathrm{Nilp}_W$) then morphisms can be understood as equivalence classes of diagrams of the form
$$\mathbb{D} \xleftarrow{p^n} \mathbb{D} \xrightarrow{\iota'} \mathbb{D}'$$
where the equivalence relation is defined by "calculus of fractions".

An *F-isocrystal* over $\mathfrak{X}$ is a pair $(\mathbb{D}[\frac{1}{p}], F)$ where $\mathbb{D}[\frac{1}{p}]$ is an isocrystal over $\mathfrak{X}$ and $F : \sigma^*\mathbb{D}[\frac{1}{p}] \xrightarrow{\sim} \mathbb{D}[\frac{1}{p}]$ is an isomorphism of isocrystals.

For a p-divisible group X, $\mathbb{D}(X)[\frac{1}{p}]$ can be naturally viewed as an F-isocrystal. Note also that the notion of F-isocrystals is closed under direct sums, tensor products, and duality. So $\mathbb{D}(X)^*[\frac{1}{p}]$ is an F-isocrystal although the Frobenius structure may not be defined on $\mathbb{D}(X)^*$.

We will often let $\mathbf{1}(-c)$ denote the isocrystal associated to $\mathbf{1}(-c)$, which is also an F-isocrystal by the discussion above.

[8] See [35], [34], or [4] for the construction.

Note that for any morphism up to isogeny $\iota : X \dashrightarrow X'$ of p-divisible groups over $\mathfrak{X}$, we obtain a morphism of isocrystals $\mathbb{D}(\iota) : \mathbb{D}(X')[\frac{1}{p}] \to \mathbb{D}(X)[\frac{1}{p}]$. If ι is a quasi-isogeny, then $\mathbb{D}(\iota)$ is an isomorphism of isocrystals.

We now define $\mathbb{D}(X)^{\otimes}$ by setting $\mathcal{C}$ to be the category of (integral) crystals of quasi-coherent $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$-modules and $\mathcal{D} \subset \mathcal{C}$ to be the full subcategory of finitely generated locally free objects (*cf.* Definition 2.1.2). Then the Hodge filtration on $\mathbb{D}(X)_{\mathfrak{X}}$ induces a natural filtration on $\mathbb{D}(X)^{\otimes}_{\mathfrak{X}}$, and the Frobenius morphism on $\mathbb{D}(X)$ induces an isomorphism of isocrystals $F : \sigma^*\mathbb{D}(X)^{\otimes}[\frac{1}{p}] \xrightarrow{\sim} \mathbb{D}(X)^{\otimes}[\frac{1}{p}]$. More generally, for any quasi-isogeny $\iota : X \dashrightarrow X'$ of p-divisible groups over R, $\mathbb{D}(\iota)$ extends to

$$\mathbb{D}(\iota) : \mathbb{D}(X')^{\otimes}[1/p] \xrightarrow{\sim} \mathbb{D}(X)^{\otimes}[1/p].$$

For example, $[p] : X \to X$ induces the multiplication by p on $\mathbb{D}(X)[1/p]$, and the multiplication by $1/p$ on $\mathbb{D}(X)^*[1/p]$.

Definition 2.3.4. Let X be a p-divisible group over R, and $t : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ a morphism of crystals. For a PD thickening $S \twoheadrightarrow R$ where p is nilpotent in S, we define the *section* $t(S)$ of t over S (or the S-*section* of t) to be the image of 1 under the map

$$S = \mathbf{1}(S) \xrightarrow{t} \mathbb{D}(X)(S)^{\otimes}.$$

Later, we will implicitly extend the definition to the case when S is a "p-adic PD thickening" in the sense of Definition 5.3.1.

2.4. **F-isocrystals with G-structure over κ.** We review some basic definitions and results on "F-isocrystals with G-structure" [31]. See [39, Ch.1] for a more detailed overview.

Recall that the category of quasi-coherent crystals of $\mathcal{O}_{\operatorname{Spec}\kappa/W}$-modules is equivalent to the category of W-modules by taking sections over $W = W(\kappa)$. Therefore, an F-isocrystal over κ can be regarded as a pair (D, F), where D is a K_0-vector space and $F : \sigma^*D \xrightarrow{\sim} D$ is an isomorphism. By abuse of notation, we also call (D, F) an *F-isocrystal* over κ.

Let (D, F) be a rank-n F-isocrystal over κ. By choosing a basis of D (or equivalently, by choosing a faithful algebraic action of GL_n on D), we can find $b \in \mathrm{GL}_n(K_0)$ which is the matrix representation of F (i.e., F is the linearisation of $b\sigma$). Choosing a different basis of D, b is replaced by a suitable "σ-twisted conjugate". Motivated by this, we make the following definition (*cf.* [31], [39, §1.7]):

Definition 2.4.1. Let G be a linear algebraic group over $\mathbb{Q}_p$. We say that $b, b' \in G(K_0)$ are *σ-conjugate in $G(K_0)$* if there exists $g \in G(K_0)$ such that $b' = gb\sigma(g)^{-1}$. Let $[b] \subset G(K_0)$ denote the set of σ-conjugates of $b \in G(K_0)$ in $G(K_0)$.

Let $\mathrm{Rep}_{\mathbb{Q}_p}(G)$ denote the category of finite-dimensional $\mathbb{Q}_p$-vector spaces with algebraic G-action. Then for any $b \in G(K_0)$ we can functorially associate, to any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$, an F-isocrystal $\beta_b(V) = (K_0 \otimes_{\mathbb{Q}_p} V, F)$, where F is defined as follows:

$$F : \sigma^*(K_0 \otimes_{\mathbb{Q}_p} V) \cong K_0 \otimes_{\mathbb{Q}_p} V \xrightarrow{\rho_V(b)} K_0 \otimes_{\mathbb{Q}_p} V. \tag{2.4.2}$$

Here, $\rho_V : G \to \mathrm{GL}(V)$ is the homomorphism of algebraic groups defining the G-action on V. For $b, b' \in G(K_0)$, we have $\beta_b \cong \beta_{b'}$ if b and b' are σ-conjugate in $G(K_0)$. Note that if $G = \mathrm{GL}_n/\mathbb{Q}_p$ and $V = \mathbb{Q}_p^n$ is the standard representation of GL_n, then $b \mapsto \beta_b(V)$ induces a bijection between the set of σ-conjugacy classes in $\mathrm{GL}_n(K_0)$ and the set of isomorphism classes F-isocrystals of rank n over κ.

We define the following group valued functor $J_b = J_{G,b}$ on $\mathrm{Alg}_{\mathbb{Q}_p}$:

$$J_b(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} K_0) |\; gb\sigma(g)^{-1} = b\},\ \forall R \in \mathrm{Alg}_{\mathbb{Q}_p}. \tag{2.4.3}$$

Proposition 2.4.4 ([39, Corollary 1.14]). *If G is a connected reductive group over $\mathbb{Q}_p$ then J_b can be represented by an inner form of some Levi subgroup of G.*

2.5. **Affine Deligne-Lusztig set.** From now on, we let G be a connected reductive group which is *unramified*[9] over $\mathbb{Q}_p$. We fix a reductive model $G_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$, and will often write $G = G_{\mathbb{Z}_p}$ if there is no risk of confusion. For any $\mathbb{Z}_p$-algebra R, we write G_R the base change of G over $\operatorname{Spec} R$.

Note that we have a bijection

(2.5.1)

$$\operatorname{Hom}_W(\mathbb{G}_m, G_W)/G(W) \cong \operatorname{Hom}_{K_0}(\mathbb{G}_m, G_{K_0})/G(K_0) \xrightarrow{\sim} G(W)\backslash G(K_0)/G(W)$$

induced by $\{\nu\} \mapsto G(W)p^\nu G(W)$, where $p^\nu := \nu(p) \in G(K_0)$; indeed, the first bijection is because G is split over W and the second bijection is the Cartan decomposition.

Definition 2.5.2. To $b \in G(K_0)$ and a $G(W)$-conjugacy class $\{\nu\}$ of cocharacters $\nu : \mathbb{G}_m \to G_W$, we associate the *affine Deligne-Lusztig set* as follows:

$$X^G_{\{\nu\}}(b) := \big\{g \in G(K_0) \text{ such that } g^{-1}b\sigma(g) \in G(W)p^\nu G(W)\big\}/G(W) \subset G(K_0)/G(W).$$

In the intended application, we will choose $\{\nu\}$ so that $b \in G(W)p^\nu G(W)$. If $\{\nu\}$ is chosen this way, then we write $X^G(b) := X^G_{\{\nu\}}(b)$ since it only depends on (G, b).

For $\gamma \in G(K_0)$, the left translation $gG(W) \to \gamma gG(W)$ induces

$$X^G_{\{\nu\}}(\gamma^{-1}b\sigma(\gamma)) \xrightarrow{\sim} X^G_{\{\nu\}}(b). \tag{2.5.3}$$

In particular, we obtain a natural action $J_b(\mathbb{Q}_p)$ on $X^G_{\{\nu\}}(b)$ (since $J_b(\mathbb{Q}_p) \subset G(K_0)$), and $X^G_{\{\nu\}}(b)$ only depends on the tuple $(G, [b], \{\nu\})$ up to bijection, where $[b]$ is the σ-conjugacy class of b in $G(K_0)$.

The following properties are straightforward to verify from the definition:

Lemma 2.5.4.
(1) *For any morphism $f : G \to G'$ of connected reductive group over $\mathbb{Z}_p$, we have a natural map $X^G_{\{\nu\}}(b) \to X^{G'}_{\{f\circ\nu\}}(f(b))$ induced by $gG(W) \mapsto f(g)G'(W)$. Furthermore, if f is a closed immersion, then the induced map on the affine Deligne-Lusztig sets is injective.*
(2) *For another connected reductive group G' over $\mathbb{Z}_p$, $b' \in G'(W)$ and a conjugacy class of cocharacters $\nu' : \mathbb{G}_m \to G'_W$, we have an isomorphism*

$$X^{G\times G'}_{\{(\nu,\nu')\}}(b, b') \xrightarrow{\sim} X^G_{\{\nu\}}(b) \times X^{G'}_{\{\nu'\}}(b')$$

induced by the natural projections.

In particular, $f : G \to G'$ induces $X^G(b) \to X^{G'}(f(b))$ and we have a natural isomorphism $X^{G\times G'}(b, b') \xrightarrow{\sim} X^G(b) \times X^{G'}(b')$.

Proof. The only possibly non-trivial assertion is the injectivity of the map $X^G_{\{\nu\}}(b) \to X^{G'}_{\{f\circ\nu\}}(f(b))$ when f is a closed immersion. By embedding G' into some GL_n, we may assume that $G' = \mathrm{GL}_n$. Then by Proposition 2.1.3, the following map

$$G(K_0)/G(W) \to G'(K_0)/G'(W),$$

induced by f, is injective, from which the desired injectivity follows. □

Later, we will consider pairs (G, b) which can be related to some p-divisible group in the following sense.

[9] I.e., quasi-split and split over $\mathbb{Q}_p^{\mathrm{ur}}$; or equivalently, G admits a reductive model over $\mathbb{Z}_p$.

Definition 2.5.5. We fix $b \in G(K_0)$. Assume that there exists a faithful G-representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ such that we have the following W-lattice

$$\mathbf{M}_b^\Lambda := W \otimes_{\mathbb{Z}_p} \Lambda^* \subset \beta_b(\Lambda^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

satisfies $p\mathbf{M}_b^\Lambda \subset F(\sigma^*\mathbf{M}_b^\Lambda) \subset \mathbf{M}_b^\Lambda$. (The existence of such Λ is a restrictive condition on G and b. See Example 2.5.11 for the reason to dualise Λ in the definition of $\mathbf{M}_b^\Lambda$.) Then for such b and Λ, we let $\mathbb{X}_b^\Lambda$ denote the p-divisible group over κ such that $\mathbb{D}(\mathbb{X}_b^\Lambda)(W) \cong \mathbf{M}_b^\Lambda$.

Note that if $G = \mathrm{GL}_n$ and $\Lambda = \mathbb{Z}_p^n$ is the standard representation, then b is the transpose-inverse of the matrix representation of the Frobenius operator on $\mathbf{M}_b^\Lambda$ (since $\mathbf{M}_b^\Lambda := W \otimes_{\mathbb{Z}_p} \Lambda^*$).

Just like the abelian variety which appears as a complex point of some Shimura variety of Hodge type has certain Hodge cycles, the p-divisible group $\mathbb{X}$ as in Definition 2.5.5 has certain crystalline tensors from the fact that $b \in G(K_0)$. The following lemma is straigntforward.

Lemma 2.5.6. *Let (G, b) and $\mathbb{X} := \mathbb{X}_b^\Lambda$ be as in Definition 2.5.5. Let $s \in \Lambda^\otimes$ be such that for any $\mathbb{Z}_p$-algebra R, $1 \otimes s \in R \otimes \Lambda^\otimes$ is fixed by $G(R)$. (For example, we may take $s = s_\alpha$ for s_α as in Proposition 2.1.3.) Consider $1 \otimes s \in W \otimes \Lambda^\otimes = \mathbb{D}(\mathbb{X})(W)^\otimes$. Then $1 \otimes s \in \mathbb{D}(\mathbb{X})(W)^\otimes[\frac{1}{p}]$ is fixed by F, where we extend F naturally to $\mathbb{D}(\mathbb{X})(W)^\otimes[\frac{1}{p}]$.*

We fix finitely many tensors $(s_\alpha) \subset \Lambda^\otimes$ which defines G in the sense of Proposition 2.1.3. We set $(\mathbf{t}_\alpha) := (1 \otimes s_\alpha) \in W \otimes_{\mathbb{Z}_p} \Lambda^\otimes \cong \mathbb{D}(\mathbb{X})(W)^\otimes$.

Lemma 2.5.7. *Let (G, b) and Λ be as in Definition 2.5.5, and set $\mathbb{X} := \mathbb{X}_b^\Lambda$. Let $\{\mu\}$ be a $G(W)$-conjugacy class of cocharacters $\mathbb{G}_m \to G_W$. Then the following hold*

1. *If $\mu : \mathbb{G}_m \to G_W$ is such that $b \in G(W)p^{\sigma^*\mu^{-1}}$, then the Hodge filtration $\mathrm{Fil}^1_{\mathbb{X}} \subset \mathbb{D}(\mathbb{X})(\kappa) = \kappa \otimes_{\mathbb{Z}_p} \Lambda^*$ is induced by μ. (Recall that we identified $\mathbb{D}(\mathbb{X})(W) = \mathbf{M}_b^\Lambda = W \otimes_{\mathbb{Z}_p} \Lambda^*$.)*
2. *The Hodge filtration $\mathrm{Fil}^1_{\mathbb{X}} \subset \mathbb{D}(\mathbb{X})(\kappa)$ is a $\{\mu\}$-filtration with respect to the image of $(\mathbf{t}_\alpha)$ in $\mathbb{D}(\mathbb{X})(\kappa)^\otimes$ (*cf.* Definition 2.2.3) if and only if we have $b \in G(W)p^{\sigma^*\mu^{-1}}G(W)$.*
3. *The image of $(\mathbf{t}_\alpha)$ in $\mathbb{D}(\mathbb{X})(\kappa)^\otimes$ lies in the 0th filtration with respect to the Hodge filtration.*

Proof. Let us first show (1). Assume that $b \in G(W)p^\nu$ for some $\nu : \mathbb{G}_m \to G_W$. Since the Hodge filtration $\mathrm{Fil}^1_{\mathbb{X}} \subset \mathbb{D}(\mathbb{X})(\kappa)$ is the kernel of $b(\sigma \otimes \mathrm{id}_{\Lambda^*})$ acting on $\kappa \otimes_{\mathbb{Z}_p} \Lambda^* = \mathbb{D}(\mathbb{X})(\kappa)$, we have

$$(b\sigma)^{-1}(p\mathbf{M}_b^\Lambda) = \sigma^{-1}((p^{-1})^\nu \cdot p\mathbf{M}_b^\Lambda) = \mathrm{gr}^1_{(\sigma^{-1})^*\nu^{-1}} \mathbf{M}_b^\Lambda + p\mathbf{M}_b^\Lambda,$$

whose image in $\mathbf{M}_b^\Lambda / p\mathbf{M}_b^\Lambda \cong \mathbb{D}(\mathbb{X})(\kappa)$ is the kernel of $b(\sigma \otimes \mathrm{id}_{\Lambda^*})$. This shows that $\mu := (\sigma^{-1})^*\nu^{-1}$ induces the Hodge filtration $\mathrm{Fil}^1_{\mathbb{X}}$.

Let us show (2). Choose a cocharacter $\nu : \mathbb{G}_m \to G_W$ such that $b \in G(W)p^\nu G(W)$; recall that such ν is unique up to $G(W)$-conjugate. By replacing ν by a suitable $G(W)$-conjugate, we may assume that $b = gp^\nu$ for some $g \in G(W)$; indeed, if $b = g_1 p^\nu g_2$ for $g_1, g_2 \in G(W)$, then we take $g = g_1 g_2$ and replace ν with $g_2^{-1}\nu g_2$. Now from (1) it follows that the Hodge filtration $\mathrm{Fil}^1_{\mathbb{X}}$ is a $\{\mu\}$-filtration for $\mu = (\sigma^{-1})^*\nu^{-1}$. Now, by uniqueness of $\mathrm{Hom}_W(\mathbb{G}_m, G_W)/G(W)$ is a constant sheaf on $\mathrm{Spec}\, W$ as G_W is a split reductive group, there exists exactly one $G(W)$-conjugacy class of cocharacters $\{\mu\}$ over W such that $\mathrm{Fil}^1_{\mathbb{X}}$ is a $\{\mu\}$-filtration. So if $\mathrm{Fil}^1_{\mathbb{X}}$ is a $\{\mu\}$-filtration, then we have $\{\mu\} = \{(\sigma^{-1})^*\nu^{-1}\}$, which proves (2).

By Lemma 2.2.7, (3) follows from (2). □

Remark 2.5.8. We use the notation in Lemma 2.5.7. Then Lemma 2.5.7 asserts that for any $\mathbb{X} := \mathbb{X}_b^\Lambda$ with $b \in G(K_0)$, there exists a unique conjugacy class of cocharacters $\{\mu\}$ such that the Hodge filtration $\mathrm{Fil}^1_{\mathbb{X}}$ is a $\{\mu\}$-filtration.

Since there is no obstruction of lifting $\{\mu\}$-filtrations (*cf.* Lemma 2.2.6), there exists a $\{\mu\}$-filtration $\mathrm{Fil}^1_{\widetilde{\mathbb{X}}} \subset \mathbf{M}_b^\Lambda$ lifting $\mathrm{Fil}^1_{\mathbb{X}}$. If $p > 2$, then such a lift $\mathrm{Fil}^1_{\widetilde{\mathbb{X}}}$ gives rise to a p-divisible group $\widetilde{\mathbb{X}}$ over W. For such a lift $\widetilde{\mathbb{X}}$, the tensors $(\mathbf{t}_\alpha) \subset \mathbb{D}(\widetilde{\mathbb{X}})(W)^\otimes = (\mathbf{M}_b^\Lambda)^\otimes$ lie in the 0th filtration with respect to the Hodge filtration for $\widetilde{\mathbb{X}}$; *cf.* Lemma 2.2.7.

We consider (G,b) and Λ as in Definition 2.5.5, and choose finitely many tensors $(s_\alpha) \subset \Lambda^\otimes$ defining $G \subset \mathrm{GL}(\Lambda)$ as in Proposition 2.1.3. Then to (G,b) and Λ, we can associate $(\mathbb{X},(\mathbf{t}_\alpha))$ and an isomorphism $\varsigma : W \otimes_{\mathbb{Z}_p} \Lambda^* \cong \mathbb{D}(\mathbb{X})(W)$, where $\mathbb{X} := \mathbb{X}_b^\Lambda$ is a p-divisible group over κ, $(\mathbf{t}_\alpha) \subset \mathbb{D}(\mathbb{X})(W)^\otimes$ are F-invariant tensors (*cf.* Lemma 2.5.6), and ς is a W-linear isomorphism which matches $(1\otimes s_\alpha)$ and $(\mathbf{t}_\alpha)$. Note that we can recover (G,b) from $(\mathbb{X},(\mathbf{t}_\alpha),\varsigma)$. In the setting of Example 2.1.4 (when $G = \mathrm{GSp}_{2g}$ and $\Lambda = \mathbb{Z}_p^{2g}$ is the standard representation), we can interpret $(\mathbb{X},(\mathbf{t}_\alpha))$ as a principally polarised p-divisible group.

We will now interpret $X^G(b) = X^G_{\{\sigma^*\mu^{-1}\}}(b)$ in terms of quasi-isogenies of p-divisible groups with F-invariant tensors over κ. For $gG(W) \in X^G(b)$, we pick a representative $g \in gG(W)$ and set $b' := g^{-1}b\sigma(g)$. Consider $\mathbf{M}_{b'}^\Lambda = W \otimes_{\mathbb{Z}_p} \Lambda^*$ with F given by $b' \in G(K_0)$, and F-invariant tensors $(\mathbf{t}'_\alpha) = (1\otimes s_\alpha) \subset (\mathbf{M}_{b'}^\Lambda)^\otimes$. The condition $b' \in G(W)p^{\sigma^*\mu^{-1}}G(W)$ implies that $\mathbf{M}_{b'}^\Lambda$ corresponds to a p-divisible group $\mathbb{X}' := \mathbb{X}_{b'}^\Lambda$, whose Hodge filtration is a $\{\mu\}$-filtration with respect to $(\mathbf{t}'_\alpha)$ by Lemma 2.5.7. We also obtain a quasi isogeny $\iota : \mathbb{X} \dashrightarrow \mathbb{X}'$ corresponding to

$$\mathbf{M}_{b'}^\Lambda[\tfrac{1}{p}] = K_0 \otimes_{\mathbb{Z}_p} \Lambda^* \xrightarrow{g} K_0 \otimes_{\mathbb{Z}_p} \Lambda^* = \mathbf{M}_b^\Lambda[\tfrac{1}{p}],$$

which matches $(\mathbf{t}'_\alpha) \subset (\mathbf{M}_{b'}^\Lambda)^\otimes$ with $(\mathbf{t}_\alpha) \subset (\mathbf{M}_b^\Lambda)^\otimes$. The tuple $(\mathbb{X}',(\mathbf{t}'_\alpha),\iota)$ only depends on $gG(W)$ up to (the natural notion of) isomorphism.

Proposition 2.5.9. *The map defined above gives a bijection from $X^G(b)$ to the set of isomorphism classes of tuples $(\mathbb{X}',(\mathbf{t}'_\alpha),\iota)$ which satisfies the following*

- *$\mathbb{X}'$ is a p-divisible group over κ and $(\mathbf{t}'_\alpha) \subset \mathbb{D}(\mathbb{X}')(W)^\otimes$ are F-invariant tensors, such that there exists an isomorphism $\varsigma' : W \otimes_{\mathbb{Z}_p} \Lambda^* \xrightarrow{\sim} \mathbb{D}(\mathbb{X}')(W)$ that matches $(1\otimes s_\alpha)$ and $(\mathbf{t}'_\alpha)$, and the Hodge filtration $\mathrm{Fil}^1_{\mathbb{X}'}$ is a $\{\mu\}$-filtration with respect to (t'_α).*
- *$\iota : \mathbb{X} \dashrightarrow \mathbb{X}'$ is a quasi-isogeny such that $\mathbb{D}(\iota) : \mathbb{D}(\mathbb{X}')(W)[\frac{1}{p}] \xrightarrow{\sim} \mathbb{D}(\mathbb{X})(W)[\frac{1}{p}]$ matches $(\mathbf{t}'_\alpha)$ with $(\mathbf{t}_\alpha)$.*

Proof. We will define the inverse map. Let $(\mathbb{X}',(\mathbf{t}'_\alpha),\iota)$ be a tuple as in the statement. Then choosing $\varsigma' : W \otimes_{\mathbb{Z}_p} \Lambda^* \xrightarrow{\sim} \mathbb{D}(\mathbb{X}')(W)$ which matches $(1\otimes s_\alpha)$ and $(\mathbf{t}'_\alpha)$, we can obtain $b' \in G(K_0)$ such that $\mathbb{X}' \cong \mathbb{X}_{b'}^\Lambda$, and $g \in G(K_0)$ such that $\mathbb{D}(\iota)$ coincides with

$$\mathbb{D}(\mathbb{X}')(W)[\tfrac{1}{p}] \xrightarrow[\sim]{(\varsigma')^{-1}} K_0 \otimes_{\mathbb{Z}_p} \Lambda^* \xrightarrow[\sim]{g} K_0 \otimes_{\mathbb{Z}_p} \Lambda^* \xrightarrow[\sim]{\varsigma} \mathbb{D}(\mathbb{X})(W)[\tfrac{1}{p}].$$

It then follows that $b' = g^{-1}b\sigma(g)$, and since the Hodge filtration for $\mathbb{X}'$ is a $\{\mu\}$-filtration, we have $b' \in G(W)p^{\sigma^*\mu^{-1}}G(W)$.

The choice of ς' is not canonical, but any other choice of ς' is of the form $\varsigma' \circ h$ for some $h \in G(W)$, which would replace g by gh. Since any tuple $(\mathbb{X}',(\mathbf{t}'_\alpha),\iota)$ is isomorphic to the one associated to $\mathbb{X}' := \mathbb{X}_{b'}^\Lambda$, one can easily check that map sending the isomorphism class of $(\mathbb{X}',(\mathbf{t}'_\alpha),\iota)$ to $gG(W)$ is well defined, and is the inverse map as desired. $\square$

The following notion, which is the local analogue of Hodge-type Shimura data, turns out to provide a group-theoretic invariant associated to the equivalence class of $X^G(b)$ given by (2.5.3):

Definition 2.5.10. Let G be a connected reductive group over $\mathbb{Z}_p$, and $b \in G(K_0)$. We associate, to any (G, b), a tuple $(G, [b], \{\mu^{-1}\})$, where $[b]$ is the σ-conjugacy class of b in $G(K_0)$ and $\{\mu^{-1}\}$ is the unique $G(W)$-conjugacy class of cocharacters $\mathbb{G}_m \to G_W$ such that $b \in G(W)p^{\sigma^*\mu^{-1}}G(W)$. (The unique existence of such $\{\mu\}$ is by the Cartan decomposition.)

If there is a faithful representation $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G)$ as in Definition 2.5.5 (i.e., there exists a p-divisible group $\mathbb{X}_b^\Lambda$, then we call the associated triple $(G, [b], \{\mu^{-1}\})$ an *(unramified) integral local Shimura datum of Hodge type*. We take the obvious notion of morphisms.

To an unramified integral Hodge-type local Shimura datum, one can easily associate a local Shimura datum as defined in Rapoport and Viehmann by replacing G with $G_{\mathbb{Q}_p}$ [38, Definition 5.1]. (Since G is split over W, geometric conjugacy classes of cocharacters can be viewed as $G(W)$-conjugacy classes of cocharacters defined over W.)

If $(G, [b], \{\mu^{-1}\})$ is an unramified integral Hodge-type local Shimura datum via Λ, then the natural inclusion $G \hookrightarrow \mathrm{GL}(\Lambda)$ induces a morphism $(G, [b], \{\mu^{-1}\}) \to (\mathrm{GL}(\Lambda), [b], \{\mu^{-1}\})$ of unramified integral Hodge-type local Shimura data.

Let $(G', [b'], \{\mu'^{-1}\})$ be another unramified integral local Shimura datum of Hodge type induced by (G', b') and a faithful G-representation Λ' (giving rise to a p-divisible group $\mathbb{X}_{b'}^{\Lambda'}$). Then the product $(G \times G', [(b, b')], \{(\mu^{-1}, \mu'^{-1})\})$ is again an integral unramified Hodge-type local Shimura datum. (Indeed, we can associate the following p-divisible group $\mathbb{X}_{(b,b')}^{\Lambda\times\Lambda'} \cong \mathbb{X}_b^\Lambda \times \mathbb{X}_{b'}^{\Lambda'}$.)

Example 2.5.11. Assume that G comes from a reductive group over $\mathbb{Z}_{(p)}$, which we also denote by G. Assume that there exists a Hodge-type Shimura datum $(G_{\mathbb{Q}}, \mathfrak{H})$. Let $\mathsf{K}_p := G(\mathbb{Z}_p)$ be the hyperspecial maximal subgroup of $G(\mathbb{Q}_p)$. By construction, an integral canonical model $\mathscr{S}_{\mathsf{K}_p}(G_{\mathbb{Q}}, \mathfrak{H})$, when it exists, carries a "universal" abelian scheme depending on some auxiliary choices. See [44, §1.4] or [29, §2.3] for more details on the construction. Pick any point valued in $W := W(\overline{\mathbb{F}}_p)$, and let $\widetilde{\mathbb{X}}$ be the p-divisible group associated to the corresponding abelian scheme over W. Let $\Lambda := T(\widetilde{\mathbb{X}})$ denote the (integral) Tate module. Then by construction there exist finitely many "étale Tate tensors" $(s_\alpha) \subset \Lambda^\otimes$ whose pointwise stabiliser is $G_{\mathbb{Z}_p}$. By a conjecture of Milne (proved independently in [45, Main Theorem 1.2] and [29, Proposition 1.3.4]) there exists a W-isomorphism

$$W \otimes_{\mathbb{Z}_p} \Lambda^* \cong \mathbb{D}(\widetilde{\mathbb{X}})(W)$$

which takes $(1 \otimes s_\alpha)$ to the crystalline Tate tensors $(\mathbf{t}_\alpha)$ corresponding to (s_α) via crystalline comparison isomorphism. Choosing such an isomorphism, we can extract $b \in G(K_0)$ from the matrix representation of F.

As $\mathbf{t}_\alpha : \mathbf{1} \to (\mathbf{M}_b^\Lambda)^\otimes$ are morphisms of "strongly divisible modules", we may apply Wintenberger's theory of canonical splitting [47, Théorème 3.1.2] and obtain a unique cocharacter $\mu : \mathbb{G}_m \to G_W$ such that μ gives the Hodge filtration for $\widetilde{\mathbb{X}}$ and $b \in G(W)p^\nu$ with $\nu = \sigma^*\mu^{-1}$; *cf.* Lemma 2.5.7. Therefore, the triple $(G_{\mathbb{Z}_p}, [b], \{\mu^{-1}\})$ obtained from $\mathscr{S}_{\mathsf{K}_p}(G_{\mathbb{Q}}, \mathfrak{H})(W)$ is an unramified Hodge-type local Shimura datum in the sense of Definition 2.5.5. Note that the geometric conjugacy class $\{\mu\}$ corresponds to the geometric conjugacy class associated to the Shimura datum $(G_{\mathbb{Q}}, \mathfrak{H})$.

3. Faltings's construction of universal deformation

In this section we review Faltings' explicit constructions of a "universal" deformation of p-divisible groups with Tate tensors (depending on some auxiliary choices). We will crucially use this formal local construction to obtain the natural closed formal subscheme of a Rapoport-Zink space where some natural crystalline Tate tensors are defined. All the results in this section (except Proposition 3.8) can be found in [17, §7] and [37, §4].

Let κ be an algebraically closed field of characteristic $p > 2$, with $W := W(\kappa)$. We consider a p-divisible group $\mathbb{X}$ over κ. Recall that $\mathfrak{AR}_W$ is the category of artin local W-algebra with residue field κ.

Definition 3.1. We define a functor $\mathtt{Def}_{\mathbb{X}} : \mathfrak{AR}_W \to (\mathbf{Sets})$ by

$$\mathtt{Def}_{\mathbb{X}}(R) := \{(X_{/R}, X_\kappa \cong \mathbb{X})\}_{/\cong}$$

for any $R \in \mathfrak{AR}_W$. We will often suppress $X_\kappa \cong \mathbb{X}$, and write $X \in \mathtt{Def}_{\mathbb{X}}(R)$.

3.2. **Explicit construction in characteristic p.** The functor $\mathtt{Def}_{\mathbb{X}}$ can be prorepresented by the formal power series ring over W with $d \cdot d^\vee$ variables, where d and $d^\vee$ are respectively the dimensions of $\mathbb{X}$ and $\mathbb{X}^\vee$; *cf.* [24, Corollaire 4.8(i)]. Faltings made such an identification via explicitly describing a "universal Dieudonné crystal" when $p > 2$; *cf.* [37, §4.8], [29, §1.5], which we recall now.

For a p-divisible group $\mathbb{X}$ over κ, we write $(\mathbf{M}, F) := \mathbb{D}(\mathbb{X})(W)$ for the contravariant Dieudonné module. Choosing a lift $\widetilde{\mathbb{X}}$ over $\operatorname{Spf} W$, we obtain a direct factor $\operatorname{Fil}^1_{\widetilde{\mathbb{X}}} \subset \mathbb{D}(\widetilde{\mathbb{X}})(W) \cong \mathbf{M}$ from the Hodge filtration for $\widetilde{\mathbb{X}}$. We also fix a splitting of this filtration; or equivalently, we choose a cocharacter $\mu : \mathbb{G}_m \to \mathrm{GL}_W(\mathbf{M})$ which induces $\operatorname{Fil}^1_{\widetilde{\mathbb{X}}}$. Using the choice of splitting, we can define the "opposite unipotent subgroup" U^μ.[10] Let A^μ_{GL} be such that $\operatorname{Spf} A^\mu_{\mathrm{GL}} \cong \widehat{U}^\mu$, where $\widehat{U}^\mu$ is the formal completion of U^μ at the identity section. We also choose a lift of Frobenius $\sigma : A^\mu_{\mathrm{GL}} \to A^\mu_{\mathrm{GL}}$ by choosing "coordinates" u_{ij} and setting $\sigma(u_{ij}) = u_{ij}^p$.

Let $u_t \in \widehat{U}^\mu(A^\mu_{\mathrm{GL}})$ be the tautological point. We consider the following object:

$$\mathbf{M}^\mu_{\mathrm{GL}} := A^\mu_{\mathrm{GL}} \otimes_W \mathbf{M}; \quad \operatorname{Fil}^1 \mathbf{M}^\mu_{\mathrm{GL}} := A^\mu_{\mathrm{GL}} \otimes_W \operatorname{Fil}^1_{\widetilde{\mathbb{X}}}; \quad F := u_t^{-1} \circ (A^\mu_{\mathrm{GL}} \otimes F).$$

More concretely, if we choose $\mathbf{M} \cong \mathbf{M}^\Lambda_b$ (with the notation of Definition 2.5.5 for $G = \mathrm{GL}(\Lambda)$) then the matrix representation of F on $\mathbf{M}^\mu_{\mathrm{GL}} \cong A^\mu_{\mathrm{GL}} \otimes_{\mathbb{Z}_p} \Lambda^*$ is $u_t^{-1} b$.

As discussed in [37, §4.5], Faltings showed that there exists a unique integrable connection ∇ on $\mathbf{M}^\mu_{\mathrm{GL}}$ which commutes with F. In particular, the tuple $(\mathbf{M}^\mu_{\mathrm{GL}}, F, \nabla)$ is a *crystalline Dieudonné module* in the sense of [11, Definition 2.3.4] and gives rise to a p-divisible group $\overline{X}^\mu_{\mathrm{GL}}$ over $\operatorname{Spec} A^\mu_{\mathrm{GL}}/(p)$ by [11, Main Theorem 1].[11] By construction, it is clear that $\sigma^*(\operatorname{Fil}^1 \mathbf{M}^\mu_{\mathrm{GL}}/(p))$ coincides with the kernel of F on $\mathbf{M}^\mu_{\mathrm{GL}}/(p)$; in other words, $\operatorname{Fil}^1 \mathbf{M}^\mu_{\mathrm{GL}}/(p)$ is the Hodge filtration of $\overline{X}^\mu_{\mathrm{GL}}$.

Faltings also showed that $\overline{X}^\mu_{\mathrm{GL}}$ is a universal mod p deformation of $\mathbb{X}$ via the Kodaira-Spencer theory. Implicit in the proof is the following lemma, which compares the tangent space of $A^\mu_{\mathrm{GL}}/(p)$ and the deformations over $\kappa[\epsilon]/(\epsilon^2)$ given by the Grothendieck-Messing deformation theory. We give a proof of the lemma as we will need it later (*cf.* Proposition 3.8).

---

[10] I.e., U^μ is the unipotent radical of the parabolic subgroup opposite to the stabiliser of $\operatorname{Fil}^1_{\widetilde{\mathbb{X}}^\Lambda_b}$.

[11] Crystalline Dieudonné modules over A^μ_{GL} correspond to Dieudonné crystals over $\operatorname{Spec} A^\mu_{\mathrm{GL}}/(p)$, not $\operatorname{Spf} A^\mu_{\mathrm{GL}}/(p)$. The distinction between Spec and Spf will be important, especially for verifying the effectivity property §6.1(3).

Lemma 3.2.1. *Let $B := \kappa[\epsilon]/(\epsilon^2)$, and we give the square-zero PD structure on $\mathfrak{b} := \epsilon B$. For any $\gamma \in \widehat{U}^\mu(B) = \mathrm{Hom}_W(A^\mu_{\mathrm{GL}}, B)$, we set ${}^\gamma X := \gamma^* X^\mu_{\mathrm{GL}}$. Then ${}^\gamma X$ is the lift of $\mathbb{X}$ which corresponds, via the Grothendieck-Messing deformation theory, to the lift of the Hodge filtration $\gamma(B \otimes_\kappa \mathrm{Fil}^1_{\mathbb{X}}) \subset B \otimes_\kappa \mathbb{D}(\mathbb{X})(\kappa) = \mathbb{D}(\mathbb{X})(B)$, where $\gamma(B \otimes_\kappa \mathrm{Fil}^1_{\mathbb{X}})$ is the filtration translated by $\gamma \in \widehat{U}^\mu(B)$.*

Proof. Let $\widetilde{B} := W[\epsilon]/(\epsilon^2)$, and we give the square-zero PD structure on $\epsilon\widetilde{B}$, which is compatible with the standard PD structure on (p). We also define a lift of Frobenius $\sigma : \widetilde{B} \to \widetilde{B}$ by letting $\sigma(\epsilon) := \epsilon^p = 0$. Since $\widetilde{B}$ is a (compatible) PD thickening of both κ and B, there exists a natural Frobenius-equivariant isomorphism $\mathbb{D}({}^\gamma X)(\widetilde{B}) \xrightarrow{\sim} \mathbb{D}(\mathbb{X})(\widetilde{B})$ such that after reducing modulo p the Hodge filtration for ${}^\gamma X$ on the left hand side maps to the lift of the Hodge filtration for $\mathbb{X}$ which corresponds to ${}^\gamma X$ via the Grothendieck-Messing deformation theory. Note that choosing a lift $\tilde{\gamma} \in \widehat{U}^\mu(\widetilde{B})$ of γ, we obtain natural isomorphisms

$$\mathbb{D}({}^\gamma X)(\widetilde{B}) \cong \widetilde{B} \otimes_{\tilde{\gamma}, A^\mu_{\mathrm{GL}}} \mathbf{M}^\mu_{\mathrm{GL}} \cong \widetilde{B} \otimes_W \mathbb{D}(\mathbb{X})(W) \tag{3.2.2}$$

and the crystalline Frobenius action on the left hand side corresponds to $\tilde{\gamma}^{-1} \circ (\widetilde{B} \otimes F)$ on the right hand side. (Recall that we used the *inverse* of the tautological point to define Frobenius-action on $\mathbf{M}^\mu_{\mathrm{GL}}$.) Thus, the natural Frobenius-equivariant isomorphism $\mathbb{D}({}^\gamma X)(\widetilde{B}) \xrightarrow{\sim} \mathbb{D}(\mathbb{X})(\widetilde{B})$ can be translated as $g \in 1 + \epsilon \,\mathrm{End}_W(\mathbf{M})$ which makes the following diagram commute:

$$\begin{tikzcd}
\mathbb{D}({}^\gamma X)(\widetilde{B}) \arrow[r, "\sim", "(3.2.2)"'] & \widetilde{B} \otimes_W \mathbb{D}(\mathbb{X})(W) \arrow[r, "\sim", "g"'] \arrow[d, "\tilde{\gamma}^{-1}\circ(\widetilde{B}\otimes F)"'] & \widetilde{B} \otimes_W \mathbb{D}(\mathbb{X})(W) \arrow[d, "\widetilde{B}\otimes F"] & \mathbb{D}(\mathbb{X})(\widetilde{B}) \arrow[l, "\sim"'] . \\
& \widetilde{B} \otimes_W \mathbb{D}(\mathbb{X})(W) \arrow[r, "\sim", "g"'] & \widetilde{B} \otimes_W \mathbb{D}(\mathbb{X})(W) &
\end{tikzcd}$$

By chasing the top row, it follows that $\mathrm{Fil}^1_{\tilde{\gamma}X} \subset \mathbb{D}({}^{\tilde{\gamma}}X)(\widetilde{B})$ maps to $g(\widetilde{B} \otimes_W \mathrm{Fil}^1_{\mathbb{X}}) \subset \mathbb{D}(\mathbb{X})(\widetilde{B})$. So to prove the lemma, it suffices to prove that $g = \tilde{\gamma}$. Indeed, we have $(\widetilde{B} \otimes F) \circ g = \widetilde{B} \otimes F$, as $\sigma(\epsilon) = 0$. Now the commutative square forces $g = \tilde{\gamma}$, since $\widetilde{B} \otimes F$ becomes an isomorphism after inverting p. $\square$

3.3. **Explicit construction: lifting Hodge filtration.** Lemma 3.2.1 also shows that any p-divisible group X^μ_{GL} over $\mathrm{Spf}(A^\mu_{\mathrm{GL}}, (p))$ which lifts $\overline{X}^\mu_{\mathrm{GL}}$ is a universal deformation of $\mathbb{X}$. Note that the Frobenius endomorphism and ∇ on $\mathbb{D}(X^\mu_{\mathrm{GL}})(A^\mu_{\mathrm{GL}})$ only depends on $\overline{X}^\mu_{\mathrm{GL}}$; i.e., we naturally have a Frobenius-equivariant horizontal isomorphism $\mathbb{D}(X^\mu_{\mathrm{GL}})(A^\mu_{\mathrm{GL}}) \cong \mathbf{M}^\mu_{\mathrm{GL}}$.

We now turn to the Hodge filtration for X^μ_{GL}. Since the Hodge filtration for $\overline{X}^\mu_{\mathrm{GL}}$ is the image of $\mathrm{Fil}^1 \mathbf{M}^\mu_{\mathrm{GL}}$ in $\mathbf{M}^\mu_{\mathrm{GL}}/(p)$, the Grothendieck-Messing deformation theory gives us a A^μ_{GL}-lift X^μ_{GL} of $\overline{X}^\mu_{\mathrm{GL}}$ with Hodge filtration $\mathrm{Fil}^1 \mathbf{M}^\mu_{\mathrm{GL}}$ if $p > 2$.

To sum up, we have proved the following result:

Theorem 3.4 (Faltings)**.** *Assume that $p > 2$. Then the p-divisible group X^μ_{GL} over A^μ_{GL}, corresponding to the filtered crystalline Dieudonné module $(\mathbf{M}^\mu_{\mathrm{GL}}, \mathrm{Fil}^1 \mathbf{M}^\mu_{\mathrm{GL}}, F, \nabla)$, is a universal deformation of $\mathbb{X}$.*

3.5. **Deformation with Tate tensors.** In this section, we return to the "unramified Hodge-type" setting (*cf.* Definition 2.5.5): namely, we let $G \subset \mathrm{GL}(\Lambda)$ be a connected reductive subgroup over $\mathbb{Z}_p$, and assume that there exists an isomorphism $\mathbb{X} \cong \mathbb{X}^\Lambda_b$ for some $b \in G(K_0)$. (In particular, we assume that there exists an isomorphism $\mathbf{M} \cong \mathbf{M}^\Lambda_b$ of Dieudonné modules.) For finitely many tensors $(s_\alpha) \subset \Lambda^\otimes$ whose stabiliser is G, we let $(\mathbf{t}_\alpha) \subset \mathbf{M}^\otimes$ denote the image of $(1 \otimes s_\alpha) \in (\mathbf{M}^\Lambda_b)^\otimes$ via the isomorphism $\mathbf{M} \cong \mathbf{M}^\Lambda_b$. Note that $(\mathbf{t}_\alpha)$ are F-invariant up to isogeny.

Let us now recall Faltings' construction of the "universal" deformation of $(\mathbb{X}, (\mathbf{t}_\alpha))$ when $p > 2$. We can choose a W-lift $\widetilde{\mathbb{X}}$ whose Hodge filtration is induced by some cocharacter $\mu : \mathbb{G}_m \to G_W$; *cf.* Remark 2.5.8. Then $(\mathbf{t}_\alpha)$ lies in the 0th filtration with respect to the Hodge filtration for $\widetilde{\mathbb{X}}$ by Lemma 2.2.7.

Let $U^\mu_G := U^\mu \cap G_W$ be the scheme-theoretic intersection, which turns out to be a smooth unipotent W-group (being the unipotent radical of some parabolic subgroup of G_W). Let A^μ_G be the quotient of A^μ_{GL} corresponding to the formal subgroup $\widehat{U}^\mu_G \subset \widehat{U}^\mu$. Then, A^μ_G is a formal power series over W. We also choose a "coordinate" for A^μ_{GL} so that the kernel of $A^\mu_{\mathrm{GL}} \twoheadrightarrow A^\mu_G$ is stable under σ. (In particular, we get a lift of Frobenius σ on A^μ_G induced by σ on A^μ_{GL}.)

Let X^μ_G denote the pull-back of X^μ_{GL} over $\mathrm{Spf}(A^\mu_G, (p))$. Then, $\mathbb{D}(X^\mu_G)(A^\mu_G)$ with the Hodge filtration and Frobenius action corresponds to the following quotient of $(\mathbf{M}^\mu_{\mathrm{GL}}, \mathrm{Fil}^1 \mathbf{M}^\mu_{\mathrm{GL}}, F)$:

$$\mathbf{M}^\mu_G := A^\mu_G \otimes_W \mathbf{M}; \quad \mathrm{Fil}^1 \mathbf{M}^\mu_G := A^\mu_G \otimes_W \mathrm{Fil}^1_{\widetilde{\mathbb{X}}}; \quad F := u_t^{-1} \circ (A^\mu_G \otimes F), \tag{3.5.1}$$

where $(\mathbf{M}, F) := \mathbb{D}(\mathbb{X})(W)$ and $u_t \in U^\mu_G(A^\mu_G)$ is the tautological point.

From this explicit description, it is immediate that the tensors $(1 \otimes \mathbf{t}_\alpha) \subset (\mathbf{M}^\mu_G)^\otimes[\frac{1}{p}]$ are F-invariant, and the pointwise stabiliser of $(1 \otimes \mathbf{t}_\alpha)$ is isomorphic to $G_{A^\mu_G}$. Since $\mathrm{Fil}^1 \mathbf{M}^\mu_G$ is a $\{\mu\}$-filtration, the tensors $(1 \otimes \mathbf{t}_\alpha)$ lie in the 0th filtration by Lemma 2.2.7. It is also known that $(1 \otimes \mathbf{t}_\alpha)$ are horizontal (*cf.* [29, §1.5.4]). So for each α we obtain a morphism

$$t^{\mathrm{univ}}_\alpha : \mathbf{1} \to \mathbb{D}(X^\mu_G)^\otimes \tag{3.5.2}$$

of crystals over $\mathrm{Spf}(A^\mu_G, (p))$ such that $t^{\mathrm{univ}}_\alpha(A^\mu_G) = 1 \otimes \mathbf{t}_\alpha$ on the A^μ_G-sections by the usual dictionary [11, Corollary 2.2.3].

Now we can rephrase the theorem of Faltings as follows (*cf.* [17, §7], [37, Theorem 4.9]):

Theorem 3.6 (Faltings). *Let A be either $W[[u_1, \cdots, u_N]]$ or $W[[u_1, \cdots, u_N]]/(p^m)$, and choose a p-divisible group X over A which lifts $\mathbb{X}$. Let $f : A^\mu_{\mathrm{GL}} \to A$ be the morphism induced by X (via $\mathrm{Spf}\, A^\mu_{\mathrm{GL}} \cong \mathtt{Def}_{\mathbb{X}}$). Then f factors through $A^\mu_G = \mathbb{D}(\mathbb{X})(W)^\otimes$ if and only if the map $\mathbf{1} \to \mathbf{M}^\otimes$, sending 1 to $\mathbf{t}_\alpha$, has a (necessarily unique) lift to a morphism of crystals over $\mathrm{Spf}(A, (p))$*

$$t_\alpha : \mathbf{1} \to \mathbb{D}(X)^\otimes$$

which is Frobenius-equivariant up to isogeny and has the property that its A-section $t_\alpha(A) \in \mathbb{D}(X)(A)^\otimes$ lies in the 0th filtration with respect to the Hodge filtration. If this holds, then we necessarily have $f^* t^{\mathrm{univ}}_\alpha = t_\alpha$.

Furthermore, the image of the closed immersion $\mathrm{Spf}\, A^\mu_G \hookrightarrow \mathtt{Def}_{\mathbb{X}}$, given by X^μ_G, is independent of the choice of $(\mathbf{t}_\alpha)$ and $\mu \in \{\mu\}$.

Proof. The universal property for A^μ_G for test rings of the form $A = W[[u_1, \cdots, u_N]]$ was proved by Faltings (*cf.* [37, Theorem 4.9]). For the case when $A = W[[u_1, \cdots, u_N]]/(p^m)$, we choose a lift $\widetilde{X}$ over $\widetilde{A} := W[[u_1, \cdots, u_N]]$ corresponding to a $\{\mu\}$-filtration (with respect to $(t_\alpha(\widetilde{A}))$) in $\mathbb{D}(X)(\widetilde{A})$ lifting the Hodge filtration of X. Then t_α also defines a unique morphism $\mathbf{1} \to \mathbb{D}(\widetilde{X})^\otimes$ (as it only depends on the mod p fibre of $\widetilde{X}$), and we obtain the desired claim by applying [37, Theorem 4.9] to $(\widetilde{X}, (t_\alpha))$.

The closed immersion $\mathrm{Spf}\, A^\mu_G \hookrightarrow \mathtt{Def}_{\mathbb{X}}$ is clearly independent of the choice of $(\mathbf{t}_\alpha)$, and the independence of the choice of $\mu \in \{\mu\}$ follows from the universal property. □

3.7. **Functoriality of deformation spaces.** We identify the deformation functor $\mathtt{Def}_{\mathbb{X}}$ with the formal spectrum of complete local noetherian ring which prorepresents $\mathtt{Def}_{\mathbb{X}}$.

Definition 3.7.1. Using the notation from Theorem 3.6, we define $\mathtt{Def}_{\mathbb{X},G}$ to be the formally smooth closed formal subscheme of $\mathtt{Def}_{\mathbb{X}}$ which classify deformations of $(\mathbb{X}, (\mathbf{t}_\alpha))$ over formal power series rings over W or $W/(p^m)$ in the sense of Theorem 3.6. Note that for any cocharacter $\mu : \mathbb{G}_m \to G_W$ giving rise to the Hodge filtration of $\mathbb{X}$, we get an isomorphism $\widehat{U}^\mu_G \xrightarrow{\sim} \mathtt{Def}_{\mathbb{X},G}$ induced by X^μ_G, and the closed formal subscheme $\mathtt{Def}_{\mathbb{X},G} \subset \mathtt{Def}_{\mathbb{X}}$ is independent of the choice of (t_α).

Note that for any isomorphism $(\mathbb{X}, (\mathbf{t}_\alpha)) \xrightarrow{\sim} (\mathbb{X}^\Lambda_b, (1\otimes s_\alpha))$, we have a natural isomorphism $\mathtt{Def}_{\mathbb{X},G} \xrightarrow{\sim} \mathtt{Def}_{\mathbb{X}^\Lambda_b,G}$. Therefore, we fix an identification $\mathbb{X} = \mathbb{X}^\Lambda_b$ for the moment, and show that $\mathtt{Def}_{\mathbb{X},G}$ only depends on (G, b), not on Λ, in a canonical way, and that it is functorial with respect to (G, b). (See Remark 3.7.4 for the discussion on the choice of isomorphism $(\mathbb{X}, (\mathbf{t}_\alpha)) \xrightarrow{\sim} (\mathbb{X}^\Lambda_b, (1\otimes s_\alpha))$.)

We consider another pair (G', b') and Λ' as in Definition 2.5.5, and consider $\mathbb{X}' := \mathbb{X}^{\Lambda'}_{b'}$. We also obtain a subfunctor $\mathtt{Def}_{\mathbb{X}',G'} \subset \mathtt{Def}_{\mathbb{X}'}$, such that for any cocharacter $\mu' : \mathbb{G}_m \to G'_W$ that induces the Hodge filtration of $\mathbb{X}'$ we have a natural isomorphism $\widehat{U}^{\mu'}_{G'} \xrightarrow{\sim} \mathtt{Def}_{\mathbb{X}',G'}$ induced by $X^{\mu'}_{G'}$. We do not assume the existence of any morphism between $\mathbb{X}$ and $\mathbb{X}'$.

Proposition 3.7.2. *In the above setting, the natural monomorphism* $\mathtt{Def}_{\mathbb{X}} \times \mathtt{Def}_{\mathbb{X}'} \to \mathtt{Def}_{\mathbb{X}\times\mathbb{X}'}$*, defined by taking the product of deformations, induces an isomorphism*

$$\mathtt{Def}_{\mathbb{X},G} \times \mathtt{Def}_{\mathbb{X}',G'} \xrightarrow{\sim} \mathtt{Def}_{\mathbb{X}\times\mathbb{X}',G\times G'}.$$

Let $f : G_W \to G'_W$ be a homomorphism over W such that $f(b) = b'$. We choose a cocharacter $\mu : \mathbb{G}_m \to G_W$ inducing the Hodge filtration of $\mathbb{X} := \mathbb{X}^\Lambda_b$. Then the morphism $\mathtt{Def}_{\mathbb{X},G} \to \mathtt{Def}_{\mathbb{X}',G'}$, corresponding to $f|_{\widehat{U}^\mu_G} : \widehat{U}^\mu_G \to \widehat{U}^{f\circ\mu}_{G'}$, depends only on f, not on the choice of μ.

Before we begin the proof, let us make some remarks on the statement.

Remark 3.7.3. One can apply this proposition to the identity map on (G, b) with different choice of Λ and Λ' to obtain a natural functorial isomorphism $\mathtt{Def}_{\mathbb{X}^\Lambda_b,G} \xrightarrow{\sim} \mathtt{Def}_{\mathbb{X}^{\Lambda'}_b,G}$. Under this identification, the morphism $\mathtt{Def}_{\mathbb{X}^\Lambda_b,G} \to \mathtt{Def}_{\mathbb{X}^{\Lambda'}_{b'},G'}$ associated to $f : (G, b) \to (G', b')$ depends only on f, not on the choice of Λ and Λ'.

Proof of Proposition 3.7.2. For the first assertion on the product decomposition, observe that we have $X^{(\mu,\mu')}_{G\times G'} \cong X^\mu_G \times X^{\mu'}_{G'}$, which follows from the explicit description (§3.5). The claim now follows.

Let us now show that for a fixed choice of Λ and Λ' the map $\mathtt{Def}_{\mathbb{X},G} \to \mathtt{Def}_{\mathbb{X}',G'}$ induced by $f|_{\widehat{U}^\mu_G}$ is independent of the choice of μ. For this, we factor the map $\mathtt{Def}_{\mathbb{X},G} \to \mathtt{Def}_{\mathbb{X}',G'}$ as follows, and show that each arrow on the top row is independent of the choice of μ:

$$\begin{tikzcd}
\mathtt{Def}_{\mathbb{X},G} \arrow[d, "\cong"] & \mathtt{Def}_{\mathbb{X}\times\mathbb{X}',G} \arrow[l, "\mathrm{pr}_1"'] \arrow[r, hook] \arrow[d, "\cong"] & \mathtt{Def}_{\mathbb{X}\times\mathbb{X}',G\times G'} \arrow[d, "\cong"] & \mathtt{Def}_{\mathbb{X},G} \times \mathtt{Def}_{\mathbb{X}',G'} \arrow[l, "\cong"'] \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, "\cong"] \arrow[lll, bend right, "\mathrm{pr}_1"'] & \mathtt{Def}_{\mathbb{X}',G'} \arrow[d, "\cong"] \\
\widehat{U}^\mu_G \arrow[rrrr, bend right, "f"'] & \widehat{U}^\mu_G \arrow[l, "\mathrm{id}"'] \arrow[r, hook, "(\mathrm{id},f)"] & \widehat{U}^{(\mu,f\circ\mu)}_{G\times G'} & \widehat{U}^\mu_G \times \widehat{U}^{f\circ\mu}_{G'} \arrow[l, "\cong"'] \arrow[r, two heads] & \widehat{U}^{f\circ\mu}_{G'}
\end{tikzcd}.$$

Here, we view $\Lambda\times\Lambda'$ as a faithful G-representation by $G \xrightarrow{(\mathrm{id},f)} G\times G'$, so we have $\mathbb{X}\times\mathbb{X}' = \mathbb{X}^{\Lambda\times\Lambda'}_b$.

Note that the third arrow on the top is the isomorphism defined by taking the product of deformations, which is independent of the choice of μ since the subspaces $\mathtt{Def}_{\mathbb{X},G} \subset \mathtt{Def}_{\mathbb{X}}$ and $\mathtt{Def}_{\mathbb{X}',G'} \subset \mathtt{Def}_{\mathbb{X}'}$ are independent of the choice of cocharacters (*cf.* Theorem 3.6). Similarly, it follows that the projection maps on the top row (pr_1 and pr_2) are independent of the choice of μ, as they are the restrictions of the natural projections $\mathtt{Def}_{\mathbb{X}} \times \mathtt{Def}_{\mathbb{X}'} \twoheadrightarrow \mathtt{Def}_{\mathbb{X}}$ and $\mathtt{Def}_{\mathbb{X}} \times \mathtt{Def}_{\mathbb{X}'} \twoheadrightarrow \mathtt{Def}_{\mathbb{X}'}$ to a closed subspace independent of the choice of cocharacter.

The second arrow on the top row can be obtained from the universal property for $\mathtt{Def}_{\mathbb{X}\times\mathbb{X}',G\times G'}$, hence it is independent of the choice of μ. This shows that the first arrow does not depend on the choice of μ as it can be obtained as the compositions of maps independent of μ. Furthermore, it is an isomorphism as it corresponds to the identity map of $\widehat{U}^{\mu}_G$. Now, chasing the diagram, we conclude that the map $\mathtt{Def}_{\mathbb{X},G} \to \mathtt{Def}_{\mathbb{X}',G'}$ does not depend on the choice of μ. □

Remark 3.7.4. We remark on the effect of different choice of isomorphism $\mathbb{X} \cong \mathbb{X}^{\Lambda}_b$ in Proposition 3.7.2. For $g \in G(W)$, we write $b' := g^{-1}b\sigma(g)$ and $\mathbb{X}' := \mathbb{X}^{\Lambda}_{b'}$. Then g induces an F-equivariant isomorphism $g : \mathbf{M}^{\Lambda}_{b'} \to \mathbf{M}^{\Lambda}_b$, so we get an isomorphism

$$(\mathbb{X}, (1\otimes s_\alpha)) \xrightarrow{\sim} (\mathbb{X}', (1\otimes s_\alpha)),$$

preserving tensors, where $\mathbb{X} := \mathbb{X}^{\Lambda}_b$ as before. This induces an isomorphism $\mathtt{Def}_{\mathbb{X},G} \xrightarrow{\sim} \mathtt{Def}_{\mathbb{X}',G}$. We want to give a group-theoretic interpretation of this isomorphism via the explicit construction of universal deformations with Tate tensors.

Choose a cocharacter $\mu : \mathbb{G}_m \to G_W$ which induces the Hodge filtration of $\mathbb{X}$. Then, $\mu' := g^{-1}\mu g$ induces the Hodge filtration of $\mathbb{X}'$, so we have an isomorphism $\widehat{U}^{\mu'}_G = \operatorname{Spf} A^{\mu'}_G \xrightarrow{\sim} \mathtt{Def}_{\mathbb{X}',G}$ defined by the deformation $X^{\mu'}_G$ of $\mathbb{X}'$. Let $u_t \in \widehat{U}^{\mu}_G(A^{\mu}_G) \subset G(A^{\mu}_G)$ and $u'_t \in \widehat{U}^{\mu'}_G(A^{\mu'}_G) \subset G(A^{\mu'}_G)$ be the tautological points.

We have an isomorphism $j_g : \widehat{U}^{\mu}_G \xrightarrow{\sim} \widehat{U}^{\mu'}_G$, defined by the conjugation by g^{-1}, and we have $u'_t = g^{-1}(j_g^* u_t)g$ as an elements in $G(A^{\mu'}_G)$. So we have

$$(u'_t)^{-1}(g^{-1}b\sigma(g)) = (g^{-1}j_g^*(u_t^{-1})g)\cdot(g^{-1}b\sigma(g)) = g^{-1}j_g^*(u_t^{-1}b)\sigma(g).$$

In particular, by identifying the underlying $A^{\mu'}_G$-modules of $\mathbf{M}^{\mu'}_G$ and $j_g^*\mathbf{M}^{\mu}_G$ with $A^{\mu'}_G \otimes_{\mathbb{Z}_p} \Lambda$, the isomorphism $g : \mathbf{M}^{\mu'}_G \xrightarrow{\sim} j_g^*\mathbf{M}^{\mu}_G$ is horizontal, filtered, and F-equivariant. In short, we obtain the following commutative diagram of isomorphisms

$$\begin{tikzcd} \widehat{U}^{\mu}_G \arrow[r, "\sim", "j_g"'] \arrow[d, "X^{\mu}_G"'] & \widehat{U}^{\mu'}_G \arrow[d, "X^{\mu'}_G"] \\ \mathtt{Def}_{\mathbb{X},G} \arrow[r, "\sim"] & \mathtt{Def}_{\mathbb{X}',G} \end{tikzcd},$$

where the bottom isomorphism is induced by $(\mathbb{X}, (1\otimes s_\alpha)) \xrightarrow{\sim} (\mathbb{X}', (1\otimes s_\alpha))$ which corresponds to $g : \mathbf{M}^{\Lambda}_{b'} \to \mathbf{M}^{\Lambda}_b$.

Let us return to the setting of Proposition 3.7.2, and consider a homomorphism $f : G \to G'$. Then it follows without difficulty that for any $g \in G(W)$ the following diagram commutes

(3.7.5)
$$\begin{tikzcd} \mathtt{Def}_{\mathbb{X}^{\Lambda}_b,G} \arrow[r] \arrow[d, "\cong"] & \mathtt{Def}_{\mathbb{X}^{\Lambda'}_{f(b)},G'} \arrow[d, "\cong"] \\ \mathtt{Def}_{\mathbb{X}^{\Lambda}_{g^{-1}b\sigma(g)},G} \arrow[r] & \mathtt{Def}_{\mathbb{X}^{\Lambda'}_{f(g^{-1}b\sigma(g))},G'} \end{tikzcd},$$

where the vertical isomorphisms are as constructed above associated to $g \in G(W)$ and $f(g) \in G'(W)$, respectively, and the horizontal arrows are associated to $f : G \to G'$ via Proposition 3.7.2.

We now study deformation theory for points of $\mathtt{Def}_{\mathbb{X},G}$ valued in artin local rings. We consider a W-morphism $f : \operatorname{Spec} R \to \mathtt{Def}_{\mathbb{X},G}$ for $R \in \mathfrak{AR}_W$, and set $(X_R, (t_\alpha)) := (f^* X^\mu_{G,b}, (f^* t^{\mathrm{univ}}_\alpha))$. Let $B \twoheadrightarrow R$ be a square-zero thickening with finitely generated kernel $\mathfrak{b}$, and give the square-zero PD structure on $\mathfrak{b}$; i.e., $a^{[i]} = 0$ for any $i > 1$ and $a \in \mathfrak{b}$. Then we can define the B-sections $(t_\alpha(B)) \subset \mathbb{D}(X_R)(B)^\otimes$ as in Definition 2.3.4. Let $\tilde{f} : \operatorname{Spec} B \to \mathtt{Def}_{\mathbb{X},G}$ be a lift of f, and set $(X_B, (\tilde{t}_\alpha)) := (\tilde{f}^* X^\mu_{G,b}, (\tilde{f}^* t^{\mathrm{univ}}_\alpha))$. Then we have a natural isomorphism $\mathbb{D}(X_B)(B) \cong \mathbb{D}(X_R)(B)$, which matches $(\tilde{t}_\alpha(B))$ with $(t_\alpha(B))$.

Proposition 3.8. *Assume that $p > 2$. Let $(X_R, (t_\alpha))$ and $B \twoheadrightarrow R$ be as above. Then a B-lift X_B of X_R defines a B-point of $\mathtt{Def}_{\mathbb{X},G}$ if and only if the Hodge filtration*

$$\mathrm{Fil}^1_{X_B} \subset \mathbb{D}(X_B)(B) \cong \mathbb{D}(X_R)(B)$$

is a $\{\mu\}$-filtration with respect to $(t_\alpha(B))$, where $\mu : \mathbb{G}_m \to G_W$ is the cocharacter in the definition of X^μ_G.

Let us outline the basic strategy of the proof. The proposition when $B = \kappa[\epsilon] \twoheadrightarrow R = \kappa$ can be deduced from Lemma 3.2.1. When $B \twoheadrightarrow R$ is a small thickening (i.e., $\mathfrak{b}$ is of B-length 1) then we prove the proposition using the fact that the set of B-lifts of f is a torsor under the reduced tangent space of $\mathtt{Def}_{\mathbb{X},G}$. The general case can be deduced by filtering $B \twoheadrightarrow R$ into successive small thickenings.

Before beginning the proof of the proposition, let us review fibre products of rings. Let $B \twoheadrightarrow R$ be a small thickening of rings in $\mathfrak{AR}_W$ with kernel $\mathfrak{b} \subset B$. Let $\kappa[\mathfrak{b}]$ denote the κ-algebra whose underlying κ-vector space is $\kappa \oplus \mathfrak{b}$, such that $\mathfrak{b}$ is the augmentation ideal. If we pick a generator $\epsilon \in \mathfrak{b}$ then we have $\kappa[\mathfrak{b}] = \kappa[\epsilon]/\epsilon^2$.

We have the following isomorphism

$$B \times_\kappa \kappa[\mathfrak{b}] \xrightarrow{\sim} B \times_R B =: B'; \quad (a, \bar{a} + a') \mapsto (a, a + a'), \tag{3.8.1}$$

where $a \in B$, $a' \in \mathfrak{b}$, and $\bar{a} \in \kappa$ is the image of a. The inverse is given by $(a, a') \mapsto (a, \bar{a} + (a' - a))$.

Let $\mathcal{F} : \mathfrak{AR}_W \to (\mathbf{Sets})$ be a pro-representable functor. (For example, $\mathcal{F} = \mathtt{Def}_{\mathbb{X}}$ or $\mathcal{F} = \mathtt{Def}_{\mathbb{X},G}$.) Then we have a natural bijection

$$\mathcal{F}(B \times_R B') \xrightarrow{\sim} \mathcal{F}(B) \times_{\mathcal{F}(R)} \mathcal{F}(B')$$

for any $B, B' \twoheadrightarrow R$. So from (3.8.1) we obtain a natural bijection

$$\mathcal{F}(B) \times \mathcal{F}(\kappa[\mathfrak{b}]) \xrightarrow{\sim} \mathcal{F}(B) \times_{\mathcal{F}(R)} \mathcal{F}(B), \tag{3.8.2}$$

which defines an $\mathcal{F}(\kappa[\mathfrak{b}])$-action on $\mathcal{F}(B)$, and makes the set of $\tilde{f} \in \mathcal{F}(B)$ lifting a fixed $f \in \mathcal{F}(R)$ into a $\mathcal{F}(\kappa[\mathfrak{b}])$-torsor.

Let us consider the case when $\mathcal{F} = \mathtt{Def}_{\mathbb{X}}$. For any $R \in \mathfrak{AR}_W$, we set $\mathbf{M}_R := R \otimes_W \mathbf{M}$ and $\mathrm{Fil}^1 \mathbf{M}_R := R \otimes_W \mathrm{Fil}^1_{\mathbb{X}} \subset \mathbf{M}_R$. Via the Grothendieck-Messing deformation theory, we have a natural bijection

$$\widehat{U}^\mu(\kappa[\mathfrak{b}]) \cong \mathtt{Def}_{\mathbb{X}}(\kappa[\mathfrak{b}]). \tag{3.8.3}$$

Indeed, we associate to $\gamma \in \widehat{U}^\mu(\kappa[\mathfrak{b}])$ the lift ${}^\gamma X \in \mathtt{Def}_{\mathbb{X}}(\kappa[\mathfrak{b}])$ which corresponds to the filtration $\gamma(\mathrm{Fil}^1 \mathbf{M}_{\kappa[\mathfrak{b}]})$; *cf.* Lemma 3.2.1.

Now we give the square-zero PD structure on $\mathfrak{b}$. (We still assume that $p > 2$.) We define the $\widehat{U}^\mu(\kappa[\mathfrak{b}])$-action on $\mathtt{Def}_{\mathbb{X}}(B)$ to be the one induced from the natural $\widehat{U}^\mu(\kappa[\mathfrak{b}])$-action on the Hodge filtration of X_B via the Grothendieck-Messing deformation theory.

Lemma 3.8.4. *In the above setting setting, the actions of $\widehat{U}^\mu(\kappa[\mathfrak{b}])$ and $\mathtt{Def}_{\mathbb{X}}(\kappa[\mathfrak{b}])$ on $\mathtt{Def}_{\mathbb{X}}(B)$, which are defined above, coincide via the isomorphism (3.8.3).*

Proof. Let us give the square-zero PD structure on the kernel of $B' := B \times_R B \twoheadrightarrow R$ so that both projections $B' \rightrightarrows B$ are PD morphisms. Let ${}^\gamma X \in \mathtt{Def}_{\mathbb{X}}(\kappa[\mathfrak{b}])$ be the deformation corresponding to $\gamma \in \widehat{U}^\mu(\kappa[\mathfrak{b}])$. Then the action of ${}^\gamma X$ maps X_B to the pull-back of $X_B \times_{\mathbb{X}} {}^\gamma X \in \mathtt{Def}_{\mathbb{X}}(B')$ via the second projection $B' \twoheadrightarrow B$. Meanwhile, $X_B \times_{\mathbb{X}} {}^\gamma X$ corresponds to the filtration

$$\mathrm{Fil}^1_{X_B} \times_{\mathrm{Fil}^1 \mathbf{M}_\kappa} \left[\gamma(\mathrm{Fil}^1 \mathbf{M}_{\kappa[\mathfrak{b}]})\right] \subset \mathbb{D}(X_R)(B) \times_{\mathbf{M}_\kappa} \mathbf{M}_{\kappa[\mathfrak{b}]} \cong \mathbb{D}(X_R)(B'), \tag{3.8.5}$$

using the notation as above. Now from the isomorphism (3.8.1) it follows that the image of the filtration (3.8.5) under the second projection is $\gamma\,\mathrm{Fil}^1_{X_B}$. □

Proof of Proposition 3.8. If $B = \kappa[\epsilon]$ then the proposition is clear from Lemma 3.2.1. Now assume that $B \twoheadrightarrow R$ is a small thickening (i.e., the B-length of $\mathfrak{b}$ is 1). We let $f : \operatorname{Spec} R \to \mathtt{Def}_{\mathbb{X},G}$ denote the map induced by X_R. For any lift $\tilde{f} : \operatorname{Spec} B \to \mathtt{Def}_{\mathbb{X},G}$ of f (with $X_B := \tilde{f}^* X^\mu_G$) the Hodge filtration

$$\mathrm{Fil}^1_{X_B} \subset \mathbb{D}(X_B)(B) \cong \mathbb{D}(X_R)(B)$$

is a $\{\mu\}$-filtration with respect to $(t_\alpha(B))$; indeed, $\mathrm{Fil}^1_{X_B}$ and $(t_\alpha(B))$ are respectively the images of $\mathrm{Fil}^1 \mathbf{M}^\mu_G$ and $(t^{\mathrm{univ}}_\alpha(A^\mu_G))$. (Note that we have a natural isomorphism $\mathbb{D}(X_B)(B) = \tilde{f}^* \mathbf{M}^\mu_G$.) So we obtain a map

$$\{\tilde{f} \in \mathtt{Def}_{\mathbb{X},G}(B) \text{ lifting } f\} \to \{\{\mu\}\text{-filtrations in } \mathbb{D}(X_R)(B) \text{ lifting } \mathrm{Fil}^1_{X_R}\},$$

sending $\tilde{f}$ to $\mathrm{Fil}^1_{\tilde{f}^* X^\mu_G}$. By Lemma 3.8.4, this map is a morphism of $\widehat{U}^\mu_G(\kappa[\mathfrak{b}])$-torsors, so it has to be a bijection. This proves the proposition when $B \twoheadrightarrow R$ is a small thickening.

Now let $B \twoheadrightarrow R$ be any square-zero thickening with $B \in \mathfrak{AR}_W$, and consider an quotient R' of B which surjects onto R. Then $\mathfrak{b}' := \ker(B \twoheadrightarrow R')$ is a square-zero ideal, so by giving the "square-zero PD structure", $\mathfrak{b}'$ is a PD subideal of $\mathfrak{b}$. We fix a lift $f' \in \mathtt{Def}_{\mathbb{X},G}(R')$ of f and set $(X_{R'}, (t'_\alpha)) := (f'^* X^\mu_{G,b}, (f'^* t^{\mathrm{univ}}_\alpha))$.

Since $\mathfrak{b}'$ is a (nilpotent) PD subideal, we have a natural isomorphism $\mathbb{D}(X_R)(B) \cong \mathbb{D}(X_{R'})(B)$, which matches $(t_\alpha(B))$ and $(t'_\alpha(B))$.[12] Therefore, the proposition for $B \twoheadrightarrow R$ is obtained by filtering it with successive small thickenings. □

4. MODULI OF p-DIVISIBLE GROUPS WITH TATE TENSORS

In this section, we state the main results on Hodge-type analogue $\mathtt{RZ}^\Lambda_{G,b}$ of Rapoport-Zink spaces (Theorem 4.9.1). This construction recovers EL and PEL Rapoport-Zink spaces if (G, b) is associated to an unramified EL and PEL datum (§4.7).

Let κ be an algebraically closed field of characteristic p, and set $W := W(\kappa)$ and $K_0 := \operatorname{Frac} W$. The most interesting case is when $\kappa = \overline{\mathbb{F}}_p$. Let $\mathbb{X}$ be a p-divisible group over κ.

We begin with the review of the formal moduli schemes classifying p-divisible groups up to quasi-isogeny [39, Ch.II], which is the starting point of the construction of $\mathtt{RZ}^\Lambda_{G,b}$.

Definition 4.1 ([39, Definition 2.15])**.** Let $\mathtt{RZ}_{\mathbb{X}} : \mathrm{Nilp}_W \to (\mathbf{Sets})$ be a covariant functor defined as follows: for any $R \in \mathrm{Nilp}_W$, $\mathtt{RZ}_{\mathbb{X}}(R)$ is the set of isomorphism

[12] This can be seen as follows. For any maps $\tilde{f} \in \mathtt{Def}_{\mathbb{X},G}(B)$ lifting f' (and hence, f) and $(X_B, (\tilde{t}_\alpha))$ pulling back the universal objects, we have natural isomorphisms $\mathbb{D}(X_R)(B) \cong \mathbb{D}(X_{R'})(B) \cong \mathbb{D}(X_B)(B)$ matching $(t_\alpha(B))$, $(t'_\alpha(B))$, and $(\tilde{t}_\alpha(B))$.

classes of (X, ι), where X is a p-divisible group over R and $\iota : \mathbb{X}_{R/p} \dashrightarrow X_{R/p}$ is a quasi-isogeny. (We take the obvious notion of isomorphism of (X, ι).)

For $h \in \mathbb{Z}$, let $\mathtt{RZ}_{\mathbb{X}}(h) : \mathrm{Nilp}_W \to$ (**Sets**) be a subfunctor of $\mathtt{RZ}_{\mathbb{X}}$ defined by requiring that the quasi-isogeny ι has height h.

Remark 4.1.1. Let $\widetilde{\mathbb{X}}$ be any p-divisible group over W which lifts $\mathbb{X} := \mathbb{X}_b^\Lambda$. Then for any $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$, ι uniquely lifts to

$$\tilde{\iota} : \widetilde{\mathbb{X}}_R \dashrightarrow X \tag{4.1.2}$$

by rigidity of quasi-isogenies; *cf.* (2.3.2). So we may regard $\mathtt{RZ}_{\mathbb{X}}(R)$ as the set of isomorphism classes of $(X, \tilde{\iota})$ where $\tilde{\iota} : \widetilde{\mathbb{X}}_R \dashrightarrow X$ is a quasi-isogeny.

Theorem 4.2 (Rapoport, Zink)**.** *The functor $\mathtt{RZ}_{\mathbb{X}}$ can be represented by a separated formal scheme which is locally formally of finite type (*cf. *Definition 2.1.1) and formally smooth over W. We also denote by $\mathtt{RZ}_{\mathbb{X}}$ the formal scheme representing $\mathtt{RZ}_{\mathbb{X}}$. For $h \in \mathbb{Z}$, the subfunctor $\mathtt{RZ}_{\mathbb{X}}(h)$ can be represented by an open and closed formal subscheme (also denoted by $\mathtt{RZ}_{\mathbb{X}}(h)$). Furthermore, any irreducible component of $(\mathtt{RZ}_b^\Lambda)_{\mathrm{red}}$ is projective.*

Proof. The representability of $\mathtt{RZ}_{\mathbb{X}}$ is proved in [39, Theorem 2.16]. It is clear that $\mathtt{RZ}_{\mathbb{X}}(h)$ is an open and closed formal subscheme of $\mathtt{RZ}_{\mathbb{X}}$. The assertion on the irreducible components of $(\mathtt{RZ}_b^\Lambda)_{\mathrm{red}}$ is proved in [39, Proposition 2.32]. □

For each $h \in \mathbb{Z}$, we can write $\mathtt{RZ}_{\mathbb{X}}(h)$ as the direct limit of subfunctors representable by closed schemes, as follows. Let $(X_{\mathtt{RZ}_{\mathbb{X}}(h)}, \iota_{\mathtt{RZ}_{\mathbb{X}}(h)})$ denote the universal p-divisible group up to height-h quasi-isogeny.

Definition 4.3. We fix a W-lift $\widetilde{\mathbb{X}}$ of $\mathbb{X}$, and view $\mathtt{RZ}_{\mathbb{X}}(R)$ as a set of $\{(X, \tilde{\iota} : \widetilde{\mathbb{X}}_R \dashrightarrow X)\}/\cong$ (*cf.* Remark 4.1.1). Then for any $m > 0$ and $n \in \mathbb{Z}$, we define $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n} \subset \mathtt{RZ}_{\mathbb{X}}(h)$ to be the subfunctor defined as follows: for any W/p^m-algebra R, $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(R)$ is the set of $(X, \tilde{\iota}) \in \mathtt{RZ}_{\mathbb{X}}(h)(R)$ such that $p^n\tilde{\iota} : \widetilde{\mathbb{X}}_R \dashrightarrow X$ is an isogeny of p-divisible group. (We set $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(R) = \emptyset$ if $p^m R \neq 0$.)

As explained in [39, §2.22], $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ can be realised as a closed subscheme of certain grassmannian, hence it can be represented by a projective scheme over W/p^m. And it is a closed subscheme of $\mathtt{RZ}_{\mathbb{X}}(h)$ such that $\mathtt{RZ}_{\mathbb{X}}(h) = \varinjlim_{m,n} \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$.

We now recall the deformation-theoretic interpretation of the completed local ring of $\mathtt{RZ}_{\mathbb{X}}$ has a deformation-theoretic interpretation.

Lemma 4.3.1. *For $x = (X_x, \iota_x) \in \mathtt{RZ}_{\mathbb{X}}(\kappa)$, the formal completion $(\mathtt{RZ}_{\mathbb{X}})^\wedge_x$ at x represents the functor $\mathtt{Def}_{X_x}$, using the notation above.*

Proof. We have a morphism $(\mathtt{RZ}_{\mathbb{X}})^\wedge_x \to \mathtt{Def}_{X_x}$ given by forgetting the quasi-isogeny. By rigidity of quasi-isogeny, we have a natural morphism of functors $\mathtt{Def}_{X_x} \to (\mathtt{RZ}_{\mathbb{X}})^\wedge_x$ defined by sending $X \in \mathtt{Def}_{X_x}(R)$ to $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$ where $\iota : \mathbb{X}_{R/p} \to X_{R/p}$ is the unique quasi-isogeny that lifts ι_x. It also follows from rigidity that the composition $(\mathtt{RZ}_{\mathbb{X}})^\wedge_x \to \mathtt{Def}_{X_x} \to (\mathtt{RZ}_{\mathbb{X}})^\wedge_x$ is an identity morphism. To finish the proof, note that both are representable by formal power series rings over W with same dimension. (The dimension of $(\mathtt{RZ}_{\mathbb{X}})^\wedge_x$ can be obtained from [39, Proposition 3.33].) □

4.4. Let us return to the setting of §2.5. Let (G, b) and Λ be as in Definition 2.5.5, so we have a p-divisible group $\mathbb{X} := \mathbb{X}_b^\Lambda$ with the contravariant Dieudonné module $\mathbf{M}_b^\Lambda$. We choose finitely many tensors $(s_\alpha) \subset \Lambda^\otimes$ which defines G as a subgroup of $\mathrm{GL}(\Lambda)$; *cf.* Proposition 2.1.3. We also choose a W-lift $\widetilde{\mathbb{X}}(= \widetilde{\mathbb{X}}_b^\Lambda)$ of $\mathbb{X}$ as in Remark 2.5.8.

Definition 4.5. Let $(X, \tilde{\iota}) \in \mathtt{RZ}_{\mathbb{X}}(R)$ with $R \in \mathrm{Nilp}_W$, where $\tilde{\iota} : \widetilde{\mathbb{X}}_R \dashrightarrow X$ is a quasi-isogeny; *cf.* Remark 4.1.1. We define $s_{\alpha,\mathbb{D}} : \mathbf{1} \to \mathbb{D}(X)^{\otimes}[\frac{1}{p}]$ to be the composition

$$\mathbf{1} \xrightarrow{1 \mapsto 1 \otimes s_\alpha} \mathbb{D}(\widetilde{\mathbb{X}}_R)^{\otimes}[\tfrac{1}{p}] \xrightarrow[\sim]{\mathbb{D}(\tilde{\iota})^{-1}} \mathbb{D}(X)^{\otimes}[\tfrac{1}{p}],$$

where the first morphism is the pull-back of the map $\mathbf{1} \to \mathbb{D}(\widetilde{\mathbb{X}})^{\otimes}$ which induces $1 \mapsto 1 \otimes s_\alpha$ on the W-sections.

Note that $s_{\alpha,\mathbb{D}} : \mathbf{1} \to \mathbb{D}(X)^{\otimes}[\frac{1}{p}]$ only depends on $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$ but not on the choice of $\widetilde{\mathbb{X}}$. Indeed, the morphism

$$\mathbf{1} \xrightarrow{1 \mapsto 1 \otimes s_\alpha} \mathbb{D}(\mathbb{X}_{R/p})^{\otimes}[\tfrac{1}{p}] \xrightarrow[\sim]{\mathbb{D}(\iota)^{-1}} \mathbb{D}(X_{R/p})^{\otimes}[\tfrac{1}{p}]$$

uniquely determines $s_{\alpha,\mathbb{D}}$.

It is clear that each $s_{\alpha,\mathbb{D}}$ is Frobenius-equivariant, but it may *not* come from a morphism of crystals $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$. Even if it does, such a morphism t_α of (integral) crystals may not be uniquely determined by $s_{\alpha,\mathbb{D}}$ due to the existence of non-zero p-torsion morphism when the base ring R is not nice enough. (See Appendix in [2] for such an example.) To deal with this problem, we work only with the morphism of (integral) crystals $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ giving rise to $s_{\alpha,\mathbb{D}}$ which is "liftable" in some suitable sense.

Let $\mathrm{Nilp}_W^{\mathrm{sm}}$ denote the full subcategory of Nilp_W consisting of formally smooth formally finitely generated W/p^m-algebras A. Then there exists a p-adically separated and complete formally smooth W-algebra $\widetilde{A}$ which lifts A; indeed, we apply [11, Lemma 1.3.3] to obtain a p-adic W-lift $\widetilde{A}$ of A/p, which is formally smooth over W by construction, so we may view $\widetilde{A}$ as a lift of A. By formal smoothness over W, any two such lifts are (not necessarily canonically) isomorphic.

Definition 4.6. For any $A \in \mathrm{Nilp}_W^{\mathrm{sm}}$, we define $\mathtt{RZ}_{\mathbb{X},G}^{(s_\alpha)}(A) \subset \mathrm{Hom}_W(\mathrm{Spf}\, A, \mathtt{RZ}_{\mathbb{X}})$ as follows: Let $f : \mathrm{Spf}\, A \to \mathtt{RZ}_{\mathbb{X}}$ be a morphism, and X a p-divisible group over $\mathrm{Spec}\, A$ which pulls back to $f^* X_{\mathtt{RZ}_{\mathbb{X}}}$ over $\mathrm{Spf}\, A$. Then we have $f \in \mathtt{RZ}_{\mathbb{X},G}^{(s_\alpha)}(A)$ if and only if there exist morphisms of integral crystals (over $\mathrm{Spec}\, A$)

$$t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$$

such that

(1) For some ideal of definition J of A (or equivalently by Lemma 4.6.3, for any ideal of definition J), the pull-back of t_α over A/J induces the map of isocrystals $s_{\alpha,\mathbb{D}} : \mathbf{1} \to \mathbb{D}(X_{A/J})^{\otimes}[\frac{1}{p}]$.

(2) We choose a formally smooth p-adic W-lift $\widetilde{A}$ of A, and let $(t_\alpha(\widetilde{A}))$ denote the $\widetilde{A}$-section of (t_α) (*cf.* Definition 2.3.4). Then the $\widetilde{A}$-scheme

$$P_{\widetilde{A}} := \underline{\mathrm{isom}}_{\widetilde{A}}\big[(\mathbb{D}(X)(\widetilde{A}), (t_\alpha(\widetilde{A}))], [\widetilde{A} \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]\big),$$

classifying isomorphisms matching $(t_\alpha(\widetilde{A}))$ and $(1 \otimes s_\alpha)$, is a G-torsor. (*Cf.* (2.2.2).) Note that this condition is independent of the choice of $\widetilde{A}$, since any choice of $\widetilde{A}$ are isomorphic.

(3) The Hodge filtration $\mathrm{Fil}^1_X \subset \mathbb{D}(X)(A)$ is a $\{\mu\}$-filtration with respect to $(t_\alpha(A)) \subset \mathbb{D}(X)(A)^{\otimes}$, where $\{\mu\}$ is the unique $G(W)$-conjugacy class of cocharacters such that $b \in G(W) p^{\sigma^* \mu^{-1}} G(W)$.

We thus obtain a functor $\mathtt{RZ}_{\mathbb{X},G}^{(s_\alpha)} : \mathrm{Nilp}_W^{\mathrm{sm}} \to (\mathbf{Sets})$. For $(X, \iota) \in \mathtt{RZ}_{\mathbb{X},G}^{(s_\alpha)})(A)$, we call (t_α) as above *crystalline Tate tensors* or *Tate tensors* on X.

If A is formally smooth and formally finitely generated over W, we write

$$\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A) := \varprojlim_m \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A/p^m) \subset \mathrm{Hom}_W(\mathrm{Spf}\, A, \mathtt{RZ}_{\mathbb{X}}).$$

Example 4.6.1. If $G = \mathrm{GL}(\Lambda)$, we may choose (s_α) to be the empty set. Then we claim that $\mathtt{RZ}_{\mathbb{X}}$ represents $\mathtt{RZ}^{\emptyset}_{\mathbb{X},G}$. Indeed, we only need to check Definition 4.6(3), which is clear since for any $(X,\iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$ the dimension of X is constant on R and consistent with $\{\mu\}$ associated to (G,b); *cf.* Remark 2.2.5.

Remark 4.6.2. Let A be a formally smooth formally finitely generated algebra over either W or W/p^m. Then we may view $\mathrm{Hom}_W(\mathrm{Spf}\, A, \mathtt{RZ}_{\mathbb{X}})$ as the set of isomorphism classes of (X,ι), where X is a p-divisible group over A and ι is a quasi-isogeny defined over $\mathrm{Spf}\, A/p$ (*not* necessarily defined over $\mathrm{Spec}\, A/p$). By rigidity of quasi-isogeny (2.3.2), giving such ι is equivalent to giving an quasi-isogeny $\iota_{A/J} : \mathbb{X}_{A/J} \dashrightarrow X_{A/J}$ for some ideal of definition $J \subset A$ containing p.

Lemma 4.6.3. *Let $R \in \mathrm{Nilp}_W$ and I be a nilpotent ideal of R. Let X be a p-divisible group and consider Frobenius-equivalent morphisms of isocrystals*

$$t, t' : \mathbf{1} \to \mathbb{D}(X)^{\otimes}[\tfrac{1}{p}].$$

Then we have $t = t'$ if and only if the equality holds over R/I.

In particular, Definition 4.6(1) for some ideal of definition J implies Definition 4.6(1) for any ideal of definition J'; indeed, if $J' \subset J$ then we apply the lemma to $R := A/J'$ and $I := J/J'$.

Proof. We may assume that $pR = 0$. Then by Frobenius-equivariance, one can replace X by $\sigma^{n*}X$ for some n, while $\sigma^{n*}X$ only depends on $X_{R/I}$ if $\sigma^n(I) = 0$; *cf.* the proof of [11, Corollary 5.1.2]. □

Lemma 4.6.4. *Let $(X,\iota) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A)$ for some $A \in \mathrm{Nilp}^{\mathrm{sm}}_W$. Then the Tate tensors $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ in Definition 4.6 are uniquely determined by (X,ι).*

Proof. The lemma is known when $A = W[[u_1, \cdots, u_N]]/(p^m)$, since the tensors (t_α) are obtained as the parallel transports of their fibres at the closed point. (This was implicitly stated in Theorem 3.6.) The general case now follows since each of (t_α) is uniquely determined by its pull-back over $\widehat{A}_x$ for each closed point $x \in \mathrm{Spf}\, A$. □

4.7. **Example: unramified EL and PEL cases.** To a (not necessarily unramified) EL or PEL datum, Rapoport and Zink formulated a suitable moduli problem of p-divisible groups with extra structure, and constructed a representing formal scheme [39, Theorem 3.25].[13] In this section, we show that when (G,b) comes from an unramified EL or PEL datum, the formal moduli schemes constructed by Rapoport and Zink represents $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ for some suitable choice of $(\Lambda, (s_\alpha))$.

Let us first recall the setting of [39, Ch.3] in the unramified case. Let $\mathscr{O}_B$ be a product of matrix algebras over finite unramified extensions of $\mathbb{Z}_p$, and Λ be a faithful $\mathscr{O}_B$-module which is finite flat over $\mathbb{Z}_p$. We consider the following data (*cf.* [39, §1.38], [18, Ch.2])[14]:

unramified EL case: $(\mathscr{O}_F, \mathscr{O}_B, \Lambda, G)$, where $(\mathscr{O}_B, \Lambda)$ is as before, $\mathscr{O}_F$ is the centre of $\mathscr{O}_B$, and $G = \mathrm{GL}_{\mathscr{O}_B}(\Lambda)$ is a reductive group over $\mathbb{Z}_p$.

[13] See also [18] for the exposition that is just focused on the unramified case of type A and C.

[14] To make the comparison with Definition 4.6 more direct, we work over $\mathbb{Z}_p$ instead of over $\mathbb{Q}_p$ as in the aforementioned references.

unramified PEL case: $(\mathscr{O}_F, \mathscr{O}_B, *, \Lambda, (\,,\,), G)$, where $(\mathscr{O}_F, \mathscr{O}_B, \Lambda, G)$ is an unramified EL-type Rapoport-Zink datum, $(\,,\,)$ is a *perfect* alternating $\mathbb{Z}_p$-bilinear form on Λ, $* : a \mapsto a^*$ is an involution on $\mathscr{O}_B$ such that $(av, w) = (v, a^*w)$ for any $v, w \in \Lambda$, and
$$G = \mathrm{GU}_{\mathscr{O}_B}(\Lambda, (\,,\,))) := \mathrm{GL}_{\mathscr{O}_B}(\Lambda) \cap \mathrm{GU}_{\mathbb{Z}_p}(\Lambda, (\,,\,))$$
where the (scheme-theoretic) intersection takes place inside $\mathrm{GL}_{\mathbb{Z}_p}(\Lambda)$.

For G as above, consider $b \in G(K_0)$ such that we have a p-divisible group $\mathbb{X} := \mathbb{X}_b^\Lambda$ (*cf.* Definition 2.5.5), and choose a conjugacy class $\{\mu\}$ so that $b \in G(W)p^{\sigma^*\mu^{-1}}G(W)$ (*cf.* Definition 2.5.10). In the PEL case, we additionally assume that $\mathrm{ord}_p(c(b)) = -1$, where $c : G \to \mathbb{G}_m$ is the similitude character. This assumption is to ensure that the pairing $(\,,\,)$ induces a polarisation of $\mathbb{X}$ via crystalline Dieudonné theory and $a \mapsto a^*$ corresponds to the Rosati involution; *cf.* [39, §3.20].[15]

Under this setting, Rapoport and Zink formulated a concrete moduli problem for p-divisible groups, and constructed a formal moduli scheme $\breve{\mathcal{M}} := \breve{\mathcal{M}}_{G,b}$, which turns out to be a formally smooth closed formal subscheme of $\mathtt{RZ}_{\mathbb{X}}$. See [39, Definition 3.21, Theorem 3.25, §3.82] for more details. Note that in the PEL case we always assume that $p > 2$.

For the unramified EL case, we choose a $\mathbb{Z}_p$-basis (s_α) of $\mathscr{O}_B$ and view them as elements in $\Lambda \otimes \Lambda^* \subset \Lambda^\otimes$. For the unramified PEL-type case, we additionally include a tensor $s_0 \in \Lambda^{\otimes 2} \otimes \Lambda^{*\otimes 2} \subset \Lambda^\otimes$ associated to the pairing $(\,,\,)$ up to similitude, as in Example 2.1.4. Now we consider the subfunctor $\mathtt{RZ}_{\mathbb{X},G}^{(s_\alpha)}$ of $\mathtt{RZ}_{\mathbb{X}}$ using this choices $(\Lambda, (s_\alpha))$; *cf.* Definition 4.6.

Proposition 4.7.1. *Assume that $p > 2$. In the unramified EL and PEL cases, the subfunctor $\mathtt{RZ}_{\mathbb{X},G}^{(s_\alpha)}$ of $\mathtt{RZ}_{\mathbb{X}}$ can be represented by the formal moduli scheme $\breve{\mathcal{M}}$ constructed by Rapoport and Zink (*cf. *[39, Theorem 3.25, §3.82]).*

Proof. Let us first handle the unramified EL case. By considering simple factors of B and applying the Morita equivalence, it suffices to handle the case when $\mathscr{O}_B = \mathscr{O}_F = W(\kappa_0)$ and $\Lambda = \mathscr{O}_F^n$, where κ_0 is a finite extension of $\mathbb{F}_p$. Let us assume this.

Let (X, ι) denote a p-divisible group over $A \in \mathrm{Nilp}_W^{\mathrm{sm}}$ with quasi-isogeny defined over A/J for some (or any) ideal of definition J containing p. We choose (s_α) corresponding to a $\mathbb{Z}_p$-basis of $\mathscr{O}_B$. If (X, ι) corresponds to $\mathrm{Spf}\, A \to \breve{\mathcal{M}}$, then the induced $\mathscr{O}_B$-action on X gives rise to Tate tensors (t_α). If we have $(X, \iota) \in \mathtt{RZ}_{\mathbb{X},G}^{(s_\alpha)}(A)$, then the Tate tensors (t_α) should correspond to an $\mathscr{O}_B$-action by the full faithfulness of the Dieudonné theory over A/p (*cf.* [11]) and the Grothendieck-Messing deformation theory.

From now on, we assume that X is equipped with an $\mathscr{O}_B$-action corresponding to the Tate tensors (t_α), and we will translate Definition 4.6 in terms of endomorphism, polarisation, and Kottwitz determination condition. For any ideal of definition $J \subset A$ containing p, the crystalline Dieudonné functor over A/J is fully faithful up to isogeny by [11, Corollary 5.1.2]. Therefore, $\mathbb{D}(\iota)$ matches (t_α) for $\mathbb{X}$ and $X_{A/J}$ if and only if ι is a B-linear quasi-isogeny. If these equivalent properties hold, then we claim that the scheme $P_{\widetilde{A}}$ (with the notation as in Definition 4.6(2)) is a $\mathrm{GL}_{\mathscr{O}_B}(\Lambda)$-torsor. Indeed, it suffices to handle the case when $\widetilde{A}$ is a formal power series ring over W by considering the completions of $\widetilde{A}$ at maximal ideals. Since $\mathscr{O}_B \otimes_{\mathbb{Z}_p} \widetilde{A} = \prod_{\tau:\kappa_0 \hookrightarrow \overline{\mathbb{F}}_p} \widetilde{A}$, the existence of B-linear quasi-isogeny ι implies that $\mathbb{D}(X)(\widetilde{A})$ is a free $\mathscr{O}_B \otimes_{\mathbb{Z}_p} \widetilde{A}$-module with the same rank as Λ, which implies that $P_{\widetilde{A}}$ is a trivial $\mathrm{GL}_{\mathscr{O}_B}(\Lambda)$-torsor.

---

[15] Note that our choice of b is the transpose-inverse of the Frobenius matrix (*cf.* the remark below Definition 2.5.5), so in our convention we have $\mathrm{ord}_p(c(b)) = -1$, which differs by sign from [39, §3.20].

Let us show that Fil^1_X is a $\{\mu\}$-filtration with respect to $(t_\alpha(A))$ (Definition 4.6(3)) if and only if the "Kottwitz determinant condition" holds for X_{A/J^n} for each n [39, Definition 3.21(iv)]. For this, it suffices to show that for a fixed n and $R := A/J^n$, $\mathrm{Fil}^1_{X_R}$ is a $\{\mu\}$-filtration if and only if the "Kottwitz determination condition" holds. Since the claim is étale-local on $\mathrm{Spec}\, R$,[16] we may assume that the torsor P_R is trivial. Since $\mathscr{O}_B = W(\kappa_0)$ for some finite extension κ_0 of $\mathbb{F}_p$, we can decompose

$$\mathbb{D}(X_R)(R) = \prod_{\tau\in\mathrm{Hom}(\kappa_0,\overline{\mathbb{F}}_p)} \mathbb{D}(X_R)(R)_\tau$$
$$\mathrm{Fil}^1_{X_R} = \prod_{\tau\in\mathrm{Hom}(\kappa_0,\overline{\mathbb{F}}_p)} \mathrm{Fil}^1_{X_R,\tau}\,.$$

It follows that the $G(R)$-conjugacy classes of (minuscule) cocharacters μ exactly correspond to certain integer tuples $(a_\tau)_\tau$ with $a_\tau \in [0, \mathrm{rk}_{R\otimes_{\mathbb{Z}_p}\mathscr{O}_B}\mathbb{D}(X_R)(R)_\tau]$, and $\mathrm{Fil}^1_{X_R}$ is a $\{\mu\}$-filtration if and only if $a_\tau = \mathrm{rk}_R \mathrm{Fil}^1_{X_R,\tau}$, which is exactly the Kottwitz determinant condition.

Let us turn to the unramified PEL case. Since $\mathrm{ord}_p(c(b)) = 1$, there exists $u \in W^\times$ such that $c(b) = p^{-1}\sigma(u)^{-1}u$. Then we obtain an F-equivariant perfect pairing

$$u(,): \mathbf{M}^\Lambda_b \otimes \mathbf{M}^\Lambda_b \xrightarrow{(,)} \mathbf{M}_{c(b)^{-1}} \xrightarrow{u} \mathbf{M}_p = \mathbf{1}(-1),$$

where $\mathbf{M}_z$ for $z \in \mathbb{G}_m(K_0)$ denotes the F-crystal on W with F given by multiplication by z. (Recall that $\mathbf{M}^\Lambda_b = W\otimes_{\mathbb{Z}_p}\Lambda^*$, so the similitude character for the pairing $(,)$ on $\mathbf{M}^\Lambda_b$ is c^{-1}.) Then $u(,)$ induces a principal polarisation $\lambda_0 : \mathbb{X} \to \mathbb{X}^\vee$, and the $\mathbb{Z}_p^\times \cdot \lambda_0$ is well defined independent of the choice of u.

Conversely, given an $\mathscr{O}_B$-linear principal polarisation $\lambda : X \to X^\vee$ of a p-divisible group over $A \in \mathrm{Nilp}^{\mathrm{sm}}_W$, we obtain a tensor $t_\lambda : \mathbf{1} \to \mathbb{D}(X)^{\otimes 2}\otimes\mathbb{D}(X)^{*\otimes 2}$ only depending on $\mathbb{Z}_p^\times\cdot\lambda$; *cf.* Example 2.1.4. By the full faithfulness result as before, the existence of a principal polarisation $\lambda : X \to X^\vee$ is equivalent to the existence of a certain Tate tensor t_λ. If $s_0 \in \Lambda^\otimes$ is the tensor corresponding to $\mathbb{Z}_p^\times\cdot(,)$ and t_0 the Tate tensor on $\mathbb{X}$ corresponding to s_0, then we have $t_0 = t_{\lambda_0}$.

We choose a W-lift $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ which lifts the $\mathscr{O}_B$-action and λ_0. We let λ_0 and t_0 also denote their lifts over W. For any $(X', \iota') \in \mathtt{RZ}_{\mathbb{X}}(R')$ with $R' \in \mathrm{Nilp}_W$, we choose the unique lift $\tilde\iota' : \mathbb{X}_{R'} \dashrightarrow X'$.

We now claim the following are equivalent:

(1) For any n, the quasi-isogeny $\tilde\iota : \widetilde{\mathbb{X}}_{A/J^n} \dashrightarrow X_{A/J^n}$ matches the Tate tensors t_0 for $\widetilde{\mathbb{X}}$ and t_λ for X. Furthermore, the scheme $P_{\widetilde{A}}$, constructed as in Definition 4.6(2) using $(t_\alpha(\widetilde{A}))$ and $t_\lambda(\widetilde{A})$, is a $\mathrm{GU}_{\mathscr{O}_B}(\Lambda)$-torsor;

(2) The polarisation λ is $\mathscr{O}_B$-linear. For any n, the following diagram commutes up to the multiple by some Zariski-locally constant function $c : \mathrm{Spec}\, A/J^n \to \mathbb{Q}_p^\times$:

(4.7.2)
$$\begin{tikzcd} X_{A/J^n} \arrow[r, "\lambda"] & X^\vee_{A/J^n} \arrow[d, dashed, "\tilde\iota^\vee"'] \\ \widetilde{\mathbb{X}}_{A/J^n} \arrow[u, dashed, "\tilde\iota"'] \arrow[r, "\lambda_0"] & \widetilde{\mathbb{X}}^\vee_{A/J^n} \end{tikzcd}.$$

[16]The notion of $\{\mu\}$-filtration is étale-local on the base. The Kottwitz determinant condition can be phrased in terms of the ranks of certain vector bundles, and ranks can be computed étale-locally.

Granting this claim, it follows that $\breve{\mathscr{M}}$ represents $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ in the unramified PEL case; indeed, the "Kottwitz determinant condition" and Definition 4.6(3) can be matched in the identical way as in the unramified EL case.

Let us first show that (1)⇒(2). For this, we may replace $\operatorname{Spf} A$ by some étale covering to assume that the torsor $P_{\widetilde{A}}$ is trivial. Then the $\mathscr{O}_B$-linearity of λ follows from the full faithfulness of the Dieudonné theory over A/p (*cf.* [11]) and Grothendieck-Messing deformation theory.

Now, let $(,)_0 : \mathbb{D}(\widetilde{\mathbb{X}})^{\otimes 2} \to \mathbf{1}(-1)$ and $(,)_\lambda : \mathbb{D}(X)^{\otimes 2} \to \mathbf{1}(-1)$ denote the perfect symplectic F-equivariant pairing induced by the principal polarisations. To simplify the notation, set $R := A/J^n$. By definition of t_0 and t_λ, the quasi-isogeny $\tilde{\iota}$ over R matches t_0 and t_λ if and only if there exists a Zariski-locally constant function $c : \operatorname{Spec} R \to \mathbb{Q}_p^\times$ such that $c(,)_{\lambda,R} = (,)_{0,R} \circ (\mathbb{D}(\tilde{\iota})^{\otimes 2})$ as pairings on $\mathbb{D}(X_R)[\frac{1}{p}]$. (Indeed, c is $\mathbb{Q}_p^\times$-valued because the automorphism of the F-isocrystal $\mathbf{1}(-1)$ over a finite-type κ-scheme is $\mathbb{Q}_p^\times$ – a very degenerate case of [11, Main Theorem 2].) On the other hand, $\mathbb{D}(\lambda_0^{-1}\tilde{\iota}^\vee\lambda) : \mathbb{D}(\widetilde{\mathbb{X}}_R)[\frac{1}{p}] \to \mathbb{D}(X_R)[\frac{1}{p}]$ is the "transpose" of $\mathbb{D}(\tilde{\iota})$ in the sense that

$$(,)_{0,R} \circ (\mathbb{D}(\tilde{\iota}) \otimes \mathrm{id}) = (,)_{\lambda,R} \circ (\mathrm{id} \otimes \mathbb{D}(\lambda_0^{-1}\tilde{\iota}^\vee\lambda)) : \mathbb{D}(X_R)[\tfrac{1}{p}] \otimes \mathbb{D}(\widetilde{\mathbb{X}}_R)[\tfrac{1}{p}] \to \mathbf{1}(-1).$$

Therefore, we have $c(,)_{\lambda,R} = (,)_{0,R} \circ (\mathbb{D}(\tilde{\iota})^{\otimes 2})$ if and only if we have $\mathbb{D}(\lambda_0^{-1}\tilde{\iota}^\vee\lambda\tilde{\iota}) = c\,\mathrm{id}$. By full faithfulness of Dieudonné theory up to isogeny over R, this is equivalent to $\tilde{\iota}^\vee\lambda_R\tilde{\iota} = c\lambda_{0,R}$. This shows that (1)⇒(2).

To show (2)⇒(1), it remains to show that (2) implies that $P_{\widetilde{A}}$ is a $\mathrm{GU}_{\mathscr{O}_B}(\Lambda)$-torsor. Since $\breve{\mathscr{M}}$ is formally smooth, we may lift (X,ι) so that $A = \widetilde{A}$ is formally smooth and formally finitely generated over W. Then, it suffices to show that P_{A/J^n} is a $\mathrm{GU}_{\mathscr{O}_B}(\Lambda)$-torsor for each n, which follows from [39, Theorem 3.16]. □

4.8. **"Closed points" and deformation theory for $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$.** We choose (G,b), Λ, and $(s_\alpha) \subset \Lambda^\otimes$ as before. Let $(G,[b],\{\mu^{-1}\})$ denote the unramified integral Hodge-type local Shimura datum associated to (G,b); *cf.* Definition 2.5.10. In this section, we show that the moduli functor $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G} \subset \mathtt{RZ}_{\mathbb{X}}$ interpolates $X^G(b) \subset X^{\mathrm{GL}(\Lambda)}(b)$ on κ-points and $\mathtt{Def}_{X_x,G} \subset \mathtt{Def}_{X_x}$ on "formal completions".

By Proposition 2.5.9, we have a natural bijection

$$X^G(b) \cong \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa). \tag{4.8.1}$$

Recall that $X^G(b)$ satisfies functorial properties with respect to (G,b) (Lemma 2.5.4). We also have the following cartesian diagram:

$$\begin{tikzcd} X^G(b) \arrow[r, hook] \arrow[d, "\cong"] & X^{\mathrm{GL}(\Lambda)}(b) \arrow[d, "\cong"] \\ \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa) \arrow[r, hook] & \mathtt{RZ}_{\mathbb{X}}(\kappa) \end{tikzcd}, \tag{4.8.2}$$

where the vertical isomorphisms are given by Proposition 2.5.9, and the horizontal map on the bottom row is the forgetful map, which is injective by Lemma 4.6.4.

Consider $(X_x, \iota_x) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa)$ corresponding to a closed point $x \in \mathtt{RZ}_{\mathbb{X}}(\kappa)$. We define the "formal completion" $(\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^\wedge_x$ of $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ at x to be the set-valued functor on the category of formal power series rings over $W/(p^m)$ (for some m) defined as follows: for any formal power series ring A over $W/(p^m)$, $(\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^\wedge_x(A)$ is the subset of elements in $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A)$ which lift x. Equivalently, we have the following description

$$(\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^{\widehat{}}_x \cong \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G} \times_{\mathtt{RZ}_{\mathbb{X}}} (\mathtt{RZ}_{\mathbb{X}})^{\widehat{}}_x, \tag{4.8.3}$$

where the right hand side is viewed as the fibre product of functors on the category of formal power series rings over $W/(p^m)$. If $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ can be represented by a "nice" formal scheme, then $(\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^{\widehat{}}_x$ can be represented by its formal completion at x.

Choose a coset representative $g_x \in G(K_0)$ of $x := g_xG(W) \in X^G(b)$, and write $b_x := g_x^{-1}b\sigma(g_x)$ (so that we have $(X_x, (t_{\alpha,x})) \cong (\mathbb{X}^\Lambda_{b_x}, (1\otimes s_\alpha))$). Then for a suitable choice of $\mu \in \{\mu\}$, the universal deformation with Tate tensors $(X^\mu_G, (t^{\mathrm{univ}}_\alpha))$ of $(X_x, (t_{\alpha,x}))$, together with the quasi-isogeny $\iota_x : \mathbb{X} \dashrightarrow X_x$, defines an element $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A^\mu_G)$. (This can be verified by the explicit construction of M^μ_G.) Now, Theorem 3.6 shows that the isomorphism $\mathtt{Def}_{X_x} \xrightarrow{\sim} (\mathtt{RZ}_{\mathbb{X}})^{\widehat{}}_x$ (*cf.* Lemma 4.3.1) induces the following isomorphism

$$\mathtt{Def}_{X_x,G} \xrightarrow{\sim} (\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^{\widehat{}}_x \tag{4.8.4}$$

of functors on formal power series rings over $W/(p^m)$; note that the deformations coming from $\mathtt{Def}_{X_x,G}$ automatically satisfy the condition on the Hodge filtrations (Definition 4.6(3)) thanks to the explicit construction of the universal deformation over $\mathtt{Def}_{X_x,G}$. In particular, $(\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^{\widehat{}}_x$ can be pro-represented by a formal power series ring over W. If $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ can be represented by a formal scheme which is formally smooth and locally formally of finite type over W, then the above isomorphism gives an identification of the formal completion at a closed point x with $\mathtt{Def}_{X_x,G}$.

Let us record some functorial properties that $\mathtt{Def}_{X_x,G}$ enjoys. To explain, let (G', b') and Λ' be as in Definition 2.5.5. (We do *not* assume the existence of any "equivariant" morphism $\Lambda \to \Lambda'$.) We write $\mathbb{X}' := \mathbb{X}^{\Lambda'}_{b'}$.

For any $x' \in X^{G'}(b')$ and the corresponding element $(X'_{x'}, \iota) \in \mathtt{RZ}_{\mathbb{X}'}(\kappa)$ (via the embedding $X^{G'}(b') \hookrightarrow X^{\mathrm{GL}(\Lambda')}(b')$, we have a natural isomorphism

$$\mathtt{Def}_{X_x,G} \times \mathtt{Def}_{X_{x'},G'} \xrightarrow{\sim} \mathtt{Def}_{X_x\times X_{x'},G\times G'}, \tag{4.8.5}$$

defined by taking the product of deformations. This isomorphism is compatible with the morphism $\mathtt{RZ}_{\mathbb{X}}\times\mathtt{RZ}_{\mathbb{X}'} \to \mathtt{RZ}_{\mathbb{X}\times\mathbb{X}'}$ defined by taking the product of p-divisible groups and quasi-isogenies.

Let $f : G \to G'$ be a homomorphism over $\mathbb{Z}_p$ which takes b to b', and consider the map $X^G(b) \to X^{G'}(b')$ associated to f by Lemma 2.5.4. We choose $x \in X^G(b)$ and let $x' \in X^{G'}(b')$ denote its image by this natural map. We choose a coset representative $g_x \in G(K_0)$ of $x = g_xG(W) \in X^G(b)$, and write $b_x := g_x^{-1}b\sigma(g_x)$ and $b'_{x'} := f(b_x)$. Then by the choice of g_x (and $f(g_x)$), we obtain isomorphisms

$$\mathtt{Def}_{X_x,G} \cong \mathtt{Def}_{X^\Lambda_{b_x},G}, \quad \mathtt{Def}_{X'_{x'},G'} \cong \mathtt{Def}_{X^{\Lambda'}_{b'_{x'}},G'}.$$

So by Proposition 3.7.2 we obtain a morphism

$$\mathtt{Def}_{X_x,G} \to \mathtt{Def}_{X'_{x'},G'} \tag{4.8.6}$$

By Remark 3.7.4 it follows that the morphism above does not depend on the choice of the coset representative g_x of $x = g_xG(W) \in X^G(b)$.

4.9. **Main Statements.** We are ready to state the main result, which asserts that the subset $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa) \subset \mathtt{RZ}_{\mathbb{X}}(\kappa)$ and the subspaces $(\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^{\widehat{}}_x \subset (\mathtt{RZ}^\Lambda_b)^{\widehat{}}_x$ for $x \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa)$ patch to give a closed formal subscheme $\mathtt{RZ}^\Lambda_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$, and the functorial properties enjoyed by $X^G(b) \cong \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa)$ and $\mathtt{Def}_{X_x,G} \cong (\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G})^{\widehat{}}_x$ patch to give the corresponding functorial properties for $\mathtt{RZ}^\Lambda_{G,b}$.

Theorem 4.9.1. *Let $p > 2$. Then there exists a closed formal subscheme $\mathtt{RZ}^{\Lambda}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$ which is formally smooth over W and represents the functor $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ for any choice of $(s_\alpha) \subset \Lambda^{\otimes}$ with pointwise stabiliser G. More precisely, for any s_α there exist "universal Tate tensors"*

$$t^{\mathrm{univ}}_{\alpha} : \mathbf{1} \to \mathbb{D}((X_{\mathtt{RZ}_{\mathbb{X}}})|_{\mathtt{RZ}^{\Lambda}_{G,b}})^{\otimes},$$

such that for $A \in \mathrm{Nilp}^{\mathrm{sm}}_{W}$, a map $f : \mathrm{Spf}\, A \to \mathtt{RZ}_{\mathbb{X}}$ factors through $\mathtt{RZ}^{\Lambda}_{G,b}$ if and only if $f \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A)$, in which case (t_α) as in Definition 4.6 recovers $(f^* t^{\mathrm{univ}}_{\alpha})$.

For another pair (G', b') and $\Lambda' \in \mathrm{Rep}_{\mathbb{Z}_p}(G')$ that give rise to a p-divisible group (as in Definition 2.5.5), we consider the closed formal subscheme $\mathtt{RZ}^{\Lambda'}_{G',b'} \subset \mathtt{RZ}_{\mathbb{X}'}$ which was just constructed. Then the following properties hold:

(1) *The morphism $\mathtt{RZ}_{\mathbb{X}} \times_{\mathrm{Spf}\, W} \mathtt{RZ}_{\mathbb{X}'} \to \mathtt{RZ}_{\mathbb{X}\times\mathbb{X}'}$, defined by the product of p-divisible groups with quasi-isogeny, induces an isomophism*

$$\mathtt{RZ}^{\Lambda}_{G,b} \times_{\mathrm{Spf}\, W} \mathtt{RZ}^{\Lambda'}_{G',b'} \xrightarrow{\sim} \mathtt{RZ}^{\Lambda\times\Lambda'}_{G\times G',(b,b')}$$

such that it induces the product decomposition of affine Deligne-Lusztig sets (Lemma 2.5.4) and the deformation spaces (Proposition 3.7.2) via the isomorphisms (4.8.1) and (4.8.4), respectively.

(2) *Let $f : G \to G'$ be a homomorphism which takes b to b'. Then there exists a (necessarily unique) morphism $\mathtt{RZ}^{\Lambda}_{G,b} \to \mathtt{RZ}^{\Lambda'}_{G',b'}$ which induces the maps that fit in the following cartesian diagrams:*

$$\begin{tikzcd} X^G(b) \arrow[r, "\textit{Lem 2.5.4}"] \arrow[d, "\cong"', "(4.8.1)"] & X^{G'}(b') \arrow[d, "\cong"', "(4.8.1)"] \\ \mathtt{RZ}^{\Lambda}_{G,b}(\kappa) \arrow[r] & \mathtt{RZ}^{\Lambda'}_{G',b'}(\kappa) \end{tikzcd} \qquad \begin{tikzcd} \mathtt{Def}_{X_x,G} \arrow[r, "(4.8.6)"] \arrow[d, "\cong"', "(4.8.4)"] & \mathtt{Def}_{X_{x'},G'} \arrow[d, "\cong"', "(4.8.4)"] \\ (\mathtt{RZ}^{\Lambda}_{G,b})^{\wedge}_{x} \arrow[r] & (\mathtt{RZ}^{\Lambda'}_{G',b'})^{\wedge}_{x'} \end{tikzcd},$$

where $x' \in \mathtt{RZ}^{\Lambda'}_{G',b'}(\kappa)$ is the image of x, and the arrows on the top row are the natural maps induced by f.

Furthermore, if $G' = \mathrm{GL}(\Lambda)$, then the natural inclusion $(G, b) \to (\mathrm{GL}(\Lambda), b)$ induces the natural inclusion $\mathtt{RZ}^{\Lambda}_{G,b} \to \mathtt{RZ}^{\Lambda}_{\mathrm{GL}(\Lambda),b} = \mathtt{RZ}_{\mathbb{X}}$ (cf. Example 4.6.1).

We will prove this theorem in §5 and §6. Let us now record immediate consequences of Theorem 4.9.1.

Remark 4.9.2. When the pair (G, b) and the choice of $(\Lambda, (s_\alpha))$ correspond to the unramified EL or PEL case (*cf.* §4.7), then $\mathtt{RZ}^{\Lambda}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$ coincides with the formal moduli subscheme $\breve{\mathcal{M}} \subset \mathtt{RZ}_{\mathbb{X}}$ constructed by Rapoport and Zink, often referred to as an EL or PEL Rapoport-Zink space.

Remark 4.9.3. Choose $g \in G(K_0)$ such that $b' := g^{-1} b \sigma(g)$ also defines a p-divisible group $\mathbb{X}' := \mathbb{X}^{\Lambda}_{b'}$. We let $\iota_g : \mathbb{X} \dashrightarrow \mathbb{X}'$ denote the quasi-isogeny induced by $g : \mathbf{M}^{\Lambda}_{b'}[\frac{1}{p}] \xrightarrow{\sim} \mathbf{M}^{\Lambda}_{b}[\frac{1}{p}]$. Then we have a natural isomorphism $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X}',G} \xrightarrow{\sim} \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ by post-composing ι_g (or rather, the suitable base change of it). In particular, we obtain an isomorphism $\mathtt{RZ}^{\Lambda}_{G,b'} \xrightarrow{\sim} \mathtt{RZ}^{\Lambda}_{G,b}$.

Remark 4.9.4. Applying Theorem 4.9.1(2) to $f = \mathrm{id} : (G, b) \to (G, b)$, we obtain a canonical isomorphism $\mathtt{RZ}^{\Lambda}_{G,b} \xrightarrow{\sim} \mathtt{RZ}^{\Lambda'}_{G,b}$ which respects the identifications of κ-points with $X^G(b)$ and the formal completion at a κ-point x with $\mathtt{Def}^{\Lambda}_{G,b_x}$.

Combining this with Remark 4.9.3, it follows that $\mathtt{RZ}^{\Lambda}_{G,b}$ only depends on the associated unramified integral Hodge-type local Shimura datum $(G, [b], \{\mu^{-1}\})$ up to (usually non-canonical) isomorphism. Furthermore, if $(G', [b'], \{\mu'^{-1}\})$ is another unramified integral Hodge-type local Shimura datum, then for any map f :

$(G, [b], \{\mu^{-1}\}) \to (G', [b'], \{\mu'^{-1}\})$, we have a morphism $\mathtt{RZ}^{\Lambda}_{G,b} \to \mathtt{RZ}^{\Lambda'}_{G',b'}$ for some suitable choice of (b, Λ) and (b', Λ').

Remark 4.9.5. In the setting of Theorem 4.9.1(2), the association

$$[(G, b) \to (G', b')] \rightsquigarrow [\mathtt{RZ}^{\Lambda}_{G,b} \to \mathtt{RZ}^{\Lambda'}_{G',b'}]$$

respects compositions and products of morphisms (in the obvious sense), since the analogous statement is true for $X^G(b)$ and $\mathtt{Def}^{\Lambda}_{G,b_x}$.

Remark 4.9.6. If $f : G \to G'$ is a closed immersion (so we write $b' = b$), then the associated map $\mathtt{RZ}^{\Lambda}_{G,b} \to \mathtt{RZ}^{\Lambda'}_{G',b}$ is a closed immersion. Indeed, by Remark 4.9.4 we may take $\Lambda = \Lambda'$, and the natural inclusion $\mathtt{RZ}^{\Lambda}_{G,b} \hookrightarrow \mathtt{RZ}_{\mathbb{X}}$ factors as $\mathtt{RZ}^{\Lambda}_{G,b} \to \mathtt{RZ}^{\Lambda}_{G',b} \hookrightarrow \mathtt{RZ}_{\mathbb{X}}$ by Remark 4.9.5.

Remark 4.9.7. If (G, b) comes from an unramified EL or PEL datum, then it follows from Proposition 4.7.1 that our construction of $\mathtt{RZ}^{\Lambda}_{G,b}$ is compatible with the original construction of Rapoport and Zink. On the other hand, the "functoriality" aspect of Theorem 4.9.1 produces morphisms between EL and PEL Rapoport-Zink spaces which cannot be constructed as a formal consequence of the (P)EL moduli problem. For example, consider an exceptional isomorphism $\mathrm{GSp}(4) \cong \mathrm{GSpin}(3, 2)$ of split reductive $\mathbb{Z}_{(p)}$-groups, and set $G := \mathrm{GSp}(4)_{\mathbb{Z}_p}$. Let $(G, [b], \{\mu^{-1}\})$ be a unramified Hodge-type local Shimura datum, coming from a (global) Shimura datum for $\mathrm{GSp}(4)_{/\mathbb{Q}}$ as in Example 2.5.11. Recall that we have another faithful G-representation Λ; namely, the even Clifford algebra for a split rank-5 quadratic space over $\mathbb{Z}_p$. Assume that there exists a perfect alternating form ψ such that the natural G-action on Λ induces a closed immersion $f : G = \mathrm{GSpin}(3, 2) \hookrightarrow \mathrm{GSp}(\Lambda, \psi) =: G'$ and the local Shimura datum $(G', [f(b)], \{f \circ \mu^{-1}\})$ is of PEL type.[17] Then Theorem 4.9.1 produces a closed immersion $\mathtt{RZ}_{G,b} \hookrightarrow \mathtt{RZ}_{G',f(b)}$ (without studying how the spin representation sits in the Clifford algebra).

5. Descent and Extension of Tate tensors from a complete local ring to a global base

In this section, we prove the technical results (especially, Propositions 5.2 and 5.6) which allow us to "globalise" the Faltings deformation spaces.

Let $\mathrm{Nilp}^{\mathrm{ft}}_W$ be the category of finitely generated W/p^m-algebras for some m. We first define the following subfunctor $\mathtt{RZ}^{\Lambda}_{G,b}$ of $\mathtt{RZ}_{\mathbb{X}}$. We will show in §6 that this subfunctor can be represented by a closed formal scheme of $\mathtt{RZ}_{\mathbb{X}}$ which satisfies the desired properties stated in Theorem 4.9.1.

Definition 5.1. In the setting of §4.4, we define a functor $\mathtt{RZ}^{\Lambda}_{G,b} : \mathrm{Nilp}^{\mathrm{ft}}_W \to (\mathbf{Sets})$ as follows: for any $R \in \mathrm{Nilp}^{\mathrm{ft}}_W$, $\mathtt{RZ}^{\Lambda}_{G,b}(R) \subset \mathtt{RZ}_{\mathbb{X}}(R)$ consists of $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$ such that for any $x : \mathrm{Spec}\, \kappa \to \mathrm{Spec}\, R$, we have $(X_x, \iota_x) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa)$ and the map $\mathrm{Spf}\, \widehat{R}_x \to \mathtt{Def}_{X_x}$, induced by $X_{\widehat{R}_x}$, factors through $\mathtt{Def}_{X_x,G}$.

For $(X, \iota) \in \mathtt{RZ}_G(R)$ with $R \in \mathrm{Nilp}^{\mathrm{ft}}_W$, we set

$$\hat{t}_{\alpha,x} : \mathbf{1} \to \mathbb{D}(X_{\widehat{R}_x})^{\otimes} \tag{5.1.1}$$

to be the pull-back of the universal Tate tensors $\hat{t}^{\mathrm{univ}}_{\alpha,x}$ (3.5.2) via $\mathrm{Spf}\, \widehat{R}_x \to \mathtt{Def}_{X_x,G}$.

We extend the definition of the subfunctor $\mathtt{RZ}^{\Lambda}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$ on Nilp_W by direct limit; namely, for any $R' \in \mathrm{Nilp}_W$, $(X', \iota') \in \mathtt{RZ}_{\mathbb{X}}(R')$ lies in $\mathtt{RZ}^{\Lambda}_{G,b}(R')$ if and only if for some finitely generated W-subalgebra $R \subset R'$, there exists $(X, \iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ which descends (X', ι').

[17] If ψ is not perfect, ψ induces a perfect alternating form on $\Lambda^{\oplus 4} \oplus \Lambda^*(-1)^{\oplus 4}$ by Zarhin's trick, so we still obtain a morphism of PEL local Shimura data of type C.

Lemma 5.1.2. *The functor* $\mathtt{RZ}^{\Lambda}_{G,b}$ *on* Nilp_W *commutes with arbitrary filtered direct limits.*

Proof. Let $\{R_\xi\}$ be a filtered direct system of rings in Nilp_W, and set $R := \varinjlim_\xi R_\xi$. We want to show that for any $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ there exists $(X_\xi, \iota_\xi) \in \mathtt{RZ}^{\Lambda}_{G,b}(R_\xi)$ which pulls back to (X,ι). Let us first handle the case when R and R_ξ are finitely generated over W. In that case, there exists R_ξ such that the natural map $R_\xi \to R$ is surjective and admits a section $R \to R_\xi$.[18] Now, we set $(X_\xi, \iota_\xi) \in \mathtt{RZ}^{\Lambda}_{G,b}(R_\xi)$ to be the pull back of (X,ι), which has the desired property.

The general case can be reduced to this special case; indeed, we may replace R with some finitely generated W-subalgebra $R_0 \subset R$, and work with a filtered direct system of finitely generated W-subalgebras of R_ξ whose direct limit is R_0. $\square$

From the definition of $\mathtt{RZ}^{\Lambda}_{G,b}$, it is not clear whether $\mathtt{RZ}^{\Lambda}_{G,b}$ is formally smooth, and whether we have $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A) \cong \varprojlim_n \mathtt{RZ}^{\Lambda}_{G,b}(A/J^n)$ for $A \in \mathrm{Nilp}^{\mathrm{sm}}_W$ with ideal of definition J (*cf.* Corollary 5.2.3). Also it is unclear how to rule out the possibility that $\mathtt{RZ}^{\Lambda}_{G,b}$ is the disjoint union of $\mathtt{Def}_{X_x,G}$ when we do not expect this to be the case (*cf.* Proposition 5.6). These issues will be resolved by the technical results proved in this section.

Let us first state our descent result:

Proposition 5.2. *Assume that* $p > 2$ *and let* $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ *for* $R \in \mathrm{Nilp}^{\mathrm{ft}}_W$. *Then for each* α, *there exists a unique morphism of crystals*

$$t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$$

which induces $s_{\alpha,\mathbb{D}}$ *on the isocrystals (*cf. *Definition 4.5), and pulls back to* $\hat{t}_{\alpha,x}$ *for each closed point* x *in* $\operatorname{Spec} R$ *(*cf. *(5.1.1)).*

Before we prove the proposition (in §5.3–§5.5), let us record a few direct consequences. We begin with the following definition:

Definition 5.2.1. Let $R \in \mathrm{Nilp}_W$, and assume that for a finitely generated W-subalgebra $R_0 \subset R$ $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ descends to $(X_{R_0},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R_0)$. Then we define

$$t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$$

by pulling back $t_\alpha : \mathbf{1} \to \mathbb{D}(X_{R_0})^{\otimes}$, constructed in Proposition 5.2. By the uniqueness part of Proposition 5.2, it follows that t_α on X is independent of the choice of $R_0 \subset R$ and (X_{R_0},ι).

Corollary 5.2.2. *Assume that* $p > 2$. *Let* $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ *for* $R \in \mathrm{Nilp}_W$, *and* $B \twoheadrightarrow R$ *be a square-zero thickening, viewed as a PD thickening via "square-zero PD structure". Consider the morphisms of crystals* $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ *as in Definition 5.2.1, and let* $(t_\alpha(B)) \subset \mathbb{D}(X)(B)^{\otimes}$ *denote their sections.*

(1) *The following* B*-scheme is a* G*-torsor:*
$$P_B := \underline{\mathrm{isom}}_B\big([\mathbb{D}(X)(B),(t_\alpha(B))],[B \otimes_{\mathbb{Z}_p} \Lambda^*,(1\otimes s_\alpha)]\big),$$
(2) *The Hodge filtration* $\mathrm{Fil}^1_X \subset \mathbb{D}(X)(R)$ *is a* $\{\mu\}$*-filtration with respect to* $(t_\alpha(R)) \subset \mathbb{D}(X)(R)^{\otimes}$.
(3) *Let* $(\widetilde{X},\iota) \in \mathtt{RZ}_{\mathbb{X}}(B)$ *be a lift of* (X,ι). *Then we have* $(\widetilde{X},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(B)$ *if and only if the Hodge filtration* $\mathrm{Fil}^1_{\widetilde{X}} \subset \mathbb{D}(\widetilde{X})(B) \cong \mathbb{D}(X)(B)$ *is a* $\{\mu\}$*-filtration with respect to* $(t_\alpha(B))$.

[18] Indeed, we choose R_{ξ_0} which surjects to R. Then for some $\xi \geqslant \xi_0$, the natural map $R_{\xi_0} \to R_\xi$ should factor through R in order to have $R = \varinjlim_\xi R_\xi$.

Furthermore, the functor $\mathtt{RZ}^{\Lambda}_{G,b}$ on Nilp_W is formally smooth.

Proof. To prove the proposition, we may replace R with a finitely generated W-subalgebra $R_0 \subset R$ such that (X,ι) descends to $\mathtt{RZ}^{\Lambda}_{G,b}(R_0)$, and B with a finitely generated W-subalgebra of $B_0 \subset B$ surjecting onto R_0. (For (3), we enlarge B_0 so that $(\widetilde{X},\iota)$ descends to $\mathtt{RZ}_{\mathbb{X}}(B_0)$.) Therefore we assume that B and R are finitely generated over W.

For a closed point $x \in \operatorname{Spec} R$, let $\widehat{B}_x$ denote the completion at x viewing it as a point of $\operatorname{Spec} B$. Then $P_{\widehat{B}_x}$ is a trivial G-torsor, as it is the pull back of the trivial G-torsor obtained from the universal deformation of X_x with Tate tensors. Since the natural map $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} B$ is a homeomorphism on the underlying topological spaces, we obtain (1).

Claim (2) follows from Lemma 2.2.6 since $\mathrm{Fil}^1_X \otimes_R \widehat{R}_x$ are $\{\mu\}$-filtrations with respect to $(\hat{t}_{\alpha,x}(\widehat{R}_x))$. Finally, (3) is a consequence of Proposition 3.8, and formal smoothness of $\mathtt{RZ}^{\Lambda}_{G,b}$ is clear from (3). □

Corollary 5.2.3. *Assume that $p>2$, and let A be a formally smooth formally finitely generated algebra over W or W/p^m with ideal of definition J. We consider a projective system $(X^{(i)},\iota) \in \mathtt{RZ}_{\mathbb{X}}(A/J^i)$ (corresponding to a map $f:\operatorname{Spf} A \to \mathtt{RZ}_{\mathbb{X}}$), and let X denote the p-divisible group over A reducing to each of $X^{(i)}$. Then we have $(X^{(i)},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(A/J^i)$ for each i if and only if $(X,\iota) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A)$.*

Proof. It suffices to handle the case when A is formally smooth over W/p^m. The "if" direction is straightforward. To prove the "only if" direction, choose a p-adic formally smooth W-lift $\widetilde{A}$ of A, and let $\widetilde{J}$ denote the preimage of J, which is an ideal of definition of $\widetilde{A}$. Let $t_\alpha(\widetilde{A}) \in \mathbb{D}(X)(\widetilde{A})^{\otimes}$ denote the limit of the sections $t_\alpha(\widetilde{A}/\widetilde{J}^i) \in \mathbb{D}(X_{\widetilde{A}/(p^m,\widetilde{J}^i)})(\widetilde{A}/\widetilde{J}^i)^{\otimes}$, where (t_α) is constructed in Proposition 5.2. Let $(t_\alpha(A)) \subset \mathbb{D}(X)(A)^{\otimes}$ denote the image of $(t_\alpha(\widetilde{A}))$.

Let us first show that $t_\alpha(\widetilde{A})$ is horizontal with respect to the crystalline connection, which ensures that it comes from a morphism $t_\alpha:\mathbf{1}\to\mathbb{D}(X)^{\otimes}$ of crystals over $\operatorname{Spec} A$. To verify this, we may assume that $\operatorname{Spec} A$ are connected. Then for any closed point $x \in \operatorname{Spec} A$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{D}(X)(\widetilde{A})^{\otimes} \arrow[r] \arrow[d, hook] & \mathbb{D}(X)(\widetilde{A})^{\otimes} \otimes_{\widetilde{A}} \widehat{\Omega}_{\widetilde{A}} \arrow[d, hook] \\
\mathbb{D}(X_{(\widetilde{A})^{\wedge}_x})((\widetilde{A})^{\wedge}_x)^{\otimes} \arrow[r] & \mathbb{D}(X_{(\widetilde{A})^{\wedge}_x})((\widetilde{A})^{\wedge}_x)^{\otimes} \otimes_{(\widetilde{A})^{\wedge}_x} \widehat{\Omega}_{(\widetilde{A})^{\wedge}_x}
\end{tikzcd} \quad . \tag{5.2.4}$$

Note that the image of $t_\alpha(\widetilde{A})$ under the left vertical arrow is the section $\hat{t}_{\alpha,x}((\widetilde{A})^{\wedge}_x)$ of $\hat{t}_{\alpha,x}$, so it is killed by the bottom horizontal arrow. By injectivity of the vertical arrows we have the desired claim.

Since we have (t_α) on X, we may verify Definition 4.6 for (X,ι) at the completions of each closed point of $\operatorname{Spec} A$, which follows from the isomorphism (4.8.4) and the assumption that $(X_{A/J^i},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(A/J^i)$ for any i. □

Let us now begin the proof of Proposition 5.2.

5.3. **Preparation: Liftable PD thickenings.**

Definition 5.3.1. For a ring R where p is nilpotent, a compatible PD thickening $S \twoheadrightarrow R$ is called *p-adic* if $S = \varprojlim_n S/p^n$ where each S/p^n is a PD thickening.

Let $B \twoheadrightarrow R$ be a PD thickening with $B,R \in \mathrm{Nilp}_W$. We call B a *liftable* PD thickening of R if there exist a p-adic p-torsion free PD thickening $S \twoheadrightarrow R$ and a surjective PD morphism $S \twoheadrightarrow B$.

Note that there exists a ring R of characteristic p which does not admit a p-adic p-torsion free PD thickening (so the trivial thickening $R \xrightarrow{=} R$ is not liftable for such R). See [42, Remark 4.1.6] for such an example, which is the quotient of a perfect $\mathbb{F}_p$-algebra by an infinitely generated ideal. The author knows neither a proof nor a counterexample of the claim that any finitely generated W/p^m-algebra R admits a p-adic W-flat PD thickening of R.

The aim of this section is to show that "sufficiently many" rings in characteristic p admit lots of liftable PD thickenings (although we cannot cover all finitely generated κ-algebras).

Recall that a ring R of characteristic p is called *semiperfect* if the Frobenius map $\sigma : R \to R$ is surjective. To a semiperfect ring R we can associate a perfect ring

$$R^\flat := \varprojlim_{\sigma} R, \tag{5.3.2}$$

complete with respect to the natural projective limit topology. Let $J \subset R^\flat$ denote the kernel of the natural projection $R^\flat \twoheadrightarrow R$. For any semiperfect ring R we define

$$A_{\mathrm{cris}}(R) \twoheadrightarrow R \tag{5.3.3}$$

to be the p-adically completed PD hull of the composition $W(R^\flat) \twoheadrightarrow R^\flat \twoheadrightarrow R$.[19] It turns out to be the universal PD thickening of R; *cf.* [42, Proposition 4.1.3].

Let us recall the following definition from [42, Definition/Proposition 4.1.2]:

Definition 5.3.4. A semiperfect ring R is called *f-semiperfect* if there is a finitely generated ideal $J_0 \subset R^\flat$ such that for some $n \gg 0$ we have $\sigma^n(J_0) \subset J \subset J_0$.

In the above setting, there is an ideal $J' \subset R^\flat$ with $\sigma(J') = (J')^p$, such that for some n' we have $\sigma^{n'}(J') \subset J \subset J'$; indeed, $J' := \bigcup_m \sigma^{-m}(J_0^{p^m})$ works.[20]

Lemma 5.3.5. *Let R be an f-semiperfect ring of characteristic p such that $J := \ker(R^\flat \twoheadrightarrow R)$ satisfies $\sigma(J) = J^p$. Set $R^\sigma := R^\flat/\sigma(J)$, and view it as a PD thickening of R by giving $f^{[p]} = 0$ for any $f \in J/\sigma(J)$, which is well-defined since $\sigma(J) = J^p$. By the universal property of $A_{\mathrm{cris}}(R)$, we obtain a unique PD morphism $A_{\mathrm{cris}}(R) \twoheadrightarrow R^\sigma$. Then there exists a p-adic p-torsion free PD thickening $W_{\mathrm{PD},1}(R) \twoheadrightarrow R$ such that there is a PD isomorphism $W_{\mathrm{PD},1}(R)/(p) \cong R^\sigma$. In particular, the natural map $A_{\mathrm{cris}}(R) \twoheadrightarrow R^\sigma$ factors through the $\mathbb{Z}_p$-flat closure of $A_{\mathrm{cris}}(R)$.*

Proof. Following the proof of [42, Lemma 4.1.7], we define

$$[J] := \{\sum_{i \geqslant 0} [r_i]p^i |\ r_i \in J\},$$

which is an ideal of $W(R^\flat)$ if $\sigma(J) = J^p$.[21] We set $W_{\mathrm{PD}}(R) \subset W(R^\flat)[\frac{1}{p}]$ to be the $W(R^\flat)$-subalgebra generated by divided powers of $[J]$. Then any element of $W_{\mathrm{PD}}(R)$ can be uniquely written as $\sum_{i \gg -\infty}[r_i]p^i$, where $r_i \in R^\flat$.. Furthermore, we have $r_i \in \sigma(J)$ for $i < 0$; indeed, for any $r \in J$, $f \in [J]$ and $n \geqslant p$, we have

$$\frac{([r]+pf)^n}{n!} = \sum_{i=0}^{n} \frac{[r^i]}{i!}\frac{p^{n-i}}{(n-i)!}f^{n-i} \equiv \sum_{i=p}^{n} \frac{\sigma[r^{(i-p)/p}\cdot r]}{i!}\frac{p^{n-i}}{(n-i)!}f^{n-i} \bmod W(R^\flat).$$

Now we define

$$W_{\mathrm{PD},1}(R) := W_{\mathrm{PD}}(R)/(W_{\mathrm{PD}}(R) \cap \sigma[J][\tfrac{1}{p}]),$$

[19] Later in §8, we will work with another ring called $A_{\mathrm{cris}}(R)$ for formally smooth W-algebra R. In this section, $A_{\mathrm{cris}}(R)$ only denotes the universal p-adic PD thickening of a semiperfect ring R.

[20] Since J_0 is finitely generated, we have $\sigma^{n_0'}(J') \subset J_0 \subset J'$, where n_0' can be explicitly obtained in terms of the number of generators of J_0.

[21] Indeed, we use $\sigma(J) = J^p$ to ensure that $[J]$ is stable under addition.

where $\sigma[J]$ is the image of $[J]$ under the Witt vector Frobenius σ. Note that $W_{\mathrm{PD},1}(R)$ is a PD quotient of $W_{\mathrm{PD}}(R)$. Furthermore, from our observation on the principal terms of $W_{\mathrm{PD}}(R)$ in the previous paragraph, it follows that the natural map $W(R^\flat)/\sigma[J] \to W_{\mathrm{PD},1}(R)$ is an isomorphism. So $W_{\mathrm{PD},1}(R)$ is a p-adic p-torsion free PD thickening of R with $W_{\mathrm{PD},1}(R)/p \cong R^\sigma$, where the induced PD structure on R^σ is as in the statement. Now by the universal property, the natural PD morphism $A_{\mathrm{cris}}(R) \to R^\sigma$ factors through a p-torsion free ring $W_{\mathrm{PD},1}(R)$, so it factors through the $\mathbb{Z}_p$-flat closure of $A_{\mathrm{cris}}(R)$. □

Corollary 5.3.6. *Let R be as in Lemma 5.3.5, and consider the following thickenings of f-semiperfect rings:*

$$R^\sigma \twoheadrightarrow R'' \twoheadrightarrow R' \twoheadrightarrow R.$$

We give the PD structure on $\ker(R'' \twoheadrightarrow R')$ *by $f^{[p]} = 0$, which is well defined. Then the natural PD morphism $A_{\mathrm{cris}}(R') \to R''$ factors through the $\mathbb{Z}_p$-flat closure of $A_{\mathrm{cris}}(R')$.*

Proof. By the universal property, the natural map $A_{\mathrm{cris}}(R') \to R''$ factors as follows:

$$A_{\mathrm{cris}}(R') \to A_{\mathrm{cris}}(R) \to R^\sigma \twoheadrightarrow R''.$$

Since the middle arrow factors through the $\mathbb{Z}_p$-flat closure of $A_{\mathrm{cris}}(R)$ by Lemma 5.3.5, the claim follows. □

The following corollary is the main conclusion of our discussion:

Corollary 5.3.7. *Let κ be an algebraically closed field of characteristic p, and set $W := W(\kappa)$. Let B be a formally finitely generated κ-algebra which is reduced as a ring, and $R := B/J$ a reduced quotient where J is a closed subideal contained in some ideal of definition of B.*[22] *Then there exists a sequence of square-zero thickenings*

$$B \twoheadrightarrow \cdots \twoheadrightarrow B_{i+1} \twoheadrightarrow B_i \twoheadrightarrow \cdots \twoheadrightarrow B_0 = R$$

such that $B = \varprojlim_i B_i$ and we have the following property. Choose a formally smooth formally finitely generated W-algebra A with $A \twoheadrightarrow B$, and let $S_i \twoheadrightarrow B_i$ be the p-adically completed PD hull of $A \twoheadrightarrow B_i$. Then given the "square-zero PD structure" on $B_{i+1} \twoheadrightarrow B_i$, the natural PD morphism $S_i \twoheadrightarrow B_{i+1}$ factors through the $\mathbb{Z}_p$-flat closure S_i^{fl} of S_i.

Similarly, for a formally smooth formally finitely generated W-algebra A such that $R := A/J$ is a reduced κ-algebra, there exists a sequence of square-zero liftable PD thickenings $\{A_i\}$ filtering $A \twoheadrightarrow R$ (i.e., satisfying the same property for $\{B_i\}$).

Proof. Let us first handle the case when B is a reduced formally finitely generated κ-algebra. We write $\widetilde{B} := \varinjlim_\sigma B$, and define $\widetilde{R}^\flat$ to be the $J\widetilde{B}$-adic completion of $\widetilde{B}$. Set $\widetilde{J} := \bigcup_n \sigma^{-n}(J^{p^n}\widetilde{R}^\flat)$ so that $\widetilde{R} := \widetilde{R}^\flat/\widetilde{J}$ is f-semiperfect and $\sigma(\widetilde{J}) = \widetilde{J}^p$. (The notation is consistent as $\widetilde{R}^\flat = \varprojlim_\sigma \widetilde{R}$.) Then R injects into $\widetilde{R}$; indeed, the kernel of $B \twoheadrightarrow R \to \widetilde{R}$ consists of elements $f \in B$ such that $\sigma^n(f) = f^{p^n} \in J^{p^n}$ for some n, but this condition forces $f \in J$ as J is its own radical.

Observe that $\{\widetilde{J}^i\}$ is a fundamental system of neighbourhoods of 0 in $\widetilde{R}^\flat$.[23] We set $\widetilde{B}_i := \widetilde{R}^\flat/\widetilde{J}^i$, and let $B_i \subset \widetilde{B}_i$ denote the image of B. As B injects into $\widetilde{R}^\flat$, it follows that $B = \varprojlim_i B_i$. To see that $S_i \to B_{i+1}$ factors through S_i^{fl}, we choose a lift $A \to W(\widetilde{R}^\flat)$ of $A \twoheadrightarrow B \to \widetilde{R}^\flat$, and naturally extend it to a PD morphism $S_i \to A_{\mathrm{cris}}(\widetilde{B}_i)$. Since $S_i \twoheadrightarrow B_{i+1} \hookrightarrow \widetilde{B}_{i+1}$ factors through $A_{\mathrm{cris}}(\widetilde{B}_i)$, the desired claim follows from Corollary 5.3.6 (which can be applied since we have $\sigma(\widetilde{J}^i) = (\widetilde{J}^i)^p$).

---

[22] In particular, R is formally finitely generated over κ, and we have $B = \varprojlim_i B/J^i$.

[23] Indeed, $\{\sigma^n(\widetilde{J}) = \widetilde{J}^{p^n}\}$ is sufficient to form a fundamental system of neighbourhoods

Let us now construct $\{A_i\}$ filtering $A \twoheadrightarrow R$ when A is formally smooth formally finitely generated over W. We set $B = A/p$, which is a reduced κ-algebra, and repeat the above argument with the following modifications. Note that the ideals $(p^m, [\widetilde{J}^n]) \subset W(\widetilde{R}^\flat)$ form a fundamental system of neighbourhoods of 0 in $W(\widetilde{R}^\flat)$ with respect to the product topology. We choose

$$W(\widetilde{R}^\flat) \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{A}_{i+1} \twoheadrightarrow \widetilde{A}_i \twoheadrightarrow \cdots \twoheadrightarrow \widetilde{A}_0 = \widetilde{R},$$

so that each $\widetilde{A}_i$ is of the form $W(\widetilde{R}^\flat)/(p^m, [\widetilde{J}^n])$ for some m, n, each of $\widetilde{A}_{i+1} \twoheadrightarrow \widetilde{A}_i$ is a square-zero thickening, and $W(\widetilde{R}^\flat) = \varprojlim_i \widetilde{A}_i$. We give the "square-zero PD structure" to each $\ker(\widetilde{A}_{i+1} \twoheadrightarrow \widetilde{A}_i)$, which is compatible with the standard PD structure on (p) since $p > 2$.

Now we fix a lift $A \to W(R^\flat)$, and let $A_i \subset \widetilde{A}_i$ denote the image of A (and define S_i and S_i^{fl} accordingly). Since $S_i \to A_{i+1} \hookrightarrow \widetilde{A}_{i+1}$ factors through $A_{\mathrm{cris}}(\widetilde{A}_i/p)$ (hence, its flat closure by Corollary 5.3.6), the PD morphism $S_i \to A_{i+1}$ factors through S_i^{fl}. □

Remark 5.3.8. Note that the construction of A_i when A is formally smooth over W only uses the property that A/pA is reduced and $A/pA \hookrightarrow R^\flat$ lifts to an injective map $A \hookrightarrow W(R^\flat)$. For example, if A is a formally finitely generated flat W-algebra with A/pA reduced such that there is a lift of Frobenius $\sigma : A \to A$, then Corollary 5.3.7 holds for such A and the reduced quotient R of A/pA.

Remark 5.3.9. Unfortunately, Corollary 5.3.7 does not show that any finitely generated κ-algebra R admits a p-adic W-flat PD thickening, as there are examples of R that cannot occur as one of R_i's as in Corollary 5.3.7.

5.4. **From reduced rings to formally smooth rings.** Assume that $p > 2$. Let R be a finitely generated *reduced* κ-algebra. We choose a formally smooth formally finitely generated W-algebra A such that for the maximal ideal of definition J we have $A/J = R$. In Corollary 5.3.7, we obtain a series of square-zero thickenings of finitely generated W-algebras $A_{i+1} \twoheadrightarrow A_i$ with $A_0 = R$, such that $A = \varprojlim A_i$ and $A_{i+1} \twoheadrightarrow A_i$ is a liftable PD thickening with respect to the "square-zero PD structure" for each i.

The goal of this section is to prove the following proposition.

Proposition 5.4.1. *Consider R' such that $A_{i+1} \twoheadrightarrow R' \twoheadrightarrow A_i$ for some i. Then any $(X_{R'}, \iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R')$ can be lifted to $(\widetilde{X}, \iota) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(A)$. In particular, for each α there exists $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ that satisfies the conclusion of Proposition 5.2.*

Let us set up the notation for the proof of Proposition 5.4.1. Since the conclusion of Corollary 5.3.7 holds for any refinement of $\{A_i\}$, we may assume that $R' = A_i$. Let $S_i \twoheadrightarrow A_i$ be the p-adically completed PD hull of $A \twoheadrightarrow A_i$. The $\mathbb{Z}_p$-flat closure S_i^{fl} of S_i is a W-flat p-adic PD thickening of A_i, since $\ker(S_i \twoheadrightarrow S_i^{\mathrm{fl}})$ is a PD subideal of $\ker(S_i \twoheadrightarrow A_i)$ by Corollary 5.3.7.

For any closed point $x \in \operatorname{MaxSpec} R$, let $\widehat{A}_{i,x}$ and $\widehat{A}_x$ be the completions at x, where we view x also as a closed point of $\operatorname{Spec} A_i$ and $\operatorname{Spec} A$. Let $\widehat{S}^{\mathrm{fl}}_{i,x} := S_i^{\mathrm{fl}} \widehat{\otimes}_A \widehat{A}_x$, where $\widehat{\otimes}$ denotes the p-adically completed tensor product. Note that $\widehat{S}^{\mathrm{fl}}_{i,x}$ is topologically flat over S_i^{fl} for the p-adic topology; i.e., $\widehat{S}^{\mathrm{fl}}_{i,x}/p^m \cong S_i^{\mathrm{fl}}/p^m \otimes_A \widehat{A}_x$ is flat over S_i/p^m for any m.[24] Since $(S_i^{\mathrm{fl}}/p)_{\mathrm{red}} = R$, it follows that $\{\widehat{S}^{\mathrm{fl}}_{i,x}\}$ is an topological fpqc covering of S_i^{fl} for the p-adic topology (or equivalently, $\{\widehat{S}^{\mathrm{fl}}_{i,x}/p^m\}$

[24]Although S_i^{fl} is usually not noetherian, one can still verify directly that $\widehat{S}_{i,x}/p^m \cong S_i^{\mathrm{fl}}/p^m \otimes_A \widehat{A}_x$. Alternatively, one can use a very general result such as [20, EGA 0_{I}, Proposition 7.2.7].

is an fpqc covering of S_i^{fl}/p^m for each m). By flatness, the PD structure on $\ker(S_i^{\mathrm{fl}} \twoheadrightarrow A_i)$ extends to a PD structure on $\ker(\widehat{S}_{i,x}^{\mathrm{fl}} \twoheadrightarrow \widehat{A}_{i,x})$ by [5, Corollary 3.22].

For $(X^{(i)}, \iota) \in \mathtt{RZ}_{\mathbb{X}}(A_i)$, we consider the following tensors

$$1 \otimes s_\alpha \in S_i^{\mathrm{fl}}[\tfrac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes \cong \mathbb{D}(X^{(i)})(S_i^{\mathrm{fl}})^\otimes[\tfrac{1}{p}], \tag{5.4.2}$$

where the isomorphism is induced by $\mathbb{D}(\iota) : \mathbb{D}(X^{(i)}_{A_i/p})[\frac{1}{p}] \xrightarrow{\sim} \mathbb{D}(\mathbb{X}_{A_i/p})[\frac{1}{p}]$. Note that $1 \otimes s_\alpha$ above coincides with the S_i^{fl}-section of $s_{\alpha,\mathbb{D}} : \mathbf{1} \to \mathbb{D}(X^{(i)})^\otimes[\frac{1}{p}]$.

For $(X^{(i)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_i)$ and $x \in \operatorname{MaxSpec} A_i$, we have a morphism of crystals $\hat{t}_{\alpha,x} : \mathbf{1} \to \mathbb{D}(X^{(i)}_{\widehat{A}_{i,x}})^\otimes$ for each α, so we obtain $\hat{t}_{\alpha,x}(\widehat{S}_{i,x}^{\mathrm{fl}}) \in \mathbb{D}(X^{(i)}_{\widehat{A}_{i,x}})(\widehat{S}_{i,x}^{\mathrm{fl}})^\otimes$. The following lemma shows that the images of $(1 \otimes s_\alpha)$ and $(\hat{t}_{\alpha,x}(\widehat{S}_{i,x}^{\mathrm{fl}}))$ coincide in $\mathbb{D}(X^{(i)}_{\widehat{A}_{i,x}})(\widehat{S}_{i,x}^{\mathrm{fl}})^\otimes[\frac{1}{p}] \cong \widehat{S}_{i,x}^{\mathrm{fl}}[\frac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes$.

Lemma 5.4.3. *Let R be a complete local noetherian W/p^m-algebra (for some m) with residue field κ, and let $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$. Assume that the map $\operatorname{Spf} R \to \mathtt{Def}_{X_\kappa}$, corresponding to the deformation X over R, factors through $\mathtt{Def}_{X_\kappa,G}$. Consider a morphism $\hat{t}_\alpha : \mathbf{1} \to \mathbb{D}(X)^\otimes$ of crystals over $\operatorname{Spec} R$, obtained from the pull back of the universal Tate tensors (3.5.2). Then $(\hat{t}_\alpha)$ induce $(s_{\alpha,\mathbb{D}})$ on the F-isocrystals (*cf. *Definition 4.5).*

Proof. Since both $(\hat{t}_\alpha)$ and $(s_{\alpha,\mathbb{D}})$ only depends on R/p, we may assume that $pR = 0$. Then the lemma is a direct consequence of [14, Lemma 4.3] (since on the special fibre $(\hat{t}_\alpha)$ induce $(s_{\alpha,\mathbb{D}})$ on the isocrystals by construction). □

Lemma 5.4.4. *Let $(X^{(i)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_i)$, and we use the notation as above. Then the tensors $(1 \otimes s_\alpha) \subset \mathbb{D}(X^{(i)})(S_i^{\mathrm{fl}})^\otimes[\frac{1}{p}]$, defined in (5.4.2), actually lie in $\mathbb{D}(X^{(i)})(S_i^{\mathrm{fl}})^\otimes$, and its image in $\mathbb{D}(X^{(i)}_{\widehat{A}_{i,x}})(\widehat{S}_{i,x}^{\mathrm{fl}})^\otimes$ coincides with $\hat{t}_{\alpha,x}(S_i^{\mathrm{fl}})$.*

Proof. By Lemma 5.4.3, the images of $(1 \otimes s_\alpha)$ in $\mathbb{D}(X^{(i)}_{\widehat{A}_{i,x}})(\widehat{S}_{i,x}^{\mathrm{fl}})^\otimes[\frac{1}{p}]$ are precisely $(\hat{t}_{\alpha,x}(S_i^{\mathrm{fl}}))$, which lie in $\mathbb{D}(X^{(i)}_{\widehat{A}_{i,x}})(\widehat{S}_{i,x}^{\mathrm{fl}})^\otimes$. By topological fpqc descent, it follows that $(1 \otimes s_\alpha) \subset \mathbb{D}(X^{(i)})(S_i^{\mathrm{fl}})^\otimes$. Indeed, by $\mathbb{Z}_p$-flatness of S_i^{fl} and $\widehat{S}_{i,x}^{\mathrm{fl}}$, the collection of morphisms $\widehat{S}_{i,x}^{\mathrm{fl}} \to \mathbb{D}(X_{\widehat{R}_x})(\widehat{S}_{i,x}^{\mathrm{fl}})^\otimes$, defined by $1 \mapsto t_\alpha(\widehat{S}_{i,x}^{\mathrm{fl}})$, should glue to a map

$$S_i^{\mathrm{fl}} \to \mathbb{D}(X_R)(S_i^{\mathrm{fl}})^\otimes,$$

and it should map 1 to $1 \otimes s_\alpha$. □

Proof of Proposition 5.4.1. By Corollary 5.3.7, for any i the projection $A \twoheadrightarrow A_{i+1}$ naturally extends to $S_i^{\mathrm{fl}} \twoheadrightarrow A_{i+1}$, inducing the square-zero PD structure on the kernel of $A_{i+1} \twoheadrightarrow A_i$. For any $(X^{(i)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_i)$, we define

$$t_\alpha(A_{i+1}) \in \mathbb{D}(X^{(i)})(A_{i+1})^\otimes \cong \mathbb{D}(X^{(i)})(S_i^{\mathrm{fl}})^\otimes \otimes_{S_i^{\mathrm{fl}}} A_{i+1}$$

to be the image of $1 \otimes s_\alpha \in \mathbb{D}(X^{(i)})(S_i^{\mathrm{fl}})^\otimes \subset S_i^{\mathrm{fl}}[\frac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes$. Note that $(t_\alpha(A_{i+1}))$ glues $\{(\hat{t}_{\alpha,x}(\widehat{A}_{i+1,x}))\}_x$; *cf.* Lemma 5.4.4.

We view any closed point $x \in \operatorname{MaxSpec} A_i$ as a point of $\operatorname{Spec} A_{i'}$ for all $i' \geqslant i$, and hence, as a point of $\operatorname{Spf} A$. For $(X^{(i)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_i)$, let $X^{(i+1)}$ be a p-divisible group over A_{i+1} lifting $X^{(i)}$, such that the Hodge filtration $\mathrm{Fil}^1_{X^{(i+1)}} \subset \mathbb{D}(X^{(i)})(A_{i+1})$ is a $\{\mu\}$-filtration with respect to $(t_\alpha(A_{i+1}))$. Then by Proposition 3.8, we have $(X^{(i+1)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_{i+1})$, where ι is the unique lift of the quasi-isogeny. By repeating this process, we obtain projective systems $\{(X^{(i')}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_{i'})\}_{i' \geqslant i}$. and $\{(t_\alpha(A_{i'}))\}_{i' \geqslant i}$.

Let $\widetilde{X}$ denote the p-divisible group over A which reduces to $X^{(i')}$ for each $i' \geqslant i$. By taking the limit of $t_\alpha(A_{i'})$, we also obtain $t_\alpha(A) \in \mathbb{D}(\widetilde{X})(A)^\otimes$. By construction, the image of $t_\alpha(A)$ in $\mathbb{D}(\widetilde{X}_{\widehat{A}_x})(\widehat{A}_x)^\otimes$ is precisely $\hat{t}_{\alpha,x}(\widehat{A}_x)$ where $\hat{t}_{\alpha,x}$ is the pull-back of $\hat{t}^{\mathrm{univ}}_{\alpha,x}$. It follows from the diagram (5.2.4) that $t_\alpha(A)$ is horizontal; indeed, it is horizontal over $\widehat{A}_x$ for any closed point x. Therefore, $(t_\alpha(A))$ induce morphisms $t_\alpha : \mathbf{1} \to \mathbb{D}(\widetilde{X})^\otimes$.

It remains to verify Definition 4.6 for $(\widetilde{X}, (t_\alpha))$. To verify Definition 4.6(1) it suffices to check that the image of $t_\alpha(A)$ in $\mathbb{D}(X^{(i)})(S_i^{\mathrm{fl}})^\otimes[\frac{1}{p}] \cong S_i^{\mathrm{fl}}[\frac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes$ is precisely $1 \otimes s_\alpha$, which holds by construction (using Lemma 5.4.3). The other two conditions can be verified over $\widehat{A}_x$ for any $x \in \operatorname{MaxSpec} A$, which clearly hold by construction. □

5.5. **Proof of Proposition 5.2.** For $R \in \mathrm{Nilp}_W^{\mathrm{ft}}$ consider $B, R' \in \mathrm{Nilp}_W^{\mathrm{ft}}$ with W-morphisms $B, R' \to R$. Assume that $B \to R$ is surjective with the kernel $\mathfrak{b}$ killed by the nil-radical of B, and $R'_{\mathrm{red}} \twoheadrightarrow R_{\mathrm{red}}$ is surjective. We set $B' := B \times_R R' \in \mathrm{Nilp}_W^{\mathrm{ft}}$. By assumption, we can regard any closed point $x \in \operatorname{Spec} R$ as a point of $\operatorname{Spec} R'$, $\operatorname{Spec} B$ and $\operatorname{Spec} B'$. Since $\mathtt{RZ}_{\mathbb{X}}$ is representable by a formal scheme, we have a natural bijection:

$$\mathtt{RZ}_{\mathbb{X}}(B') \xrightarrow{\sim} \mathtt{RZ}_{\mathbb{X}}(B) \times_{\mathtt{RZ}_{\mathbb{X}}(R)} \mathtt{RZ}_{\mathbb{X}}(R'). \tag{5.5.1}$$

Lemma 5.5.2. *In the above setting, the natural map $\mathtt{RZ}^\Lambda_{G,b}(B') \to \mathtt{RZ}^\Lambda_{G,b}(B) \times_{\mathtt{RZ}^\Lambda_{G,b}(R)} \mathtt{RZ}^\Lambda_{G,b}(R')$, obtained by restricting the isomorphism (5.5.1), is a bijection.*

Proof. The lemma follows from the isomorphism

$$\mathtt{Def}_{X_x,G}(\widehat{B}'_x) \xrightarrow{\sim} \mathtt{Def}_{X_x,G}(\widehat{B}_x) \times_{\mathtt{Def}_{X_x,G}(\widehat{R}_x)} \mathtt{Def}_{X_x,G}(\widehat{R}'_x),$$

which holds since $\mathtt{Def}_{X_x,G}$ is a formal scheme and we have $\widehat{B}'_x \xrightarrow{\sim} \widehat{B}_x \times_{\widehat{R}_x} \widehat{R}'_x$. □

Proof of Proposition 5.2. Let $R \in \mathrm{Nilp}_W^{\mathrm{ft}}$, and choose a formally smooth formally finitely generated W-algebra A with $A/J = R$ for some ideal of definition J. Let $A_0 := R_{\mathrm{red}}$ and choose $\{A_{i+1} \twoheadrightarrow A_i\}$ as in Corollary 5.3.7. We write $I_i := \ker(A \twoheadrightarrow A_i)$. We refine the sequence of thickenings so that $I_0 I_i \subseteq I_{i+1}$ for each i.

Note that for some j we have $A_j \twoheadrightarrow R$. For $i \leqslant j$, we set $R_i := A/(J + I_i)$, which is a simultaneous quotient of R and A_i. Let $(X, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(R)$, and we denote by $(X^{(0)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_0)$ the restriction of (X, ι) over $A_0 = R_{\mathrm{red}}$.

We now inductively show that the restriction $(X_{R_i}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(R_i)$ of (X, ι) can be lifted to some $(X^{(i)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_i)$ for any $i \leqslant j$. The base case $i = 0$ is already handled in Proposition 5.4.1 since $A_0 = R_0 = R_{\mathrm{red}}$.

Assume that there exists $(X^{(i-1)}, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(A_{i-1})$ lifting $(X_{R_{i-1}}, \iota)$. We also have $t_\alpha : \mathbf{1} \to \mathbb{D}(X_{R_{i-1}})^\otimes$, glueing $\{\hat{t}_{\alpha,x}\}_x$, by pulling back t_α on $X^{(i-1)}$ (*cf.* Proposition 5.4.1).

Note that we have

$$B_i := A/((J \cap I_{i-1}) + I_i) \xrightarrow{\sim} R_i \times_{R_{i-1}} A_{i-1};$$

indeed, for any ring A and ideals $\mathfrak{a}, \mathfrak{a}' \subset A$ the diagonal map $A/(\mathfrak{a} \cap \mathfrak{a}') \to A/\mathfrak{a} \times_{A/(\mathfrak{a}+\mathfrak{a}')} A/\mathfrak{a}'$ is an isomorphism,[25] and we apply it to $\mathfrak{a} = J + I_i$ and $\mathfrak{a}' = I_{i-1}$. Since we have $I_0(J + I_{i-1}) \subset J + I_i$ by assumption (that $I_0 I_{i-1} \subseteq I_i$), we can

[25] The injectivity of the map $A/(\mathfrak{a} \cap \mathfrak{a}') \to A/\mathfrak{a} \times_{A/(\mathfrak{a}+\mathfrak{a}')} A/\mathfrak{a}'$ is clear. It remains to show that for $a, a' \in A$ such that $a \equiv a' \mod \mathfrak{a} + \mathfrak{a}'$, the element $(a \bmod \mathfrak{a}, a' \bmod \mathfrak{a}')$ is in the image of $A/(\mathfrak{a} \cap \mathfrak{a}')$. For this, we may replace (a, a') with $(0, a' - a)$, where $a' - a \in \mathfrak{a} + \mathfrak{a}'$. Now the claim follows from the isomorphism $(\mathfrak{a} + \mathfrak{a}')/\mathfrak{a}' \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{a}')$.

apply Lemma 5.5.2 and obtain $(X_{B_i},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(B_i)$ which simultaneously lifts $(X_{R_i},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R_i)$ and $(X^{(i-1)},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(A_{i-1})$. Now, since we have

$$A_i = A/I_i \twoheadrightarrow B_i = A/((J\cap I_{i-1})+I_i) \twoheadrightarrow A_{i-1} = A/I_{i-1},$$

we may apply Proposition 5.4.1 to get a lift $(X^{(i)},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(A_i)$ of (X_{B_i},ι) (which lifts (X_{R_i},ι)). This completes the induction claim.

By repeating this step, we eventually obtain a lift $(X^{(j)},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(A_j)$ of $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ for $j \gg 0$. We then obtain the desired $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ by restricting t_α on $X^{(j)}$ (*cf.* Proposition 5.4.1). □

Let us now move on to the other main result of this section on "effectiveness":

Proposition 5.6. *Let $\widehat{R}$ be a formally finitely generated W/p^m-algebra, and let $I \subset \widehat{R}$ denote the Jacobson radical. (In particular, R is I-adically separated and complete, and R/I is a reduced κ-algebra.) Consider $(X_{\widehat{R}},\iota_{\widehat{R}/p}) \in \mathtt{RZ}_{\mathbb{X}}(\widehat{R})$. (In particular, $\iota_{\widehat{R}/p}$ is defined over $\operatorname{Spec}\widehat{R}/p$.) Then we have $(X_{\widehat{R}},\iota_{\widehat{R}/p}) \in \mathtt{RZ}^{\Lambda}_{G,b}(\widehat{R})$ if and only if $(X_{\widehat{R}/I^i},\iota_{\widehat{R}/(p,I^i)}) \in \mathtt{RZ}^{\Lambda}_{G,b}(\widehat{R}/I^i)$ for any i.*

Note that the "only if" direction is trivial. We will now prove the "if" direction for the rest of the section. Conceptually, the "if" direction when $\widehat{R}$ is a complete local noetherian W/p^m-algebra asserts that formal-locally defined Tate tensors extend over some finite-type base ring if the quasi-isogeny does.

5.7. **Preparation.** For $(\widehat{R},I)$ as in Proposition 5.6, let $(X_{\widehat{R}},\iota) \in \mathtt{RZ}_{\mathbb{X}}(\widehat{R})$ be such that $(X_{\widehat{R}/I^i},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(\widehat{R}/I^i)$ for any i. Then the "if" direction of Proposition 5.6 claims the existence of $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ for some finitely generated W-subalgebra $R \subset \widehat{R}$, such that (X,ι) pulls back to $(X_{\widehat{R}},\iota)$.

Let $\widehat{A}$ be any formally smooth formally finitely generated W-algebra which surjects onto $\widehat{R}$ and is J-adically separated and complete, where $J \subset \widehat{A}$ is the preimage of $I \subset \widehat{R}$. From the formal smoothness of $\mathtt{RZ}^{\Lambda}_{G,b}$ and Corollary 5.2.3, there exists $(X_{\widehat{A}},\iota) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\widehat{A})$ lifting each of $(X_{\widehat{R}/J^i},\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(\widehat{R}/J^i)$. (Indeed, we use the thickenings $\widehat{A}/J^{i+1} \twoheadrightarrow \widehat{A}/J^i \times_{\widehat{R}/I^i} \widehat{R}/I^{i+1} \twoheadrightarrow \widehat{A}/J^i$ to choose a lift over $\widehat{A}/J^{i+1}$ which simultaneously lifts the chosen lift over $\widehat{A}/J^i$ and $(X_{\widehat{R}/I^{i+1}},\iota)$.) We also let $\hat{t}_\alpha : \mathbf{1} \to \mathbb{D}(X_{\widehat{R}})^{\otimes}$ denote the pull back of the Tate tensors $\hat{t}_\alpha : \mathbf{1} \to \mathbb{D}(X_{\widehat{A}})^{\otimes}$. Then $\hat{t}_\alpha$ induces $s_{\alpha,\mathbb{D}}$ on the isocrystals over $\operatorname{Spec}\widehat{R}$.

By assumption, the following $\widehat{A}$-scheme

$$P_{\widehat{A}} := \underline{\mathrm{isom}}_{\widehat{A}}\big([\mathbb{D}(X_{\widehat{A}})(\widehat{A}),(\hat{t}_\alpha(\widehat{A}))],[\widehat{A}\otimes_{\mathbb{Z}_p}\Lambda^*,(1\otimes s_\alpha)]\big),$$

is a G-torsor, and the Hodge filtration $\mathrm{Fil}^1_{X_{\widehat{A}}} \subset \mathbb{D}(X_{\widehat{A}})(\widehat{A})$ is a $\{\mu\}$-filtration with respect to $(\hat{t}_\alpha(\widehat{A}))$. Clearly, the same holds over $\widehat{R}$ by pull back.

By standard argument, one can find a finitely generated W-subalgebra $R \subset \widehat{R}$ with the following properties:

(1) There exists $(X,\iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$ which pulls back to $(X_{\widehat{R}},\iota)$ over $\widehat{R}$; this is possible since $\mathtt{RZ}_{\mathbb{X}}$ is locally formally of finite type over W.
(2) The (finitely many) tensors $(\hat{t}_\alpha(\widehat{R})) \subset \mathbb{D}(X_{\widehat{R}})(\widehat{R})^{\otimes}$ lie in the image of $\mathbb{D}(X)(R)^{\otimes}$; indeed, this can be arranged by considering a finite-rank direct factor of $\mathbb{D}(X_{\widehat{R}})(\widehat{R})^{\otimes}$ containing $(\hat{t}_\alpha(\widehat{R}))$, and possibly by increasing R by adjoining finitely many elements in $\widehat{R}$. We let $(t_\alpha(R)) \subset \mathbb{D}(X)(R)^{\otimes}$ denote the tensors which (injectively) map to $(\hat{t}_\alpha(\widehat{R}))$.

(3) The following R-scheme is a G-torsor
$$P_R := \underline{\mathrm{isom}}_R\big([\mathbb{D}(X)(R), (t_\alpha(R))], [R \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]\big).$$
In fact, the natural G-action on P_R is already transitive, and by possibly increasing R we can ensure that P_R is smooth with non-empty fibre everywhere. (This is possible by [20, EGA IV$_4$, Proposition 17.7.8, Théorème 8.10.5] as $P_{\widehat{R}}$ has these properties).

(4) The Hodge filtration $\mathrm{Fil}^1_X \subset \mathbb{D}(X)(R)$ is a $\{\mu\}$-filtration with respect to $(t_\alpha(R))$; this follows since $\{\mu\}$-filtrations form a closed subscheme of a suitable grassmannian.

If $R \subset \widehat{R}$ is a finitely generated W-subalgebra which satisfies the above conditions, then any finitely generated W-subalgebra of $\widehat{R}$ containing R satisfies the same conditions.

To prove Proposition 5.6, it suffices to prove the following proposition.

Proposition 5.8. *There exists a finitely generated W-subalgebra $R \subset \widehat{R}$ which satisfies the properties listed in §5.7, such that $(X, \iota) \in \mathtt{RZ}^\Lambda_{G,b}(R)$ and $(t_\alpha(R))$ coincides with the R-sections of (t_α) constructed in Proposition 5.2.*

5.9. **Proof of Proposition 5.8.** Let us first handle the case when $\widehat{R}$ is *reduced*. Since the κ-subalgebra $R \subset \widehat{R}$ is also reduced, we can consider the perfections $\widetilde{R} := \varinjlim_\sigma R$ and $(\widehat{R})^\sim := \varinjlim_\sigma \widehat{R}$. We extend the injective map $R \hookrightarrow \widehat{R}$ to an injective map $\widetilde{R} \hookrightarrow (\widehat{R})^\sim$.

Using the quasi-isogeny $\iota_{\widetilde{R}} : \mathbb{X}_{\widetilde{R}} \dashrightarrow X_{\widetilde{R}}$, we get $(1 \otimes s_\alpha) \subset W(\widetilde{R})[\frac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes \cong \mathbb{D}(X_{\widetilde{R}})(W(\widetilde{R}))^\otimes[\frac{1}{p}]$. Note that they are the $W(\widetilde{R})[\frac{1}{p}]$-sections of $(s_{\alpha,\mathbb{D}})$; *cf.* Definition 4.5.

Lemma 5.9.1. *In the above setting, each of $(1 \otimes s_\alpha)$ lies in $\mathbb{D}(X_{\widetilde{R}})(W(\widetilde{R}))^\otimes$. In particular, each of $s_{\alpha,\mathbb{D}}$ comes from a unique morphism of integral crystals, which we denote by $t_\alpha : \mathbf{1} \to \mathbb{D}(X_{\widetilde{R}})^\otimes$. Furthermore, the $\widetilde{R}$-sections $(t_\alpha(\widetilde{R})) \in \mathbb{D}(X_{\widetilde{R}})(\widetilde{R})^\otimes$ of (t_α) coincide with the image of $(t_\alpha(R))$.*

Proof. Note that the isomorphism $\mathbb{D}(\hat{\iota})$ matches $(\hat{t}_\alpha(W((\widehat{R})^\sim)) \subset \mathbb{D}(X_{\widehat{R}})(W((\widehat{R})^\sim))^\otimes$ with $1\otimes s_\alpha \in W((\widehat{R})^\sim)[\frac{1}{p}]\otimes_{\mathbb{Z}_p}\Lambda^\otimes$. This shows that each of $1\otimes s_\alpha$ lies in $\mathbb{D}(X_{\widetilde{R}})(W(\widetilde{R}))^\otimes$ since we have $W(\widetilde{R}) = W(\widetilde{R})[\frac{1}{p}] \cap W((\widehat{R})^\sim)$. By standard dictionary (*cf.* [21, IV §4]), we obtain a unique map $t_\alpha : \mathbf{1} \to \mathbb{D}(X_{\widetilde{R}})^\otimes$ so that its $W(\widetilde{R})$-section is $1 \otimes s_\alpha \in W(\widetilde{R})[\frac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes$. To verify the last claim, we may compare the (injective) images of both over $(\widehat{R})^\sim$, where the claim is obvious from the construction. $\square$

For any closed point $x \in \operatorname{Spec} R$ (which may not lie in the image of $\operatorname{Spec}\widehat{R} \to \operatorname{Spec} R$), we have a natural map $R \hookrightarrow \widetilde{R} \twoheadrightarrow \kappa(x) = \kappa$. By taking the image of $t_\alpha(W(\widetilde{R}))$ in $\mathbb{D}(X_{\widetilde{R}})(W(\widetilde{R})) \otimes_{W(\widetilde{R})} W \cong \mathbb{D}(X_y)(W)$, we obtain

$$(t_{\alpha,x}) \subset \mathbb{D}(X_x)(W)^\otimes, \tag{5.9.2}$$

which induces $(s_{\alpha,\mathbb{D}})$ on the isocrystals.

Lemma 5.9.3. *For any closed point $x \in \operatorname{Spec} R$, the images of $(t_{\alpha,x})$ and $(t_\alpha(R))$ in $\mathbb{D}(X_x)(\kappa)$ coincide. Furthermore, by replacing R with a larger finitely generated W-subalgebra of $\widehat{R}$, we may ensure that $(X_x, \iota_x) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa)$ for any x.*

Proof. The only assertion which may not directly follow from the construction is to verify Definition 4.6(2) for (X_x, ι_x) (up to increasing R). We consider the following scheme

$$P_{W(\widetilde{R})} := \underline{\mathrm{isom}}_{W(\widetilde{R})}\big([\mathbb{D}(X_{\widetilde{R}})(W(\widetilde{R})),(t_\alpha(W(\widetilde{R})))],[W(\widetilde{R})\otimes_{\mathbb{Z}_p}\Lambda^*,(1\otimes s_\alpha)]\big).$$

We want to show that the pull back of $P_{W(\widetilde{R})}$ by the map $W(\widetilde{R}) \twoheadrightarrow W$ induced by $x : \widetilde{R} \twoheadrightarrow \kappa$ is a G-torsor over W. By construction, $P_{W(\widetilde{R})[\frac{1}{p}]}$ and $P_{\widetilde{R}}$ are G-torsors. The assumption on $(X_{\widehat{R}}, \iota)$ implies that the base change $P_{W((\widehat{R})^{\sim})}$ is a G-torsor. (To see this, note that $(t_\alpha(W((\widehat{R})^{\sim}))$ comes from $(\hat{t}_\alpha(\widehat{A})) \subset \mathbb{D}(X_{\widehat{A}})(\widehat{A})^{\otimes}$, where $(X_{\widehat{A}}, \iota) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\widehat{A})$ is a lift of $(X_{\widehat{R}}, \iota)$ over a formally smooth formally finitely generated W-algebra $\widehat{A}$, as in the beginning of §5.7.)

We choose a filtered direct system $\{R_\xi\}$ of finitely generated R-algebras with $\widehat{R} = \varinjlim_\xi R_\xi$.[26] Although we do not know whether $W((\widehat{R})^{\sim})$ is faithfully flat over $\varinjlim_\xi W(\widetilde{R}_\xi)$, we do have that for a generic point $\widehat{\eta} \in \mathrm{Spec}(\widehat{R})^{\sim}$ and its image $\eta_\xi \in \mathrm{Spec}\,\widetilde{R}_\xi$, the natural map $\varinjlim_\xi W(\kappa(\eta_\xi)) \to W(\kappa(\widehat{\eta}))$ is faithfully flat (with the obvious notation); indeed, the mod p reduction of the source is $\varinjlim_\xi \kappa(\eta_\xi) = \kappa(\widehat{\eta})$, so the natural map is a local map of discrete valuation rings. Therefore, by [20, EGA IV$_3$, Théorème 11.2.6], there exists ξ such that $P_{W(\widetilde{R}_\xi)}$ is flat at η_ξ. If we denote $U \subset \mathrm{Spec}\,W(\widetilde{R}_\xi)$ to be the flatness locus of $P_{W(\widetilde{R}_\xi)}$, then $U \cap \mathrm{Spec}\,\widetilde{R}_\xi$ is non-empty and the lemma holds for the closed points in $U \cap \mathrm{Spec}\,\widetilde{R}_\xi$. Now, if the preimage of $U \cap \mathrm{Spec}\,\widetilde{R}_\xi$ in $\mathrm{Spec}(\widehat{R})^{\sim}$ is the entire space, then we may increase ξ so that $U \cap \mathrm{Spec}\,\widetilde{R}_\xi = \mathrm{Spec}\,\widetilde{R}_\xi$; *cf.* [20, EGA IV$_3$, Corollaire 8.3.4].

If the preimage of $U \cap \mathrm{Spec}\,\widetilde{R}_\xi$ is a proper (open) subset of $\mathrm{Spec}(\widehat{R})^{\sim}$, then we consider the reduced complement $Z := (\mathrm{Spec}\,\widetilde{R}) \setminus (U \cap \mathrm{Spec}\,\widetilde{R})$. Since the natural projection $\mathrm{Spec}\,\widetilde{R} \to \mathrm{Spec}\,R$ is a homeomorphism on the underlying topological spaces, there exists an ideal $J \subset R$ which defines Z (with its reduced subscheme structure). We now repeat the previous step for $\widehat{R}/J\widehat{R} = \varinjlim_{\xi' \geqslant \xi} R_{\xi'}/JR_{\xi'}$ instead of $\widehat{R}$, and this process terminates after finitely many times since $\widehat{R}$ is noetherian. $\square$

Proposition 5.9.4. *Proposition 5.8 holds when $\widehat{R}$ is reduced.*

Proof. We work in the setting of §5.7 with $\widehat{R}$ reduced. We assume that $R \subset \widehat{R}$ satisfies the conclusion of Lemma 5.9.3. It remains to show that $(X, \iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ and $(t_\alpha(R)) \subset \mathbb{D}(X)(R)^{\otimes}$ coincides with the sections of (t_α) constructed in Proposition 5.2.

For any closed point $x \in \mathrm{Spec}\,R$, we want to show that the map $\mathrm{Spf}\,\widehat{R}_x \to \mathtt{Def}_{X_x}$, induced by $X_{\widehat{R}_x}$, factors through $\mathtt{Def}_{X_x,G}$. Applying Corollary 5.3.7 to $B = \widehat{R}_x \twoheadrightarrow \kappa$ we obtain a sequence of small thickenings of artin local κ-algebras:

$$\widehat{R}_x \twoheadrightarrow \cdots \twoheadrightarrow \widehat{R}_{x,i+1} \twoheadrightarrow \widehat{R}_{x,i} \twoheadrightarrow \cdots \twoheadrightarrow \widehat{R}_{x,0} = \kappa,$$

and p-adic p-torsion free PD thickenings $\widehat{S}^{\mathrm{fl}}_{x,i} \twoheadrightarrow \widehat{R}_{x,i}$ which lift to $\widehat{S}^{\mathrm{fl}}_{x,i} \twoheadrightarrow \widehat{R}_{x,i+1}$ for any i. We set $X^{(i)}_x := X_{\widehat{R}_{x,i}}$, and let $(t_\alpha(\widehat{R}_{x,i})) \subset \mathbb{D}(X^{(i)}_x)(\widehat{R}_{x,i})^{\otimes}$ denote the image of $(t_\alpha(R))$. To prove the proposition, we need to prove that for any i we have $X^{(i)}_x \in \mathtt{Def}_{X_x,G}(\widehat{R}_{x,i})$, and $(t_\alpha(\widehat{R}_{x,i})) \in \mathbb{D}(X^{(i)}_x)(\widehat{R}_{x,i})^{\otimes}$ coincides with the $\widehat{R}_{x,i}$-section of $t^{(i)}_{\alpha,x} : \mathbf{1} \to \mathbb{D}(X^{(i)}_x)^{\otimes}$, the pull-back of the "universal Tate tensor

[26] We do not require that R_ξ to be W-subalgebras of R. Although we initially choose R_ξ to be subalgebras of $\widehat{R}$, in the course of the proof we replace $\{R_\xi\}$ with suitable quotients, which may not preserve injectivity of R_ξ into the direct limit.

$\hat{t}^{\mathrm{univ}}_{\alpha,x}$. (Granting this claim, we can conclude the proof by taking the projective limit with respect to i.)

We show this claim by induction on i. The base case with $i = 0$ is exactly Lemma 5.9.3. Now, we assume the claim for i (i.e., $X^{(i)}_x \in \mathtt{Def}_{X_x,G}(\widehat{R}_{x,i})$ and $t_\alpha(\widehat{R}_{x,i}) = t^{(i)}_{\alpha,x}(\widehat{R}_{x,i})$) and want to deduce the claim for $i+1$. Let $t_\alpha(\widehat{R}_{x,i+1}) \in \mathbb{D}(X^{(i)}_x)(\widehat{R}_{x,i+1})^\otimes$ be the image of of $t_\alpha(R) \in \mathbb{D}(X)(R)^\otimes$ via $\mathbb{D}(X)(R) \otimes_R \widehat{R}_{x,i+1} \cong \mathbb{D}(X^{(i)}_x)(\widehat{R}_{x,i+1})$, and let us consider the following commutative diagram:

(5.9.5)

$$\begin{tikzcd}
1\otimes s_\alpha & t_\alpha(W(\widetilde{R})) \arrow[l, mapsto, "\text{Lemma 5.9.1}"'] \arrow[r, mapsto] & t_\alpha(\widetilde{R}) & t_\alpha(R) \arrow[l, mapsto, "\text{Lemma 5.9.1}"'] \\
W(\widetilde{R})[\frac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes & \mathbb{D}(X_{\widetilde{R}})(W(\widetilde{R}))^\otimes \arrow[l, hook'] \arrow[r, two heads] & \mathbb{D}(X_{\widetilde{R}})(\widetilde{R})^\otimes & \mathbb{D}(X_R)(R)^\otimes \arrow[l, hook'] \arrow[d, two heads, "t_\alpha(R)\mapsto t_\alpha(\widehat{R}_{x,i+1})"'] \\
\Lambda^\otimes \arrow[u] \arrow[r] & \widehat{S}^{\mathrm{fl}}_{x,i}[\frac{1}{p}] \otimes_{\mathbb{Z}_p} \Lambda^\otimes & \mathbb{D}(X^{(i)}_x)(\widehat{S}^{\mathrm{fl}}_{x,i})^\otimes \arrow[l, hook'] \arrow[r, two heads] & \mathbb{D}(X^{(i)}_x)(\widehat{R}_{x,i+1})^\otimes \\
s_\alpha \arrow[r, mapsto] & 1\otimes s_\alpha & t^{(i)}_{\alpha,x}(\widehat{S}^{\mathrm{fl}}_{x,i}) \arrow[l, mapsto, "\text{Lemma 5.4.3}"'] \arrow[r, mapsto] & t^{(i)}_{\alpha,x}(\widehat{R}_{x,i+1})
\end{tikzcd}$$

Indeed, the diagram shows that $t_\alpha(\widehat{R}_{x,i+1})$ (respectively, $t^{(i)}_{\alpha,x}(\widehat{R}_{x,i+1})$) is uniquely determined by s_α by chasing the top row and the right vertical arrow (respectively, by chasing the bottom row), so we have $t_\alpha(\widehat{R}_{x,i+1}) = t^{(i)}_{\alpha,x}(\widehat{R}_{x,i+1})$.

It now follows from Proposition 3.8 that $X^{(i+1)}_x \in \mathtt{Def}_{X_x,G}(\widehat{R}_{x,i+1})$ since the Hodge filtration of $X^{(i+1)}_x$ corresponds to a $\{\mu\}$-filtration with respect to $(t_\alpha(\widehat{R}_{x,i+1}))$ by assumption (*cf.* §5.7). The equality $t^{(i+1)}_{\alpha,x}(\widehat{R}_{x,i+1}) = t_\alpha(\widehat{R}_{x,i+1})$ follows from $t^{(i)}_{\alpha,x}(\widehat{R}_{x,i+1}) = t_\alpha(\widehat{R}_{x,i+1})$ and the fact that $t^{(i+1)}_{\alpha,x}$ lifts $t^{(i)}_{\alpha,x}$. □

Proof of Proposition 5.8. We consider a sequence of square-zero thickenings

$$\widehat{A} \twoheadrightarrow \cdots \twoheadrightarrow \widehat{A}_{n+1} \twoheadrightarrow \widehat{A}_n \twoheadrightarrow \cdots = \widehat{A}_0 := \widehat{R}/\mathfrak{n}$$

as in Corollary 5.3.7, where $\mathfrak{n} \subset \widehat{R}$ is the nil-radical and each step is a liftable PD thickening when given the "square-zero PD structure". We have seen at the beginning of §5.7 that there exists a lift $(X_{\widehat{A}}, \iota) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\widehat{A})$ of $(X_{\widehat{R}}, \iota)$. Choosing N so that $\widehat{A}_N$ surjects onto $\widehat{R}$, the pull back $(X_{\widehat{A}_N}, \iota) \in \mathtt{RZ}_{\mathbb{X}}(\widehat{A}_N)$ of $(X_{\widehat{A}}, \iota)$ lifts $(X_{\widehat{R}}, \iota)$ and satisfies all the assumptions for $(X_{\widehat{R}}, \iota)$. Furthermore, in order to prove Proposition 5.8, it suffices to prove it for $(X_{\widehat{A}_N}, \iota)$. From now on, we assume that $\widehat{R} = \widehat{A}_N$.

We choose $R \subset \widehat{R}$, $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$, and $(t_\alpha(R)) \subset \mathbb{D}(X)(R)^\otimes$ as in §5.7 such that the conclusion of Lemma 5.9.3 holds. For $n \leqslant N$, we set R_n to be the image of R in $\widehat{A}_n$, which form a finite sequence of square-zero *liftable* PD thickenings $R_{n+1} \twoheadrightarrow R_n$.[27] For each n, we choose a p-adic p-torsion free PD thickening $S^{\mathrm{fl}}_n \twoheadrightarrow R_n$ with a PD morphism $S^{\mathrm{fl}}_n \to R_{n+1}$.

Let us write $X^{(n)} := X_{R_n}$. By Proposition 5.9.4, we have $(X^{(0)}, \iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R_0)$, and the image $t_\alpha(R_0) \in \mathbb{D}(X^{(0)})(R_0)^\otimes$ of $t_\alpha(R)$ (as in §5.7) coincides with the section of $t^{(0)}_\alpha : \mathbf{1} \to \mathbb{D}(X^{(0)})^\otimes$ constructed in Proposition 5.2.

[27]Here is how we show that $R_{n+1} \twoheadrightarrow R_n$ is a liftable PD thickening. Let A is a polynomial ring over W surjecting onto R_{n+1}, and let $S_n \twoheadrightarrow R_n$ denote the p-adically completed PD hull. By liftability of $\widehat{A}_{i+1} \twoheadrightarrow \widehat{A}_i$ one can easily check that the natural PD morphism $S_n \twoheadrightarrow R_{n+1} \to \widehat{A}_{n+1}$ factors through some p-adic $\mathbb{Z}_p$-flat PD thickening of $\widehat{A}_n$.

We now proceed inductively on n. More precisely, assume that we have $(X^{(n)}, \iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R_n)$ and $t^{(n)}_\alpha(R_n) = t_\alpha(R_n)$ in $\mathbb{D}(X^{(n)})(R_n)^\otimes$, where $t_\alpha(R_n)$ is the image of $t_\alpha(R)$ and $t^{(n)}_\alpha : \mathbf{1} \to \mathbb{D}(X^{(n)})^\otimes$ is as constructed in Proposition 5.2. Then we can deduce that the same assertion holds for $n+1$ by repeating the proof of Proposition 5.8 (especially, by chasing the same diagram as (5.9.5)). Since $R = R_N$, we obtain the desired claim □

6. Construction of the moduli of p-divisible groups with Tate tensors

We give a proof of Theorem 4.9.1 in this section, applying the technical results proved in §5.

6.1. **Artin representability theorem.** Let $p > 2$. We choose a W-lift $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ as in Remark 2.5.8, and define $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n} := \mathtt{RZ}^{\Lambda}_{G,b} \times_{\mathtt{RZ}_{\mathbb{X}}} \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ as a functor on the category of W/p^m-algebras; i.e., for a W/p^m-algebra R, we set $\mathtt{RZ}^{\Lambda}_{G,b}(R) := \mathtt{RZ}^{\Lambda}_{G,b}(R) \cap \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(R)$. Similarly, we define $\mathtt{RZ}^{\Lambda}_{G,b}(h) := \mathtt{RZ}^{\Lambda}_{G,b} \times_{\mathtt{RZ}_{\mathbb{X}}} \mathtt{RZ}_{\mathbb{X}}(h)$ as a functor on Nilp_W.

We first prove (Theorem 6.1.5) that $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is a separated algebraic space locally of finite type over W/p^m using the Artin representability theorem [1, Corollary 5.4][28]. See [30, §2] for the definition of algebraic spaces.

To apply the Artin representability theorem as stated in [1, Corollary 5.4], it suffices to verify the following conditions[29], some of which will be more precisely stated when they are verified:

(1) $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is separated; i.e., the diagonal map
$$\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n} \to \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n} \times \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$$
is representable by a closed immersion. More concretely, for any W/p^m-algebra R and given two points $x, y \in \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(R)$, the locus over which x and y coincide is a closed subscheme of $\operatorname{Spec} R$. This follows since $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is a subfunctor of a separated scheme $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$.

(2) $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ commutes with filtered direct limits of W/p^m-algebras (i.e., locally of finite presentation over W/p^m); indeed, this follows because both $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ and $\mathtt{RZ}^{\Lambda}_{G,b}$ commute with filtered direct limits (*cf.* Lemma 5.1.2).

(3) $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ satisfies the "effectivity property"; namely, for any complete local noetherian W/p^m-algebra $(R, \mathfrak{m}_R)$ with residue field κ, the following natural map is bijective:
$$\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(R) \to \varprojlim_i \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(R/\mathfrak{m}_R^i).$$
This follows since $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ satisfies the effectivity property (*cf.* [39, §2.22]), and $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$ lies in $\mathtt{RZ}^{\Lambda}_{G,b}(R)$ if and only if this is the case over $R/\mathfrak{m}_R^i$ for any i (*cf.* Proposition 5.6).

(4) $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is an fppf sheaf; *cf.* Lemma 6.1.1.

(5) $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ satisfies some suitable generalisation of Schlessinger's criterion (i.e., Conditions (S1′) and (S2) in [1, §2]); namely, Lemma 6.1.2 holds and the tangent space at any $x \in \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(\kappa)$ is finite dimensional over κ. Finiteness of tangent spaces is obvious since $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is a subfunctor of $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$.

[28] See also [10] for some clarifications.

[29] The conditions that we state here are slightly stronger than the ones given in [1, Corollary 5.4].

(6) If $n \gg 0$ then for any $m \geqslant 1$ there exists an obstruction theory for $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ in the sense of [1, (2.6), (4.1)]. Indeed, we verify some variant of this (exploiting the flexibility to increase n), which would still imply the representability of $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$; *cf.* Lemma 6.1.4, the proof of Theorem 6.1.5.

We have already verified the first three conditions, so it remains to verify the remaining three conditions.

To show that $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is an fppf sheaf, it suffices to show that $\mathtt{RZ}^{\Lambda}_{G,b}$ is an fppf sheaf (as the height of a quasi-isogeny can be computed fppf-locally, and whether $p^n\tilde{\iota}$ is an isogeny can be verified fppf-locally). Since $\mathtt{RZ}^{\Lambda}_{G,b}$ is a subfunctor of $\mathtt{RZ}_{\mathbb{X}}$, which is an fppf sheaf (as it can be represented by a formal scheme), the following lemma shows that $\mathtt{RZ}^{\Lambda}_{G,b}$ is an fppf sheaf.

Lemma 6.1.1. *Let $(X,\iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$ for $R \in \mathrm{Nilp}_W$. Assume that there exists an fppf covering $\{R_\xi\}_\xi$ such that the pull-back (X_ξ, ι_ξ) lies in $\mathtt{RZ}^{\Lambda}_{G,b}(R_\xi)$ for each ξ. Then we have $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$.*

Proof. We may assume that R is finitely generated. Since whether $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ is decided by the pull back over all artinian quotients of R, we may assume that $R \in \mathfrak{AR}_W$. Then (X,ι) defines a map $\operatorname{Spec} R \to (\mathtt{RZ}_{\mathbb{X}})^{\widehat{}}_x \cong \mathtt{Def}_{X_x}$ for some closed point $x \in \mathtt{RZ}_{\mathbb{X}}(\kappa)$. This map factors through $\mathtt{Def}_{X_x,G}$ if and only if it does after precomposing with a faithfully flat map $\operatorname{Spec} R' \to \operatorname{Spec} R$, since $\mathtt{Def}_{X_x,G}$ is a closed formal subscheme of $\mathtt{Def}_{X_x}$. □

Lemma 6.1.2. *[*Cf. *Condition (S1′) in* [1, (2.2)]*] Assume that $p > 2$, and consider $B, R, R' \in \mathrm{Nilp}_W$ such that $B \twoheadrightarrow R$ is a square-zero thickening with the kernel annihilated by the nilradical of B, and $R' \to R$ is a W-algebra map that induces a surjective map $R' \to R_{\mathrm{red}}$. Set $B' := B \times_R R'$. Let $\mathcal{F}$ be one of $\mathtt{RZ}^{\Lambda}_{G,b}$, $\mathtt{RZ}^{\Lambda}_{G,b}(h)$ and $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$, where in the last case we assume $p^m B = 0$ and $p^m R' = 0$. Then the natural map*

$$\mathcal{F}(B') \to \mathcal{F}(B) \times_{\mathcal{F}(R)} \mathcal{F}(R') \tag{6.1.3}$$

is a bijection.

Proof. The map (6.1.3) is injective, because the analogous maps for $\mathtt{RZ}_{\mathbb{X}}$, $\mathtt{RZ}_{\mathbb{X}}(h)$, and $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ are bijections.

To show the surjectivity of (6.1.3) for $\mathtt{RZ}^{\Lambda}_{G,b}$, we may assume that both B and R' are finitely generated over W, in which case the claim follows from Lemma 5.5.2. To show the surjectivity of (6.1.3) for $\mathtt{RZ}^{\Lambda}_{G,b}(h)$ and $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$, we observe that a cofibre product of quasi-isogenies of height h is again of height h, and that a cofibre product of isogenies is again an isogeny. □

Lemma 6.1.4 ("Obstruction theory"). *Let $\mathfrak{U} \subset \mathtt{RZ}_{\mathbb{X}}(h)$ be a quasi-compact open formal subscheme, and choose an integer n large enough so that the natural map $\widehat{\Omega}_{\mathfrak{U}/W}|_{\mathfrak{U}\cap\mathtt{RZ}_{\mathbb{X}}(h)^{1,n}} \to \Omega_{\mathfrak{U}\cap\mathtt{RZ}_{\mathbb{X}}(h)^{1,n}/\kappa}$ is an isomorphism.*[30] *Let $B \twoheadrightarrow R$ be any square-zero thickening such that its kernel $\mathfrak{b}$ is killed by the nilradical of B, and let $(X,\tilde{\iota}) \in \mathfrak{U}(R) \cap \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(R)$.*

Then there exists a B-point $(X_B,\tilde{\iota}) \in \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(B)$ lifting $(X,\tilde{\iota})$ if and only if there exists a B-point $(X_B,\tilde{\iota}) \in \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(B)$ lifting $(X,\tilde{\iota})$.

Note that $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ has an obstruction theory that satisfies the conditions in [1, (2.6), (4.1)] (given by the theory of cotangent complex, for example). And the

[30]Such n exists as $\mathfrak{U}$ is noetherian; indeed, as $\mathfrak{U} \times_{\operatorname{Spf} W} \operatorname{Spec} \kappa = \varinjlim_n (\mathfrak{U} \cap \mathtt{RZ}_{\mathbb{X}}(h)^{1,n})$, we may choose n so that $\mathfrak{U} \cap \mathtt{RZ}_{\mathbb{X}}(h)^{1,n}$ contains the closed subscheme of $\mathfrak{U} \times_{\operatorname{Spf} W} \operatorname{Spec} \kappa$ cut out by the square of the maximal ideal of definition. Note that $\mathtt{RZ}_{\mathbb{X}}(h)$ may not be quasi-compact.

lemma asserts that any obstruction theory for the $\mathfrak{U}\cap \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ (which exists) also provides an obstruction theory for $\mathfrak{U}\cap \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ if n is large enough (depending on $\mathfrak{U}$).

Proof. It suffices to prove the "if" direction. Let $(X,\tilde{\iota}) \in \mathfrak{U}(R)$, and assume that there exists a B-point $(X_B,\tilde{\iota}) \in \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(B)$ lifting $(X,\tilde{\iota})$. Set $R_0 := R_{\mathrm{red}}$, and write $(X_{R_0},\tilde{\iota}) \in \mathfrak{U}(R_0)$ denote the pull-back.

Let $f_0 : \operatorname{Spec} R_0 \to \mathfrak{U}$ denote the map induced by $(X_{R_0},\tilde{\iota})$. Then the set of B-points of $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ lifting $(X,\tilde{\iota})$, which is non-empty by assumption, is a torsor under the R_0-module $f_0^*(\widehat{\Omega}^*_{\mathfrak{U}/W})\otimes_{R_0}\mathfrak{b}$ by the assumption on n in the statement. (Recall that any lift of $(X,\tilde{\iota})$ lies in $\mathfrak{U}$.) Therefore, any B-lift $(X'_B,\tilde{\iota}') \in \mathtt{RZ}_{\mathbb{X}}(B)$ of $(X,\tilde{\iota})$ actually lie in $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(B)$, as the set of such B-lifts is also a torsor under the same R_0-module $f_0^*(\widehat{\Omega}^*_{\mathfrak{U}/W})\otimes_{R_0}\mathfrak{b}$. Now if we also have $(X,\tilde{\iota}) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$, then Corollary 5.2.2 produces a B-point

$$(X_B,\tilde{\iota}) \in \mathtt{RZ}^{\Lambda}_{G,b}(B)\cap \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(B) = \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(B)$$

lifting $(X,\tilde{\iota})$, as desired. □

We are ready to prove the following:

Theorem 6.1.5. *The functor $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ can be represented by a separated scheme locally of finite type over $\operatorname{Spec} W/p^m$.*

Proof. Choose a quasi-compact open $\mathfrak{U}\subset \mathtt{RZ}_{\mathbb{X}}(h)$ which contains $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$. (Recall that $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ is quasi-compact.) Then we have verified the criterion in [1, Corollary 5.4] to show that $\mathfrak{U}\cap \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n'}$ is a separated algebraic space locally of finite type over W/p^m for any $n' \gg n$. Since we have

$$\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n} = (\mathfrak{U}\cap \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n'})\times_{\mathtt{RZ}_{\mathbb{X}}(h)^{m,n'}} \mathtt{RZ}_{\mathbb{X}}(h)^{m,n},$$

it follows that $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is also a separated algebraic space locally of finite type over W/p^m.

Since the natural inclusion $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}\hookrightarrow \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ is a monomorphism which is locally of finite type, it is separated and locally quasi-finite. (By looking at étale charts, this claim reduces to the case of schemes, which is standard.) By [32, Théorème (A.2)], $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is a scheme. □

6.2. Closedness.

Theorem 6.2.1. *The natural monomorphism $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}\hookrightarrow \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ is a closed immersion of schemes for any m, n, and h. Also, the functor $\mathtt{RZ}^{\Lambda}_{G,b}$ is representable by a formal scheme locally formally of finite type over W, and the natural inclusion $\mathtt{RZ}^{\Lambda}_{G,b}\hookrightarrow \mathtt{RZ}_{\mathbb{X}}$ is a closed immersion of formal schemes.*

To show that the monomorphism $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}\to \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ is a closed immersion, it suffices to show that it is proper. For this, we need to show that $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ is quasi-compact (*cf.* Corollary 6.2.5), and verify the valuative criterion for properness (Lemma 6.2.2). The remaining claims in Theorem 6.2.1 now follow straightforwardly.

It remains to show the quasi-compactness for $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ and verify the valuative criterion for $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}\to \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$. We begin with the valuative criterion.

Lemma 6.2.2. *Let R be a κ-algebra which is a discrete valuation ring, and $L := \operatorname{Frac}(R)$. Let $(X,\iota) \in \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}(R)$ be such that $(X_L,\iota_L) \in \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(L)$. Then we have $(X,\iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}(R)$.*

Proof. It suffices to show that $(X,\iota)\in \mathtt{RZ}^{\Lambda}_{G,b}(R)$ under the assumption as in the statement. As both $\mathtt{RZ}_{\mathbb{X}}$ and $\mathtt{RZ}^{\Lambda}_{G,b}$ commute with filtered direct limits in Nilp_W, we may assume that L is a finitely generated field extension of κ, and $(X,\iota)\in\mathtt{RZ}_{\mathbb{X}}(R)$ extends to $(X_{R_0},\iota_{R_0})\in\mathtt{RZ}_{\mathbb{X}}(R_0)$ for some *smooth*[31] κ-subalgebra $R_0\subset R$ with $L=\operatorname{Frac}R_0$. By smoothness, there exists a p-adic topologically smooth W-algebra A_0 with $A_0/p=R_0$. By localising A_0 at the ideal corresponding to the closed point of $\operatorname{Spec}R$ and p-adically completing it, we obtain a p-adic flat W-algebra A with $A/p=R$.

Recall that giving a map of crystals $t:\mathbf{1}\to\mathbb{D}(X)^{\otimes}$ (respectively, $t:\mathbf{1}\to\mathbb{D}(X_L)^{\otimes}$) is equivalent to giving a horizontal section $t(A)\in\mathbb{D}(X)(A)^{\otimes}$ (respectively, $t(\widehat{A}_{(p)})\in\mathbb{D}(X_L)(\widehat{A}_{(p)})^{\otimes}$); *cf.* [11, Corollary 2.2.3].

By the assumption on (X,ι), the map of isocrystals $s_{\alpha,\mathbb{D}}:\mathbf{1}\to\mathbb{D}(X)^{\otimes}[\frac{1}{p}]$ comes from a unique map of integral crystals $t_\alpha:\mathbf{1}\to\mathbb{D}(X_L)^{\otimes}$ on the generic fibre. So we obtain

$$(t_\alpha(\widehat{A}_{(p)}))\subset\mathbb{D}(X_L)(\widehat{A}_{(p)})^{\otimes}\cap\mathbb{D}(X)(A)^{\otimes}[\tfrac{1}{p}]=\mathbb{D}(X)(A)^{\otimes},$$

since we have $A=A[\frac{1}{p}]\cap\widehat{A}_{(p)}$. We rename this section as $t_\alpha(A)\in\mathbb{D}(X)(A)^{\otimes}$.

Furthermore, since (X,ι) is defined over R_0, it follows that $t_\alpha(A)\in\mathbb{D}(X)(A)^{\otimes}[\frac{1}{p}]$ lies in the image of $\mathbb{D}(X_{R_0})(A_0)^{\otimes}[\frac{1}{p}]$. Since we also have $A_0=A_0[\frac{1}{p}]\cap A$ (as $A_0/p=R_0\hookrightarrow A/p=R$ by assumption), it follows that $t_\alpha(A)$ is the image of $t_\alpha(A_0)\in\mathbb{D}(X_{R_0})(A_0)^{\otimes}$. Since $(t_\alpha(A_0))$ are horizontal (as they are after inverting p), it follows that (t_α) on X_L extend to maps of crystals $t_\alpha:\mathbf{1}\to\mathbb{D}(X_{R_0})^{\otimes}$.

Now consider the following finitely generated A_0-scheme

$$P_{A_0}:=\underline{\mathrm{isom}}_{A_0}\big[(\mathbb{D}(X_{R_0})(A_0),(t_\alpha(A_0))],[A_0\otimes_{\mathbb{Z}_p}\Lambda^*,(1\otimes s_\alpha)]\big).$$

By construction, P_{A_0} restricts to a trivial G-torsor over $A_0[\frac{1}{p}]$ since the quasi-isogeny over R_0 gives a splitting.

Let us now show that the pull-back P_A of P_{A_0} is a G-torsor over A. It suffices to show that its pull-back $P_{A'}$ is a G-torsor for some suitable faithfully flat A-algebra A', which we introduce now. Let R' be a complete discretely valued R-algebra whose residue field is an algebraic closure κ' of the residue field of R, and we identify $R'=\kappa'[[u]]$. We set $A':=W(\kappa')[[u]]$ and choose a lift $A\to A'$ of $R\to R'$.

We now show that $P_{A'}$ is a (necessarily trivial) G-torsor. We already have that $P_{A'[1/p]}$ is a trivial G-torsor. Since $(X_L,\iota_L)\in\mathtt{RZ}^{\Lambda}_{G,b}(L)$, it follows that $P_{W(\bar{L}')}$ is a G-torsor where $L':=\operatorname{Frac}(R')$, so $P_{\widehat{A}'_{(p)}}$ is a G-torsor. (Note that $\widehat{A}'_{(p)}$ is a p-adic discrete valuation ring with $\widehat{A}'_{(p)}/p=L'$, so we have a faithfully flat map $\widehat{A}'_{(p)}\to W(\overline{L}')$.) Therefore, we have that $P_{U'}$ is a G-torsor, where $U'\subset\operatorname{Spec}A'$ is the complement of the closed point.

By [9, Théorème 6.13], $P_{U'}$ extends to some G-torsor $P'_{A'}$ over A'. But since A' is strictly henselian, $P'_{A'}$ is a trivial G-torsor, which implies that $P_{U'}$ is a trivial G-torsor. Therefore there exists an isomorphism of vector bundles

$$\varsigma:\mathcal{O}_{U'}\otimes_{A'}\mathbb{D}(X_{A'})(A')\xrightarrow{\sim}\mathcal{O}_U\otimes_{\mathbb{Z}_p}\Lambda^*$$

matching $t_\alpha(A')|_{U'}$ with $1\otimes s_\alpha$. Since ς is defined away from a codimension-2 subset in a normal scheme, ς extends to an A'-section of $P_{A'}$ by taking the global section. This shows an isomorphism $G_{A'}\cong P_{A'}$.[32]

Furthermore, since P_{A_0} pulls back to a smooth scheme over A, it has to be smooth over some open formal subscheme $\operatorname{Spec}A'_0\subset\operatorname{Spec}A_0$ containing the closed

[31] Smoothness can be arranged since R is regular.

[32] This argument is adapted from "Step 5" of the proof of [29, Proposition 1.3.4].

point of $\operatorname{Spec} R$, with non-empty fibres at any point in $\operatorname{Spec} A'_0$; i.e., the restriction $P_{A'_0}$ is a G-torsor. By replacing A_0 with the p-adic completion of A'_0, we may assume that P_{A_0} is a G-torsor.

By Lemma 2.2.6, the Hodge filtration $\mathrm{Fil}^1_{X_{R_0}} \subset \mathbb{D}(X_{R_0})(R_0)$ is a $\{\mu\}$-filtration. (Indeed, $\mathrm{Fil}^1_{X_L}$ is a $\{\mu\}$-filtration so $\mathrm{Fil}^1_{X_{R_0}}$ is a $\{\mu\}$-filtration over the closure of the generic point, which is $\operatorname{Spec} R_0$.) This shows that $(X_{R_0}, \iota_{R_0}) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(R_0)$, so we have $(X, \iota) \in \mathtt{RZ}^{\Lambda}_{G,b}(R)$. $\square$

Proposition 6.2.3. *Let R be a smooth domain over κ, and let $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$. Assume that there exists a dense subset of closed points $\Sigma \subset \operatorname{Spec} R$, such that for any $x \in \Sigma$ we have $(X_x, \iota_x) \in \mathtt{RZ}^{\Lambda}_{G,b}(\kappa)$. Then there exists a dense open subscheme $\operatorname{Spec} R' \subset \operatorname{Spec} R$ such that $(X_{R'}, \iota_{R'}) \in \mathtt{RZ}^{\Lambda}_{G,b}(R')$.*

Proof. Let us choose a formally smooth p-adic W-lift A of R. By the standard dictionary [11, Corollary 2.2.3], the morphism $s_{\alpha,\mathbb{D}} : \mathbf{1} \to \mathbb{D}(X)^{\otimes}[\frac{1}{p}]$, constructed in Definition 4.5, corresponds to a horizontal section $t_\alpha(A) \in \mathbb{D}(X)(A)^{\otimes}[\frac{1}{p}]$. Let us first show that $t_\alpha(A) \in \mathbb{D}(X)(A)^{\otimes}$; i.e., $t_\alpha(A)$ is the A-section of a (unique) map of crystals $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$.

We endow the p-adic filtration with $A[\frac{1}{p}]$ and identify the associated graded algebra $\mathrm{gr}^\bullet A[\frac{1}{p}] \cong R(\!(u)\!)$ by sending p to u. Then we define $\{t^{(m)}_\alpha \in \mathbb{D}(X)(R)^{\otimes}\}_{m\in\mathbb{Z}}$ so that $\sum_{m\in\mathbb{Z}} t^{(m)}_\alpha u^m$ is the image of $t_\alpha(A)$ via the map

$$\mathbb{D}(X)(A)^{\otimes}[\tfrac{1}{p}] \to \mathrm{gr}^\bullet \left(\mathbb{D}(X)(A)^{\otimes}[\tfrac{1}{p}]\right) \cong \bigoplus_{m\in\mathbb{Z},\ m\gg-\infty} u^m \mathbb{D}(X)(R)^{\otimes},$$

where $\mathrm{gr}^\bullet$ is with respect to the p-adic filtration.

Note that $t_\alpha(A) \in \mathbb{D}(X)(A)^{\otimes}$ if and only if $t^{(m)}_\alpha = 0$ for any $m < 0$. On the other hand, if $m < 0$ then $t^{(m)}_\alpha$ vanishes at a dense set of points Σ, so $t^{(m)}_\alpha = 0$; indeed, for any $x \in \Sigma$, any map $\tilde{x} : A \to W$ lifting $R \twoheadrightarrow R/\mathfrak{m}_x \cong \kappa$ pulls back $t_\alpha(A)$ to a (p-integral) tensor in $\mathbb{D}(X_x)(W)^{\otimes}$ because $(X_x, \iota_x) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(\kappa) = \mathtt{RZ}^{\Lambda}_{G,b}(\kappa)$.

We next consider the following A-scheme (as in Definition 4.6(2)):

$$P_A := \underline{\mathrm{isom}}_A\big([\mathbb{D}(X)(A), (t_\alpha(A))], [A \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]\big).$$

By construction, each fibre of P_A at a point of $\operatorname{Spec} A$ is either a G-torsor or empty. Since the fibre P_x at $x \in \Sigma$ is a G-torsor and $P_{A[\frac{1}{p}]}$ is a trivial G-torsor, it follows that the fibre P_η at the generic point η of $\operatorname{Spec} R \subset \operatorname{Spec} A$ is a G-torsor (by semi-continuity of fibre dimensions, for example).

By generic flatness, we find a localisation R' of R such that $P_{R'}$ is a G-torsor, and we choose an A-algebra A' which lifts R'. (If $R' = R[1/f]$ then we let A' to be the p-adic completion of $A[1/\tilde{f}]$ where $\tilde{f}$ is some lift of f.) We want to show that $P_{A'}$ is a G-torsor. Indeed, $P_{R'}$ and $P_{A'[\frac{1}{p}]}$ are G-torsors so it remains to show that $P_{A'}$ is flat over A'. By local flatness criterion [33, Theorem 22.3] it suffices to show that the following surjective map of $\mathcal{O}_{P_{R'}}$-modules is an isomorphism for each m:

$$\mathcal{O}_{P_{R'}} \cong (p^m A'/p^{m+1}A') \otimes_{R'} \mathcal{O}_{P_{R'}} \twoheadrightarrow p^m \mathcal{O}_{P_{A'}}/p^{m+1}\mathcal{O}_{P_{A'}}. \tag{6.2.4}$$

Let $\Sigma' := \Sigma \cap \operatorname{Spec} R'$, which is a dense set of closed points. For any $x' \in \Sigma'$, we choose a lift $\tilde{x}' : \operatorname{Spec} W \to \operatorname{Spec} A'$. By the defining condition of Σ' it follows that the map (6.2.4) pulls back to an isomorphism

$$\mathcal{O}_{P_{x'}} \xrightarrow{\sim} p^m \mathcal{O}_{P_{\tilde{x}'}}/p^{m+1}\mathcal{O}_{P_{\tilde{x}'}}.$$

Since $\bigcup_{x'\in\operatorname{Spec} R'} P_{x'}$ is dense in $P_{R'}$, it follows that the kernel of (6.2.4) is supported in some proper closed subset of $P_{R'}$. On the other hand, $\mathcal{O}_{P_{R'}}$ is a domain, which forces (6.2.4) to be an isomorphism.

To see that $\mathrm{Fil}^1_{X_{R'}}$ is a $\{\mu\}$-filtration with respect to $(t_\alpha(R'))$, it suffices to check this at dense set of points (namely, Σ'). This shows $(X_{R'}, \iota_{R'}) \in \mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}(R')$. □

The following corollary, together with Lemma 6.2.2, concludes the proof of Theorem 6.2.1.

Corollary 6.2.5. *The scheme* $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ *is quasi-compact for any* m, n, h.

Proof. Recall that $\mathtt{RZ}_{\mathbb{X}}(h)^{m,n}$ is quasi-compact; *cf.* [39, §2.22]. Let $Z_0 \subset \mathtt{RZ}_{\mathbb{X}}(h)^{m,n}_{\mathrm{red}}$ denote the reduced closed subscheme whose underlying topological space is the Zariski closure of the image of $|\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}|$, and $Z_0' \to Z_0$ an alteration (i.e., a generically finite projective surjective morphism) such that Z_0' is a smooth projective scheme over κ; *cf.* [13]. Then by Proposition 6.2.3, we can produce a (necessarily quasi-compact) dense open subscheme $U_0 \subset Z_0$ such that the natural map $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}_{\mathrm{red}} \to Z_0$ is isomorphic over U_0. Note that the preimage $U_0 \subset \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}_{\mathrm{red}}$ is again a quasi-compact open subscheme. This also shows that $|Z_0|$ is the set-theoretic image of $|\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}|$ by the valuative criterion (Lemma 6.2.2).

Now, let $Z_1 \subset Z_0$ denote the reduced complement of U_0. By repeating the argument in the previous paragraph (by taking Σ to be the set of all closed points of Z_1), we obtain a (quasi-compact) dense open subscheme $V_1 \subset Z_1$ which is an isomorphic image of a locally closed subscheme $V_1 \subset \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}_{\mathrm{red}}$. So we can choose a quasi-compact open $U_1 \subset \mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}$ containing V_1 as a closed subscheme. Note that this process terminates after finitely many times as Z_0 is noetherian, and we obtain finitely many quasi-compact open subschemes U_i covering $\mathtt{RZ}^{\Lambda}_{G,b}(h)^{m,n}_{\mathrm{red}}$. □

6.3. **Independence of auxiliary choices and functoriality.** We now finish the proof of Theorem 4.9.1. We have constructed a closed formal scheme $\mathtt{RZ}^{\Lambda}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$ (Theorem 6.2.1), which enjoys the following properties:

(1) $\mathtt{RZ}^{\Lambda}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$ represents $\mathtt{RZ}^{(s_\alpha)}_{\mathbb{X},G}$ as in the statement of Theorem 4.9.1. The universal tensors $t^{\mathrm{univ}}_\alpha : \mathbf{1} \to \mathbb{D}(X_{\mathtt{RZ}_{\mathbb{X}}}|_{\mathtt{RZ}^{\Lambda}_{G,b}})^{\otimes}$ can be obtained by glueing the unique Tate tensors over some affine open covering of $\mathtt{RZ}^{\Lambda}_{G,b}$. This claim follows from Corollary 5.2.3.

(2) $\mathtt{RZ}^{\Lambda}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}}$ does not depend on the choice of $(s_\alpha) \in \Lambda^{\otimes}$; indeed, the subset $\mathtt{RZ}^{\Lambda}_{G,b}(\kappa) \subset \mathtt{RZ}_{\mathbb{X}}(\kappa)$ and the completion at any κ-point do not depend on (s_α).

(3) If $G = \mathrm{GL}(\Lambda)$ then $\mathtt{RZ}^{\Lambda}_{G,b} = \mathtt{RZ}_{\mathbb{X}}$; *cf.* Example 4.6.1.

(4) For any closed connected reductive $\mathbb{Z}_p$-subgroup $G' \subset G$ with $b \in G'(K_0)$, the closed formal subscheme $\mathtt{RZ}^{\Lambda}_{G',b}$ is contained in $\mathtt{RZ}^{\Lambda}_{G,b}$. Indeed, this claim amounts to verifying analogous claims on the set of κ-points and the completions thereof; *cf.* Lemma 2.5.4, Proposition 3.7.2.

It remains to verify the functoriality assertions; namely, (1) and (2) in Theorem 4.9.1. These assertions will immediately follow from Lemma 6.3.1 and Proposition 6.3.2, hence we conclude the proof of Theorem 4.9.1.

Let (G', b') and Λ' be another datum as in Definition 2.5.5, and write $\mathbb{X}' := \mathbb{X}^{\Lambda'}_{b'}$. We have constructed a natural formal closed subsheme $\mathtt{RZ}^{\Lambda'}_{G',b'} \subset \mathtt{RZ}_{\mathbb{X}'}$.

Recall that $\mathbb{X}^{\Lambda\times\Lambda'}_{(b,b')} \cong \mathbb{X}^{\Lambda}_b \times \mathbb{X}^{\Lambda'}_{b'} = \mathbb{X} \times \mathbb{X}'$. Then we have a closed immersion $\mathtt{RZ}_{\mathbb{X}} \times_{\operatorname{Spf} W} \mathtt{RZ}_{\mathbb{X}'} \hookrightarrow \mathtt{RZ}_{\mathbb{X}\times\mathbb{X}'}$, defined by the product of deformations up to quasi-isogeny, and a closed formal subscheme $\mathtt{RZ}^{\Lambda\times\Lambda'}_{G\times G',(b,b')} \subset \mathtt{RZ}_{\mathbb{X}\times\mathbb{X}'}$.

Lemma 6.3.1. *We have* $\mathtt{RZ}^{\Lambda\times\Lambda'}_{G\times G',(b,b')} = \mathtt{RZ}^{\Lambda}_{G,b}\times_{\operatorname{Spf}W}\mathtt{RZ}^{\Lambda'}_{G',b'}$ *as closed formal subschemes of* $\mathtt{RZ}_{\mathbb{X}\times\mathbb{X}'}$*; in particular, Theorem 4.9.1(1) holds.*

Proof. As both are closed formal subschemes of $\mathtt{RZ}_{\mathbb{X}\times\mathbb{X}'}$, it suffices to show the equality of the set of κ-points and the completions thereof, which follows from Lemma 2.5.4 and Proposition 3.7.2. □

Proposition 6.3.2. *Given a map $f : G \to G'$ which maps b to b', there exists a map $\mathtt{RZ}^{\Lambda}_{G,b} \to \mathtt{RZ}^{\Lambda'}_{G',b'}$ which induces the desired maps on the set of κ-points and completions thereof as described in Theorem 4.9.1(2).*

The proposition in the case when f is an identity map asserts that the formal scheme $\mathtt{RZ}^{\Lambda}_{G,b}$ depends only on (G,b), not on the auxiliary choice of $(\Lambda,(s_\alpha))$, up to canonical isomorphism.

Proof. We follow the structure of the proof of Proposition 3.7.2.

The case when f is a closed immersion and $\Lambda=\Lambda'$ was already handled at the beginning of §6.3. For a natural projection $\mathrm{pr}_2 : (G\times G',(b,b')) \to (G',b')$, the natural projection

$$\mathtt{RZ}^{\Lambda\times\Lambda'}_{G\times G',(b,b')} \cong \mathtt{RZ}^{\Lambda}_{G,b}\times_{\operatorname{Spf}W}\mathtt{RZ}^{\Lambda'}_{G',b'} \twoheadrightarrow \mathtt{RZ}^{\Lambda'}_{G',b'}$$

has the desired properties on κ-points and completions thereof. (The same holds for the first projection.)

Now, let $f : (G,b)\to(G',b')$ be any morphism, and consider the graph morphism

$$(1,f) : (G,b) \to (G\times G',(b,b')),$$

which is a closed immersion on the reductive $\mathbb{Z}_p$-groups. By letting G act via $(1,f)$ on the faithful $G\times G'$-representation $\Lambda\times\Lambda'$, we obtain the closed subspace $\mathtt{RZ}^{\Lambda\times\Lambda'}_{G,b} \subset \mathtt{RZ}_{\mathbb{X}\times\mathbb{X}'}$. We claim that we have the following commutative diagram:

(6.3.3)
$$\begin{tikzcd}
 & & & \mathtt{RZ}^{\Lambda}_{G,b}\ , \arrow[d, dashed, "\exists !"] \\
\mathtt{RZ}^{\Lambda\times\Lambda'}_{G,b} \arrow[r] \arrow[urrr, bend left=10, "\cong"] & \mathtt{RZ}^{\Lambda\times\Lambda'}_{G\times G',(b,b')} \arrow[r, phantom, "\cong"] & \mathtt{RZ}^{\Lambda}_{G,b}\times_{\operatorname{Spf}W}\mathtt{RZ}^{\Lambda'}_{G',b'} \arrow[r, two heads, "\mathrm{pr}_2"'] \arrow[ur, "\mathrm{pr}_1"] & \mathtt{RZ}^{\Lambda'}_{G',b'}
\end{tikzcd}$$

where the solid arrows are already defined. By looking at the sets of κ-points and the completions thereof (*cf.* Proposition 3.7.2; especially, the diagram in the proof), it follows that pr_1 restricts to an isomorphism $\mathtt{RZ}^{\Lambda\times\Lambda'}_{G,b} \xrightarrow{\sim} \mathtt{RZ}^{\Lambda}_{G,b}$ as claimed in the diagram. Therefore, the broken arrow is well-defined and satisfies the desired properties on κ-points and the completions thereof. □

7. Extra structures on the moduli of p-divisible groups

We assume that $p>2$, and set $\kappa=\overline{\mathbb{F}}_p$, $W=\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$, and $K_0=\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. We fix (G,b) as in Definition 2.5.5 (with associated integral Hodge-type Shimura datum $(G,[b],\{\mu^{-1}\})$). With some suitable choice of Λ (which gives rise to $\mathbb{X}:=\mathbb{X}_b^{\Lambda}$), we construct the closed formal subscheme $\mathtt{RZ}^{\Lambda}_{G,b}\subset\mathtt{RZ}_{\mathbb{X}}$. From now on, we write $\mathtt{RZ}_{G,b}:=\mathtt{RZ}^{\Lambda}_{G,b}$ as it does not depend on Λ up to canonical isomorphism.

In this section, we define a Weil descent datum, the action of $J_b(\mathbb{Q}_p)$, "étale realisations" of crystalline Tate tensors, the rigid analytic tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$, and the Grothendieck-Messing period map – in other words, we construct "local Shimura varieties" as conjectured in Rapoport and Viehmann [38, §5]. Since $\mathtt{RZ}_{G,b}$ is locally formally of finite type over $\operatorname{Spf}W$ (*cf.* Theorem 4.9.1), Berthelot's construction of rigid generic fibre $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$ can be applied; *cf.* [3], [11, §7]. We then construct the period morphism on the rigid generic fibre $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$, which is an étale morphism (highly transcendental in general). When $\mathtt{RZ}_{G,b}$ is an EL or PEL Rapoport-Zink space, the extra structure on $\mathtt{RZ}_{G,b}$ that we define is compatible with the one defined by Rapoport and Zink in [39]. (We leave readers to verify this, which is more or less straightforward.)

The category of rigid analytic varieties can naturally be viewed as a full subcategory of the category of adic spaces (*cf.* [23, §1.1.11]), so we may regard all the rigid analytic varieties as adic spaces[33].

For the EL and PEL case, Scholze and Weinstein [42] constructed an infinite-level Rapoport-Zink space. We construct an "infinite-level Rapoport-Zink space" $\mathtt{RZ}^{\infty}_{G,b}$ associated to (G,b) using the infinite-level Rapoport-Zink space for $\mathrm{GL}_{\mathbb{Q}_p}(\Lambda[\frac{1}{p}])$ and the rigid analytic tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$. This construction is rather *ad hoc*, and there should be a more natural construction, as alluded in the introduction of [42].

7.1. **More notation on adic spaces and p-divisible groups.** We will work with the notion of adic spaces in the sense of Huber. (See [42, §2] for basic definitions.) Although it is possible to work with classical rigid analytic geometry for most part of this section (except §7.6), the flexibility of the theory of adic spaces could be useful (for example, to define geometric points).

Let $\mathfrak{X}$ be a formal scheme locally formally of finite type over $\mathrm{Spf}\, W$, and let $\mathfrak{X}^{\mathrm{rig}}$ denote the rigid analytic generic fibre constructed by Berthelot (*cf.* [11, §7.1]). It is often convenient to view $\mathfrak{X}^{\mathrm{rig}}$ as an adic space, which we will do implicitly.

Let us recall the functorial characterisation of $\mathfrak{X}^{\mathrm{rig}}$; *cf.* [11, Proposition 7.1.7]. For an analytic space (or adic space) $\mathscr{Y}$ (topologically) of finite type over K_0, we have

$$\mathrm{Hom}_{K_0}(\mathscr{Y}, \mathfrak{X}^{\mathrm{rig}}) \xleftarrow{\sim} \varinjlim_{\mathfrak{Y};\ \mathfrak{Y}^{\mathrm{rig}} = \mathscr{Y}} \mathrm{Hom}_W(\mathfrak{Y}, \mathfrak{X}), \tag{7.1.1}$$

where $\mathfrak{Y}$ runs through formal models of $\mathscr{Y}$. Note that this property uniquely determines $\mathfrak{X}^{\mathrm{rig}}$ by the rigid analytic Yoneda lemma [11, Lemma 7.1.5], and (the adic space associated to) $\mathfrak{X}^{\mathrm{rig}}$ coincides with the generic fibre of the adic space associated to the formal scheme $\mathfrak{X}$; *cf.* [42, Proposition 2.2.2].

Remark 7.1.2. Let K be a complete extension of K_0 (with rank 1 valuation), and let $\mathscr{O}_K$ denote its valuation. (We will often use C instead of K for algebraically closed complete extension of $\mathbb{Q}_p$.) Then one can check without difficulty that any point $x : \mathrm{Spa}(K, \mathscr{O}_K) \to \mathfrak{X}^{\mathrm{rig}}$ comes from a unique map $x : \mathrm{Spf}\, \mathscr{O}_K \to \mathfrak{X}$ of formal schemes, also denoted by x.

Let X be a p-divisible group over $\mathfrak{X}$, and write $\mathscr{X} := \mathfrak{X}^{\mathrm{rig}}$. Then, $X[p^n]^{\mathrm{rig}}$ is a finite étale covering of $\mathscr{X}$, which is also an abelian group object in the category of adic spaces.

Definition 7.1.3. Let $T(X)$ denote the lisse $\mathbb{Z}_p$-sheaf on $\mathscr{X}$ defined by the projective system $\{X[p^n]^{\mathrm{rig}}\}$, and define $V(X)$ to be the the lisse $\mathbb{Q}_p$-sheaf associated to $T(X)$; i.e., $T(X)$ viewed in the isogeny category. (See [40, Definition 8.1] for the definition of lisse $\mathbb{Z}_p$-sheaf on an adic space.)

As lisse $\mathbb{Z}_p$- or $\mathbb{Q}_p$- sheaves, it is possible to form tensor products, symmetric and alternating products, and duals (so $T(X)^{\otimes}$ and $V(X)^{\otimes}$ make sense). The formation of $T(X)$ and $V(X)$ commutes with any base change $\mathfrak{Y} \to \mathfrak{X}$ for reasonable formal scheme $\mathfrak{Y}$. In particular, for any geometric point $\bar{x} : \mathrm{Spa}(C, \mathscr{O}_C) \to \mathscr{X}$ the fibre

[33] In light of the recent work on "infinite-level Rapoport-Zink spaces" in [42] and [41, §6], the theory of adic spaces is the most natural framework to study the rigid analytic tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$.

$T(X)_{\bar{x}}$, as a $\mathbb{Z}_p$-module, only depends on the pull-back $X_{\bar{x}}$ of X by $\bar{x} : \mathrm{Spf}\,\mathscr{O}_C \to \mathfrak{X}$. (Here, we use the convention as in Remark 7.1.2.)

Remark 7.1.4. Let us make explicit the adic space generic fibre $\mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$ of $\mathtt{RZ}_{\mathbb{X}}$. For any analytic space (or adic space) $\mathscr{X}$ topologically of finite type over K_0, the set $\mathrm{Hom}_{K_0}(\mathscr{X}, \mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}})$ can be interpreted as the set of equivalence classes of $(X,\iota) \in \mathtt{RZ}_{\mathbb{X}}(\mathfrak{X})$ for any formal model $\mathfrak{X}$ of $\mathscr{X}$, where for any morphism $\mathfrak{X}' \to \mathfrak{X}$ of formal models of $\mathscr{X}$, $(X,\iota) \in \mathtt{RZ}_{\mathbb{X}}(\mathfrak{X})$ is equivalent to $(X_{\mathfrak{X}'}, \iota_{\mathfrak{X}'_{\mathbb{F}_p}}) \in \mathtt{RZ}_{\mathbb{X}}(\mathfrak{X}')$.

One can easily see that $T(X)$ is independent of the choice of formal model $\mathfrak{X}$ over which X is defined. If we set $X_{\mathtt{RZ}_{\mathbb{X}}}$ to be the universal p-divisible group over $\mathtt{RZ}_{\mathbb{X}}$ and $f : \mathscr{X} \to \mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$ to be the map corresponding to X, then we have $T(X) \cong f^*(T(X_{\mathtt{RZ}_{\mathbb{X}}}))$. A similar discussion holds for $\mathtt{RZ}_{G,b}^{\mathrm{rig}}$ in the place of $\mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$.

Let $\mathscr{X}$ be a connected component of $\mathtt{RZ}_{G,b}^{\mathrm{rig}}$. For a geometric point $\bar{x}$ of $\mathscr{X}$ (i.e., $\bar{x} : \mathrm{Spa}(C, \mathscr{O}_C) \to \mathscr{X}$ for some algebraically closed complete extension C of K_0), let $\pi_1^{\text{fét}}(\mathscr{X}, \bar{x})$ denote the algebraic fundamental group[34] of $\mathscr{X}$ with base point $\bar{x}$ in the terminology of [12].

Quite formally, one obtains a natural equivalence of categories from the category of lisse $\mathbb{Z}_\ell$-sheaves on $\mathscr{X}$ to the category of finitely generated $\mathbb{Z}_\ell$-modules with continuous $\pi_1^{\text{fét}}(\mathscr{X}, \bar{x})$-action, where the equivalence is defined by $\mathscr{F} \rightsquigarrow \mathscr{F}_{\bar{x}}$. (*Cf.* [12, §4], [40, Proposition 3.5].) Similarly, one obtains a natural equivalence of categories from the category of lisse $\mathbb{Q}_\ell$-sheaves on $\mathscr{X}$ (i.e., lisse $\mathbb{Z}_\ell$ sheaves viewed up to isogeny[35]) to the category of finite-dimensional $\mathbb{Q}_\ell$-vector spaces with continuous $\pi_1^{\text{fét}}(\mathscr{X}, \bar{x})$-action. Here, ℓ can be any prime (including $\ell = p$).

Let $\mathbf{1}$ denote either the constant rank-1 $\mathbb{Z}_p$-sheaf, or the constant 1-dimensional $\mathbb{Q}_p$-sheaf.

Definition 7.1.5. An *étale Tate tensor* on X is a morphism $t_{\text{ét}} : \mathbf{1} \to V(X)^{\otimes}$ of lisse $\mathbb{Q}_p$-sheaves on $\mathscr{X}$. An étale Tate tensor is called *integral* if it restricts to a map $\mathbf{1} \to T(X)^{\otimes}$ of lisse $\mathbb{Z}_p$-sheaves on $\mathscr{X}$.

It follows that when $\mathscr{X}$ is connected, giving an étale Tate tensor $t_{\text{ét}}$ is equivalent to giving an $\pi_1^{\text{fét}}(\mathscr{X}, \bar{x})$-invariant element $t_{\text{ét},\bar{x}} \in V(X)^{\otimes}_{\bar{x}}$, and $t_{\text{ét}}$ is integral if and only if $t_{\text{ét},\bar{x}} \in T(X)^{\otimes}_{\bar{x}}$ for a single geometric point $\bar{x}$.

We now claim that crystalline Tate tensors have " étale realisations".

Theorem 7.1.6. *Assume that that $\mathfrak{X}$ is formally smooth and locally formally of finite type over $\mathrm{Spf}\,W$, and let $t : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ be a morphism of crystals which is Frobenius-equivariant up to isogeny and such that $t(R) \in \mathrm{Fil}^0\,\mathbb{D}(X)(R)^{\otimes}$. Then there exists a unique morphism $t_{\text{ét}} : \mathbf{1} \to T(X)^{\otimes}$ of lisse $\mathbb{Z}_p$-sheaves on $\mathscr{X}$, such that at each geometric point $\bar{x}$ supported at a classical point x with residue field K, the (classical) crystalline comparison isomorphism matches $t_{\text{ét},\bar{x}} \in T(X_{\bar{x}})^{\otimes}$ with $t_x : \mathbf{1} \to \mathbb{D}(X_x)^{\otimes}$, obtained as the pull-back of t by $x : \mathrm{Spf}\,\mathscr{O}_K \to \mathfrak{X}$. (Here $X_x := x^*X$ is a p-divisible group over $\mathscr{O}_K$.)*

If $\bar{x}$ is as in the theorem, then $t_{\text{ét},\bar{x}} \in T(X)^{\otimes}_{\bar{x}}$ is invariant under the $\pi_1^{\text{fét}}(\mathscr{X}', \bar{x})$-action, not just the $\mathrm{Gal}(\overline{K}_0/K)$-action, where $\mathscr{X}' \subset \mathscr{X}$ is the connected component containing $\bar{x}$. Indeed, one can see that the requirement for $t_{\text{ét}}$ in Theorem 7.1.6 uniquely determines $t_{\text{ét}}$.

---

[34] In [40, §3] the algebraic fundamental group is called the "pro-finite fundamental group".

[35] Lisse $\mathbb{Q}_\ell$-sheaves are more restrictive objects than "local systems of $\mathbb{Q}_\ell$-vector spaces" as in [12, Definition 4.1], which involve "analytic étale coverings" of $\mathscr{X}$ (not just finite étale coverings). In particular, the geometric fibre $\mathscr{F}_{\bar{x}}$ of a lisse $\mathbb{Q}_\ell$-sheaf has an action of $\pi_1^{\text{fét}}(\mathscr{X}, \bar{x})$, not just (the adic space version of) the analytic fundamental group defined in [12, §2].

The main idea of the proof of Theorem 7.1.6 is to construct $t_{\text{ét}}$ using the (relative) crystalline comparison for p-divisible groups over $\mathfrak{X}$, and show that it is integral using the theory of Kisin modules (over a p-adic discrete valuation ring). Although the proof is quite "standard", it takes a long digression to set up the notation. We give a proof in §8.

It should be possible to compare the fibres $t_{\text{ét},\bar{x}} \in T(X)^{\otimes}_{\bar{x}}$ and $t_{\bar{x}} : \mathbf{1} \to \mathbb{D}(X_{\bar{x}})^{\otimes}$ at any geometric point $\bar{x}$ of $\mathscr{X}$, using the theory of vector bundles over the Fargues-Fontaine curve. We will not work in this generality, as geometric points supported at classical points are sufficient to uniquely determine $t_{\text{ét}}$.

7.2. **The action of $J_b(\mathbb{Q}_p)$.** Recall from (2.4.3) that $J_b(\mathbb{Q}_p)$ is the group of quasi-isogenies $\gamma : \mathbb{X} \dashrightarrow \mathbb{X}$ which preserves the tensors $(s_{\alpha,\mathbb{D}})$. Then $\mathtt{RZ}_{G,b}$ has a natural left $J_b(\mathbb{Q}_p)$-action defined as follows: for any $(X,\iota) \in \mathtt{RZ}_{G,b}(R)$ for $R \in \mathrm{Nilp}_W$ and $\gamma \in J_b(\mathbb{Q}_p)$, we have $\gamma(X,\iota) = (X, \iota\circ\gamma^{-1}) \in \mathtt{RZ}_{\mathbb{X}}(R)$. To see $\gamma(X,\iota) \in \mathtt{RZ}_{G,b}(R)$, it suffices to observe that for $R = \overline{\mathbb{F}}_p$ we recovers the natural $J_b(\mathbb{Q}_p)$-action on $X^G(b) \cong \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$ (*cf.* Proposition 2.5.9), and γ induces $\gamma : (\mathtt{RZ}_{\mathbb{X}})^{\widehat{}}_{x} \xrightarrow{\sim} (\mathtt{RZ}_{\mathbb{X}})^{\widehat{}}_{\gamma x}$ (as γ does not modify the underlying p-divisible group.) By functoriality of adic space generic fibre, $J_b(\mathbb{Q}_p)$ naturally acts on $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$.

The $J_b(\mathbb{Q}_p)$-action on $\mathtt{RZ}_{G,b}$ has a kind of "continuity" property in the sense of [18, Définition 2.3.10]; indeed, the proof of [18, Proposition 2.3.11] works in the more general setting of ours.

7.3. **Weil descent datum.** Let $(G,[b],\{\mu^{-1}\})$ denote the integral Hodge-type local Shimura datum associated to (G,b). The following definition is the local analogue of the reflex field for a Shimura datum. (*Cf.* [39, §1.31].)

Definition 7.3.1. The *(local) reflex field* or *(local) Shimura field* for $(G,[b],\{\mu^{-1}\})$ is the subfield $E = E(\mu) \subset K_0$ which is the field of definition of the $G(K_0)$-conjugacy class of the cocharacter μ. Note that E is a finite unramified extension of $\mathbb{Q}_p$ (as μ descends over some finite subextension of $\mathbb{Q}_p$ in K_0).

Put $d := [E:\mathbb{Q}_p]$, and let $q = p^d$ be the cardinality of the residue field of E. Let $\tau = \sigma^d \in \mathrm{Gal}(K_0/E)$ denote the q-Frobenius element (i.e., the lift of the qth power map on $\overline{\mathbb{F}}_p$).

Remark 7.3.2. Since G is split over W, it follows that the sheaf of conjugacy classes of cocharacters is a constant sheaf on $\operatorname{Spec} W$ (via the interpretation in terms of the associated root datum). In particular, for $\mu \in \{\mu\}$, μ^τ is $G(W)$-conjugate of μ, as this is the case over K_0.

For any formal scheme $\mathfrak{X}$ over $\operatorname{Spf} W$, we write $\mathfrak{X}^\tau := \mathfrak{X} \times_{\operatorname{Spf} W,\tau} \operatorname{Spf} W$. We similarly define $\mathscr{X}^\tau$ for an adic space over (K_0, W).

Definition 7.3.3. Let $\mathfrak{X}$ be a formal scheme over $\operatorname{Spf} W$. A *Weil descent datum* on $\mathfrak{X}$ over $\mathscr{O}_E$ is an isomorphism over $\operatorname{Spf} W$:

$$\Phi : \mathfrak{X} \xrightarrow{\sim} \mathfrak{X}^\tau.$$

Similarly, we define a Weil descent datum over E for a rigid analytic space $\mathscr{X}$ over K_0 as an isomorphism $\Phi : \mathscr{X} \xrightarrow{\sim} \mathscr{X}^\tau$ over K_0.

For any positive integer r, let $E_r \subset K_0$ denote the (unramified) subextension of degree r over E. Then for any Weil descent datum Φ over $\mathscr{O}_E$ for $\mathfrak{X}$, we can define a Weil descent datum over $\mathscr{O}_{E_r}$ as follows:

$$\Phi^r : \mathfrak{X} \underset{\sim}{\xrightarrow{\Phi}} \mathfrak{X}^\tau \underset{\sim}{\xrightarrow{\Phi^\tau}} \cdots \underset{\sim}{\xrightarrow{\Phi^{\tau^{r-1}}}} \mathfrak{X}^{\tau^r}. \tag{7.3.4}$$

The same construction works for rigid analytic spaces over K_0.

Let $\mathfrak{X}_0$ be a formal scheme over $\mathrm{Spf}\,\mathscr{O}_E$, and $\mathfrak{X} := \mathfrak{X}_0 \times_{\mathrm{Spf}\,\mathscr{O}_E} \mathrm{Spf}\,W$. Then there exists a natural Weil descent datum over $\mathscr{O}_E$ on $\mathfrak{X}$. We say that a Weil descent datum Φ over $\mathscr{O}_E$ is *effective* if there exists a formal scheme $\mathfrak{X}_0$ over $\mathscr{O}_E$ such that Φ is isomorphic to the one naturally associated to $\mathfrak{X}_0$. We similarly define effective Weil descent data over E for adic spaces $\mathscr{X}$ over K_0.

Let $\mathfrak{X}$ be a formal scheme locally formally of finite type over $\mathrm{Spf}\,W$, equipped with a Weil descent datum Φ over $\mathscr{O}_E$. Then on the adic space generic fibre $\mathscr{X} := \mathfrak{X}^{\mathrm{rig}}$ we obtain a Weil descent datum $\Phi^{\mathrm{rig}} : \mathscr{X} \xrightarrow{\sim} \mathscr{X}^{\tau}$ induced by Φ. If Φ is effective, then so is Φ^{rig}.

We define a Weil descent datum over $\mathscr{O}_E$ on $\mathtt{RZ}_{G,b}$ by restricting the natural Weil descent datum on $\mathtt{RZ}_{\mathbb{X}}$, which we now recall. For $R \in \mathrm{Nilp}_W$ with the structure morphism $f : W \to R$, we define $R^{\tau} \in \mathrm{Nilp}_W$ to be R as a ring with structure morphism $f \circ \tau$. Then we have $\mathtt{RZ}^{\tau}_{G,b}(R) = \mathtt{RZ}_{G,b}(R^{\tau})$. The following definition is taken from [39, §3.48].

Definition 7.3.5. For any $(X, \iota) \in \mathtt{RZ}_{\mathbb{X}}(R)$, we define $(X^{\Phi}, \iota^{\Phi}) \in \mathtt{RZ}_{\mathbb{X}}(R^{\tau})$, where X^{Φ} is X viewed as a p-divisible group over R^{τ}, and ι^{Φ} is defined as follows:

$$\iota^{\Phi} : \mathbb{X}_{R^{\tau}/p} = (\tau^*\mathbb{X})_{R/p} \overset{\mathrm{Frob}^{-d}}{\dashrightarrow} \mathbb{X}_{R/p} \overset{\iota}{\dashrightarrow} X_{R/p} = X^{\Phi}_{R/p},$$

where $\mathrm{Frob}^d : \mathbb{X} \to \tau^*\mathbb{X}$ is the relative q-Frobenius (with $q = p^d$). This defines a Weil descent datum $\Phi : \mathtt{RZ}_{\mathbb{X}} \xrightarrow{\sim} \mathtt{RZ}^{\tau}_{\mathbb{X}}$ over $\mathscr{O}_E$.

Note that for $x = (X_x, \iota_x) \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$, we have $x^{\Phi} := (X^{\phi}_x, \iota^{\Phi}_x) \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}^{\tau}_p)$; indeed, Definition 4.6(3) is satisfied for $(X^{\Phi}_x, \iota^{\Phi}_x)$ by Remark 7.3.2. Then it is clear from the construction that for $x \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$, the morphism $\Phi : (\mathtt{RZ}_{\mathbb{X}})^{\wedge}_{x} \xrightarrow{\sim} (\mathtt{RZ}^{\tau}_{\mathbb{X}})^{\wedge}_{x^{\Phi}}$ induces $\Phi : (\mathtt{RZ}_{G,b})^{\wedge}_{x} \xrightarrow{\sim} (\mathtt{RZ}^{\tau}_{G,b})^{\wedge}_{x^{\Phi}}$. Therefore we have $(X^{\Phi}, \iota^{\Phi}) \in \mathtt{RZ}_{G,b}(R^{\tau})$ for any $R \in \mathrm{Nilp}_W$ by definition of $\mathtt{RZ}_{G,b}$ (*cf.* Definition 5.1), so we get a Weil descent datum $\Phi : \mathtt{RZ}_{G,b} \xrightarrow{\sim} \mathtt{RZ}^{\tau}_{G,b}$ over $\mathscr{O}_E$ defined by sending $(X, \iota) \in \mathtt{RZ}_{G,b}(R)$ to $(X^{\Phi}, \iota^{\Phi}) \in \mathtt{RZ}_{G,b}(R^{\tau})$.

Although the Weil descent datum for $\mathtt{RZ}_{G,b}$ is *not* effective, it induces a natural action of the Weil group W_E on the ℓ-adic cohomology of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$. Alternatively, one can "complete" the components of $\mathtt{RZ}_{G,b}$ so that the Weil descent datum would become effective (*cf.* [39, Theorem 3.49]).

The Weil descent datum commutes with the natural action of $J_b(\mathbb{Q}_p)$, as the relative q-Frobenius $\mathrm{Frob}^d : \mathbb{X} \to \tau^*\mathbb{X}$ commutes with any quasi-isogenies. In particular, we have a $W_E \times J_b(\mathbb{Q}_p)$-action on the ℓ-adic cohomology of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$.

7.4. **Étale Tate tensors and rigid analytic tower.** For any open compact subgroup $\mathsf{K} \subset G(\mathbb{Z}_p)$, we will construct a finite étale cover $\mathtt{RZ}^{\mathsf{K}}_{G,b}$ of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$ that naturally fits into a $G(\mathbb{Q}_p)$-equivariant tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$ with Galois group $G(\mathbb{Z}_p)$.

For any geometric point $\bar{x}$ of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$ we let $\pi_1^{\text{fét}}(\mathtt{RZ}^{\mathrm{rig}}_{G,b}, \bar{x})$ denote the algebraic fundamental group of the connected component of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$ containing $\bar{x}$.

Let $X_{G,b}$ denote the universal p-divisible group over $\mathtt{RZ}_{G,b}$. By Theorem 7.1.6, we have a morphism of lisse $\mathbb{Z}_p$-sheaves

$$t_{\alpha,\text{ét}} : \mathbf{1} \to T(X_{G,b})^{\otimes}$$

corresponding to each t_{α}.

Proposition 7.4.1. *Let $\bar{x}$ be a geometric point of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$ supported at a classical point x. Then the following $\mathbb{Z}_p$-scheme*

$$P_{\text{ét},\bar{x}} := \underline{\mathrm{isom}}_{\mathbb{Z}_p}\big([\Lambda, (s_{\alpha})], [T(X_{G,b})_{\bar{x}}, (t_{\alpha,\text{ét},\bar{x}})]\big)$$

is a trivial G-torsor. (Here, we view Λ and $T(X_{G,b})_{\bar{x}}$ as vector bundles over $\mathrm{Spec}\,\mathbb{Z}_p$.)

Proof. Note that any G-torsor over $\mathbb{Z}_p$ is trivial; indeed, since $\mathbb{Z}_p$ is a henselian local ring, a G-torsor over $\mathbb{Z}_p$ is trivial if its special fibre is trivial. But any G-torsor over a finite field is trivial if G is connected and reductive.

It remains to show that $P_{\text{ét},\bar{x}}$ is a G-torsor. Let K be the residue field at x, and κ the residue field of K. Then, it suffices to show that $P_{\text{ét},\bar{x},W}$ is a (trivial) G-torsor over W. By [29, Proposition 1.3.4], we have a W-linear isomorphism

$$W\otimes_{\mathbb{Z}_p} T(X_{G,b})_{\bar{x}} \cong \mathbb{D}(X_{x_0})(W)^*$$

matching $(1\otimes t_{\alpha,\text{ét},\bar{x}})$ and $(t_{\alpha,x_0}(W))$, where X_{x_0} is the pull-back of $X_{G,b}$ by $x_0 : \operatorname{Spec}\kappa \to \operatorname{Spf}\mathscr{O}_K \xrightarrow{x} \mathtt{RZ}_{G,b}$. Therefore, $P_{\text{ét},\bar{x},W}$ is isomorphic to the G-torsor P_W defined using $(\mathbb{D}(X_{x_0})(W),(t_{\alpha,x_0}(W)))$. □

Let $\mathsf{K}^{(0)} := G(\mathbb{Z}_p)$ and $\mathsf{K}^{(i)} := \ker(G(\mathbb{Z}_p)\to G(\mathbb{Z}/p^i))$ for any $i>0$.

Definition 7.4.2. We set $\mathtt{RZ}^{\mathsf{K}^{(0)}}_{G,b} := \mathtt{RZ}^{\text{rig}}_{G,b}$. For any $i>0$ we define the following rigid analytic covering of $\mathtt{RZ}^{\text{rig}}_{G,b}$:

$$\mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b} = \underline{\text{isom}}_{\mathtt{RZ}^{\text{rig}}_{G,b}}\big([\Lambda/p^i\Lambda,(s_\alpha)],[X_{G,b}[p^i]^{\text{rig}},(t_{\alpha,\text{ét}})],\big);$$

i.e., for an analytic space (or adic space) $\mathscr{X}$ over K_0, its $\mathscr{X}$-point u classifies isomorphisms of $\mathbb{Z}/p^i$-local systems matching tensors after pulling back to $\mathscr{X}$. Here, we use the identification $X_{G,b}[p^i]^{\text{rig}} \cong T(X_{G,b})/(p^i)$ to view the mod p^i reduction of $(t_{\alpha,\text{ét}})$ as tensors of $X_{G,b}[p^i]^{\text{rig}}$. Since $\mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b}$ is an open and closed subspace of $\underline{\text{isom}}_{\mathtt{RZ}^{\text{rig}}_{G,b}}(\Lambda/p^i\Lambda, X_{G,b}[p^i]^{\text{rig}})$, one can see that $\mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b}$ is a finite étale Galois cover of $\mathtt{RZ}^{\text{rig}}_{G,b}$. This definition is compatible with the definition of level structure when $G=\operatorname{GL}(\Lambda)$.

We let the finite group $G(\mathbb{Z}/p^i) = \mathsf{K}^{(0)}/\mathsf{K}^{(i)}$ act on the right on $\mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b}$ as follows: an element $g\in G(\mathbb{Z}/p^i)$ acts as $\varsigma\mapsto g^{-1}\circ\varsigma$ on sections of $\mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b}$. This makes $\mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b}$ an étale Galois cover of $\mathtt{RZ}^{\text{rig}}_{G,b}$ with Galois group $G(\mathbb{Z}/p^i)$. When $G=\operatorname{GL}(\Lambda)$, this action is compatible with the natural action as defined in [39, §5.34].

For any open subgroup $\mathsf{K}\subset\mathsf{K}^{(0)}$ which contains $\mathsf{K}^{(i)}$ for some $i\geqslant 0$, we set

$$\mathtt{RZ}^{\mathsf{K}}_{G,b} = \mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b}/(\mathsf{K}/\mathsf{K}^{(i)}).$$

This definition is independent of the choice of $i\gg 0$. The $J_b(\mathbb{Q}_p)$-action and the Weil descent datum over E on $\mathtt{RZ}^{\text{rig}}_{G,b}$ pull back to $\mathtt{RZ}^{\mathsf{K}}_{G,b}$.

Let us now define the "right $G(\mathbb{Q}_p)$-action" of the rigid analytic tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$ (i.e., Hecke correspondences). We follow [39, §5.34] and [18, §2.3.9.3]. Let $g\in G(\mathbb{Q}_p)$, and choose $\mathsf{K}\subset G(\mathbb{Z}_p)$ so that $g^{-1}\mathsf{K}g\subset G(\mathbb{Z}_p)$. For a fixed g, the assumption on K can be arranged by replacing K by some finite index open subgroup; indeed, for an open compact subgroup $\mathsf{K}_0\subset G(\mathbb{Z}_p)$, $\mathsf{K}:=\mathsf{K}_0\cap g\mathsf{K}_0g^{-1}$ satisfies this assumption. By a (right) $G(\mathbb{Q}_p)$-action on the tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$, we mean a collection of isomorphisms

$$[g] : \mathtt{RZ}^{\mathsf{K}}_{G,b} \xrightarrow{\sim} \mathtt{RZ}^{g^{-1}\mathsf{K}g}_{G,b},$$

for any $g\in G(\mathbb{Q}_p)$ and $\mathsf{K}\subset G(\mathbb{Z}_p)$ with $g^{-1}\mathsf{K}g\subset G(\mathbb{Z}_p)$, which commutes with the map $\mathtt{RZ}^{\mathsf{K}'}_{G,b}\twoheadrightarrow \mathtt{RZ}^{\mathsf{K}}_{G,b}$ for $\mathsf{K}'\subset\mathsf{K}$, and we have $[g']\circ[g]=[gg']$ for any $g,g'\in G(\mathbb{Q}_p)$ whenever it makes sense.

Let us first describe the map $[g]$ on K-points, where K is a finite extension of K_0. Recall that $\mathtt{RZ}^{\text{rig}}_{G,b}(K) = \operatorname{Hom}(\operatorname{Spf}\mathscr{O}_K,\mathtt{RZ}_{G,b})$, so a point $u\in\mathtt{RZ}^{\mathsf{K}}_{G,b}(K)$ can be interpreted as a p-divisible group $X_u := u^*X_{G,b}$ over $\mathscr{O}_K$, a quasi-isogeny $\iota : \mathbb{X} \dashrightarrow X_{u,\overline{\mathbb{F}}_p}$, and a $\mathrm{Gal}(\overline{K}/K)$-stable right coset $\tilde{u}\mathsf{K}$ of isomorphisms

$$\tilde{u} : \Lambda \xrightarrow{\sim} T(X_u)_{\bar{\eta}},$$

where $\bar{\eta} : \mathrm{Spa}(\widehat{\overline{K}}, \mathscr{O}_{\widehat{\overline{K}}}) \to \mathrm{Spa}(K, \mathscr{O}_K)$ is a geometric point. Since $\tilde{u}\mathsf{K}$ is $\mathrm{Gal}(\overline{K}/K)$-stable, the $\mathrm{Gal}(\overline{K}/K)$-action on Λ via $\tilde{u}$ has its image in K.

Since we assumed that $g^{-1}\mathsf{K}g \subset G(\mathbb{Z}_p)$, it follows that $g\Lambda \subset \Lambda[\frac{1}{p}]$ is stable under the action of K, so $\tilde{u}(g\Lambda) \subset V(X_u)_{\bar{\eta}}$ is $\mathrm{Gal}(\overline{K}/K)$-stable. This means that we can find a p-divisible group $X_{u\cdot g}$ over $\mathscr{O}_K$ with quasi-isogeny $\jmath_g : X_u \dashrightarrow X_{u\cdot g}$ such that $g\Lambda$ is the image of the following map

(7.4.3) $$T(X_{u\cdot g})_{\bar{\eta}} \hookrightarrow V(X_{u\cdot g})_{\bar{\eta}} \xleftarrow[\jmath_g^*]{\sim} V(X_u)_{\bar{\eta}} \xleftarrow[\tilde{u}]{\sim} \Lambda[\tfrac{1}{p}].$$

Indeed, for n so that $p^n\Lambda \subset g\Lambda$, $g\Lambda/p^n\Lambda$ corresponds to the geometric generic fibre of some finite flat $\mathscr{O}_K$-subgroup $\mathfrak{G}$ of $X_u[p^n]$. We set $X_{u\cdot g} := X_u/\mathfrak{G}$ and

(7.4.4) $$\jmath_g : X_u \overset{p^{-n}}{\dashrightarrow} X_u \twoheadrightarrow X_u/\mathfrak{G} =: X_{u\cdot g}.$$

Then the pair $(X_{u\cdot g}, \jmath_g)$ satisfies the desired property (7.4.3). Now we obtain the following K-valued point of $\mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$:

(7.4.5) $$(X_{u\cdot g}, \jmath_{g,\overline{\mathbb{F}}_p} \circ \iota) \in \mathrm{Hom}_W(\mathrm{Spf}\,\mathscr{O}_K, \mathtt{RZ}_{\mathbb{X}}) \cong \mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}(K).$$

Lemma 7.4.6. *In the above setting, let us write* $(X,\iota) := (X_u,\iota)$ *and* $(X',\iota') := (X_{u\cdot g}, \jmath_{g,\overline{\mathbb{F}}_p} \circ \iota)$*;* cf. *(7.4.5). Then* (X',ι') *corresponds to a* $\mathrm{Spf}\,\mathscr{O}_K$*-point of* $\mathtt{RZ}_{G,b}$.

Proof. By construction, we have étale Tate tensors $(t'_{\alpha,\text{ét}}) \subset T(X')^{\otimes}$ and an isomorphism $\Lambda \xrightarrow{\sim} T(X')_{\bar{\eta}}$ matching $(t'_{\alpha,\text{ét}})$ and (s_α). Now choose $W[u] \twoheadrightarrow \mathscr{O}_K$ and let S be its p-adically completed PD hull. Then by Kisin theory, one associate $(t'_\alpha(S)) \subset \mathbb{D}(X')(S)^{\otimes}$ from $(t'_{\alpha,\text{ét}})$, such that its pointwise stabiliser is isomorphic to G_S and the image $(t_\alpha(\mathscr{O}_K)) \subset \mathbb{D}(X')(\mathscr{O}_K)$ lies in the 0th filtration with respect to the Hodge filtration; indeed, $(t'_\alpha(S))$ can be constructed using Theorems 1.2.1 and 1.4.2 in [29], and the assertion on the pointwise stabiliser follows from [29, Proposition 1.3.4].

From this, we have $x'_0 := (X'_{\overline{\mathbb{F}}_p}, \iota'_{\overline{\mathbb{F}}_p}) \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$, since the image of $(t'_\alpha(S))$ in $\mathbb{D}(X'_{\overline{\mathbb{F}}_p})(W)^{\otimes} = \mathbb{D}(X')(S) \otimes_S W$ defines Tate tensors. Then [29, Proposition 1.5.8] shows that the map $\mathrm{Spf}\,\mathscr{O}_K \to (\mathtt{RZ}_{\mathbb{X}})^{\widehat{}}_{x'_0}$, defined by (X',ι'), factors through $(\mathtt{RZ}_{G,b})^{\widehat{}}_{x'_0}$. $\square$

Now we can lift $(X_{u\cdot g}, \jmath_{g,\overline{\mathbb{F}}_p} \circ \iota) \in \mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}(K)$ to $\mathtt{RZ}_{G,b}^{g^{-1}\mathsf{K}g}(K)$ by adding the level structure corresponding to the right $g^{-1}\mathsf{K}g$-coset of the isomorphism:

(7.4.7) $$\Lambda \xrightarrow[g]{\sim} g\Lambda \xrightarrow[\text{(7.4.3)}]{\sim} T(X_{u\cdot g})_{\bar{\eta}}.$$

By construction, the associated right $g^{-1}\mathsf{K}g$-coset is $\mathrm{Gal}(\overline{K}/K)$-stable, so we obtain a map $[g] : \mathtt{RZ}_{G,b}^{\mathsf{K}}(K) \to \mathtt{RZ}_{G,b}^{g^{-1}\mathsf{K}g}(K)$. If $g \in G(\mathbb{Z}_p)$ then this action clearly recovers the natural "Galois action" of the covering.

The construction (7.4.5) and (7.4.7) can be generalised to $\mathscr{X}$-valued points in a functorial way for topologically finite-type K_0-analytic space (or adic space) $\mathscr{X}$. Then the p-divisible group X_u is defined over some formal model $\mathfrak{X}$ of $\mathscr{X}$. By replacing $\mathfrak{X}$ with some admissible blow up if necessary, we can find a finite flat group scheme $\mathfrak{G}$ of X_u whose rigid analytic generic fibre gives the local system corresponding to $g\Lambda/p^n\Lambda$; *cf.* [6]. Now by rigid analytic Yoneda lemma [11, Lemma 7.1.5], we obtain a morphism $[g] : \mathtt{RZ}_{G,b}^{\mathsf{K}} \to \mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$, which factors through

$\mathtt{RZ}^{\mathrm{rig}}_{G,b}$ by considering the image of classical points (*cf.* Lemma 7.4.6). And by considering the suitable generalisation of (7.4.7), we obtain a map $[g] : \mathtt{RZ}^{\mathsf{K}}_{G,b} \to \mathtt{RZ}^{g^{-1}\mathsf{K}g}_{G,b}$.

Assume furthermore that $g'^{-1}\mathsf{K}g' \subset G(\mathbb{Z}_p)$ for some $g' \in G(\mathbb{Q}_p)$. (This can be arranged by shrinking K further if necessary.) Then we can show that the map $[g'] : \mathtt{RZ}^{\mathsf{K}}_{G,b} \to \mathtt{RZ}^{g'^{-1}\mathsf{K}g'}_{G,b}$ is equal to the composition

$$\mathtt{RZ}^{\mathsf{K}}_{G,b} \xrightarrow{[g]} \mathtt{RZ}^{g^{-1}\mathsf{K}g}_{G,b} \xrightarrow{[g'g^{-1}]} \mathtt{RZ}^{g'^{-1}\mathsf{K}g'}_{G,b}.$$

By taking g' to be the identity, it follows that the map $[g] : \mathtt{RZ}^{\mathsf{K}}_{G,b} \to \mathtt{RZ}^{g^{-1}\mathsf{K}g}_{G,b}$ is an isomorphism.

Now the following proposition is immediate from the construction:

Proposition 7.4.8. *The assignment $g \mapsto ([g] : \mathtt{RZ}^{\mathsf{K}}_{G,b} \to \mathtt{RZ}^{g^{-1}\mathsf{K}g}_{G,b})$ defines a right $G(\mathbb{Q}_p)$-action on the tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$ extending the Galois action of $G(\mathbb{Z}_p)$, which commutes with the natural $J_b(\mathbb{Q}_p)$-action and the Weil descent datum over E.*

This shows that on the "ℓ-adic" cohomology of the tower $\{\mathtt{RZ}^{\mathsf{K}}_{G,b}\}$, we have a natural action of $W_E \times J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p)$.

7.5. **Period morphisms.** Set $\mathscr{E}_{G,b} := \mathbb{D}(X_{G,b})_{\mathtt{RZ}_{G,b}}$, which is a vector bundle on $\mathtt{RZ}_{G,b}$ equipped with a filtration $\mathrm{Fil}^1_{X_{G,b}}$. From the universal Tate tensors $t_\alpha : \mathbf{1} \to \mathscr{E}^{\otimes}_{G,b}$, we get morphisms of rigid analytic F-isocrystals $t^{\mathrm{rig}}_\alpha : \mathbf{1} \to (\mathscr{E}^{\mathrm{rig}}_{G,b})^{\otimes}$. Note that the (universal) quasi-isogeny $\iota_{\mathrm{red}} : (\mathbb{X})_{(\mathtt{RZ}_{G,b})_{\mathrm{red}}} \dashrightarrow (X_{G,b})_{(\mathtt{RZ}_{G,b})_{\mathrm{red}}}$ induces an isomorphism of vector bundles on $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$

$$\mathscr{E}^{\mathrm{rig}}_{G,b} \xrightarrow{\sim} \mathcal{O}_{\mathtt{RZ}^{\mathrm{rig}}_{G,b}} \otimes_{\mathbb{Z}_p} \Lambda$$

which matches $(t^{\mathrm{rig}}_\alpha)$ with the maps $(1 \mapsto 1 \otimes s_\alpha)$; indeed, the rigid analytic F-isocrystal $\mathscr{E}^{\mathrm{rig}}_{G,b}$ only depends on $\mathbb{D}((X_{G,b})_{(\mathtt{RZ}_{G,b})_{\mathrm{red}}})[\frac{1}{p}]$, as explained in [11, §5.3].

Let $\mathtt{Fl}_{G,\{\mu\}}$ denote the projective rigid analytic variety over K_0 obtained from the analytification of $\mathtt{Fl}^{K_0\otimes\Lambda^*,(1\otimes s_\alpha)}_{G_{K_0},\{\mu\}}$ (*cf.* §2.2). It follows that $(\mathrm{Fil}^1_{X_{G,b}})^{\mathrm{rig}} \subset \mathscr{E}^{\mathrm{rig}}_{G,b}$ defines a natural map

$$\pi : \mathtt{RZ}^{\mathrm{rig}}_{G,b} \to \mathtt{Fl}_{G,\{\mu\}}, \tag{7.5.1}$$

which we call the *period map*. By letting $J_b(\mathbb{Q}_p)$ act on $\mathtt{Fl}_{G,\{\mu\}}$ via embedding $J_b(\mathbb{Q}_p) \subset G(K_0)$ the period map π is $J_b(\mathbb{Q}_p)$-equivariant. In order to have compatibility with Weil descent data, one has to modify the target of the period map as in the case of (P)EL Rapoport-Zink spaces. Indeed, for $\Delta := \mathrm{Hom}_{\mathbb{Z}}(X^*(G)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}, \mathbb{Z})$, we obtain the map $\aleph : \mathtt{RZ}_{G,b} \to \Delta$ from [8, Lemma 2.2.9], generalising the (P)EL case [39, §3.52]; indeed, the aforementioned result gives a functorial map $\mathrm{Hom}_W(\mathrm{Spf}\, A, \mathtt{RZ}_{G,b}) \to \pi_1(G)$ for formally smooth formally finitely generated W-algebra A (noting that for a maximal torus $T \subset G_W$, the natural projection $X_*(T) \twoheadrightarrow \pi_1(G)$ defines a map $\pi_1(G) \to \Delta$ via evaluation).[36] We then define a Weil descent datum on $\mathtt{Fl}_{G,\{\mu\}} \times \Delta$ using the same formula as in [39, §5.43]. One can show that $(\pi, \aleph)$ is compatible with the Weil descent datum, generalising the (P)EL case [39, §5.46].

Proposition 7.5.2. *The period map π is étale in the sense that there exist open affinoid coverings $\{U_i\}$ of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$ and $\{V_i\}$ of $\mathtt{Fl}_{G,\{\mu\}}$ such that for each i we have $\pi(U_i) \subset V_i$*

[36]One can explicitly describe $\aleph$ on $\overline{\mathbb{F}}_p$-point as follows: it is the map that sends $gG(W) \in X^G(b)$ to the homomorphism $[\chi \mapsto \mathrm{ord}_p\, \chi(g)]$ where $\chi : G_{\mathbb{Q}_p} \to \mathbb{G}_m$ is a homomorphism over $\mathbb{Q}_p$ and g is any representative of $gG(W)$.

*and $\pi : U_i \to V_i$ is given by an étale ring map of the corresponding affinoid algebras (*cf. [39, §5.9]*).*

Proof of Proposition 7.5.2. The proof is almost the same as the proof of [39, Proposition 5.15], using Corollary 5.2.2 in the place of the Grothendieck-Messing deformation theory.

As in the case of schemes of finite type over a field, étaleness can be checked via infinitesimal lifting property for nilpotent thickenings supported at classical points by [39, Proposition 5.10]. To unwind this criterion, let $A' \twoheadrightarrow A$ be a square-zero thickenings of local K_0-algebras which are finite dimensional over K_0. Let $A^\circ \subset A$ and $A'^\circ \subset A'$ respectively denote the subrings of power-bounded elements [42, Definition 2.1.1]. Then we claim that the dotted arrow in the commutative diagram below can be uniquely filled:

$$\begin{tikzcd} \mathrm{Spa}(A, A^\circ) \arrow[r] \arrow[d, hook] & \mathtt{RZ}^{\mathrm{rig}}_{G,b} \arrow[d, "\pi"] \\ \mathrm{Spa}(A', A'^\circ) \arrow[r] \arrow[ur, dotted, "\exists!"] & \mathtt{Fl}_{G,\{\mu\}} \end{tikzcd}$$

Let us translate this diagram in more concrete terms. Let $\mathrm{Fil}^1_{A'} \subset A' \otimes_{\mathbb{Z}_p} \Lambda$ be a $\{\mu\}$-filtration such that for a finite flat W-subalgebra $R_0 \subset A$ there exists a map $f : \mathrm{Spf}\, R_0 \to \mathtt{RZ}_{G,b}$ such that the isomorphism

$$A \otimes_{R_0} \mathbb{D}(X)(R_0) \xrightarrow[\mathbb{D}(\iota)_A]{\sim} A \otimes_{\mathbb{Z}_p} \Lambda, \tag{7.5.3}$$

takes the Hodge filtration $A \otimes_{R_0} \mathrm{Fil}^1_X$ to $A \otimes_{A'} \mathrm{Fil}^1_{A'}$, where (X, ι) is the pull-back of the universal object $(X_{G,b}, \iota)$ by f, and $\mathbb{D}(\iota)_A$ is the isomorphism induced by ι. The existence of the dotted arrow translates as the existence of a finite flat W-subalgebra $R' \subset A'$ and a map $f' : \mathrm{Spf}\, R' \to \mathtt{RZ}_{G,b}$ lifting f in some suitable sense, such that the Hodge filtration $A' \otimes_{R'} \mathrm{Fil}^1_{X'}$ corresponds to $\mathrm{Fil}^1_{A'}$ by the isomorphism $\mathbb{D}(\iota')_{A'}$, where (X', ι') is the pull-back of $(X_{G,b}, \iota)$ by f'. (Note that the uniqueness of the dotted arrow follows from the Grothendieck-Messing deformation theory.)

We choose a finite flat W-subalgebra $R' \subset A'$, and let $R \subset A$ denote the image of R' in A. Assume that R contains R_0. Note that the pull-back of the universal quasi-isogeny induces an isomorphism

$$A' \otimes_{R'} \mathbb{D}(X_R)(R') \xrightarrow[\mathbb{D}(\iota)_{A'}]{\sim} A' \otimes_{\mathbb{Z}_p} \Lambda, \tag{7.5.4}$$

where we give the square-zero PD structure on $R' \twoheadrightarrow R$.

By increasing R' if necessary, we may assume that the intersection

$$\mathrm{Fil}^1_{R'} := \mathrm{Fil}^1_{A'} \cap \mathbb{D}(X_R)(R')$$

is a $\{\mu\}$-filtration with respect to $(t_\alpha(R'))$, where (t_α) is the pull-back of the universal Tate tensors over $\mathtt{RZ}_{G,b}$. To see this, note that A'° is the preimage of the valuation ring of the residue field of A'. Then by valuative criterion for properness applied to the projective R'-scheme $\mathtt{Fl}^{\mathbb{D}(X_R)(R'),(t_\alpha(R'))}_{G,\{\mu\}}$, it follows that the A'-point corresponding to $\mathrm{Fil}^1_{A'}$ uniquely extends to an A'°-point, so this A'°-point has to be defined over some finite R'-subalgebra $R'' \subset A'^\circ$. We rename R'' to be R'. Now, the existence of (X', ι') lifting (X_R, ι) follows from Corollary 5.2.2. □

Remark 7.5.5. One defines étale maps for adic spaces to be maps locally of finite presentation satisfying the usual infinitesimal lifting property for formal étale-ness using any affinoid (K_0, W)-algebras as test objects; *cf.* [23, Definition 1.6.5]. By [23, Example 1.6.6(ii)] and [39, Proposition 5.10], this definition coincides with the definition of étale morphisms given in Proposition 7.5.2.

7.6. **Infinite-level Rapoport-Zink spaces.** In this section, all the rigid analytic spaces are regarded as adic spaces in the sense of [42, Definition 2.1.5].

Since we will not directly work with the definitions of (pre)perfectoid spaces, we refer to [42, §2.1] for basic definitions. Roughly speaking, a *preperfectoid space* over $\mathrm{Spa}(K_0, W)$ is an adic space over $\mathrm{Spa}(K_0, W)$ which becomes a perfectoid space after base change over any perfectoid extension (K, K^+) of (K_0, W) and take the "p-adic completion"; *cf.* [42, Definition 2.3.9]. In particular, preperfectoid spaces may be non-reduced as explained in [42, Remark 2.3.5].

Scholze and Weinstein [42, Theorem D] constructed a preperfectoid space $\mathtt{RZ}_{\mathbb{X}}^{\infty}$ over $\mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$, which can be viewed as the "infinite-level" Rapoport-Zink space. (In [42] $\mathtt{RZ}_{\mathbb{X}}^{\infty}$ is denoted as $\mathcal{M}_\infty$.) By definition, $\mathtt{RZ}_{\mathbb{X}}^{\infty}$ parametrises $\mathbb{Z}_p$-equivariant morphism over $\mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$

$$\Lambda \to (\varprojlim X_{\mathtt{RZ}_{\mathbb{X}}}[p^n])^{\mathrm{ad}}_{(K_0,W)}$$

which induces an isomorphism $\Lambda \xrightarrow{\sim} (\varprojlim X_{\mathtt{RZ}_{\mathbb{X}}}[p^n])^{\mathrm{ad}}_{(K_0,W)}(K, K^+)$ of $\mathbb{Z}_p$-modules on the fibres at each point $\mathrm{Spa}(K, K^+) \to \mathtt{RZ}_{\mathbb{X}}^{\mathrm{rig}}$. Here, the target is the generalised adic space over (K_0, W) associated to a formal scheme over W admitting a finitely generated ideal of definition [42, §2.2], which extends the rigid analytic generic fibre construction.

For any open compact subgroup $\mathsf{K}' \subset \mathrm{GL}_{\mathbb{Z}_p}(\Lambda)(\mathbb{Z}_p)$, there exists a natural projection $\mathtt{RZ}_{\mathbb{X}}^{\infty} \to \mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}$ respecting the tower. It may not be known whether $\mathtt{RZ}_{\mathbb{X}}^{\infty}$ represents the projective limit of $\mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}$ as sheaves (or even, whether one should expect this)[37]. In other words, although any map $\mathrm{Spa}(A, A^+) \to \mathtt{RZ}_{\mathbb{X}}^{\infty}$ gives rise to an isomorphism $\Lambda \xrightarrow{\sim} T(X_{\mathtt{RZ}_{\mathbb{X}}})_{(A,A^+)}$ of lisse $\mathbb{Z}_p$-sheaves on $\mathrm{Spa}(A, A^+)$, obtained from the natural maps $\mathtt{RZ}_{\mathbb{X}}^{\infty} \twoheadrightarrow \mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}$, it is not known whether the "converse" holds.

Instead of working with a problematic notion of projective limit, Scholze and Weinstein [42, Theorem 6.3.4] showed that a weaker notion of equivalence $\mathtt{RZ}_{\mathbb{X}}^{\infty} \sim \varprojlim \mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}$ holds, where $\sim$ is defined in [42, Definition 2.4.1]. To simplify the description of the equivalence, note that any projection $\mathtt{RZ}_{\mathbb{X}}^{\infty} \to \mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}$ has the property that there exists an affinoid open cover $\{\mathrm{Spa}(A_\xi, A_\xi^+)\}$ of $\mathtt{RZ}_{\mathbb{X}}^{\infty}$ whose image in $\mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}$ is an affinoid open cover $\{\mathrm{Spa}(A_{\mathsf{K}',\xi}, A_{\mathsf{K}',\xi}^+)\}$ for each K'. (This follows from the cartesian square in the first paragraph of the proof of [42, Theorem 6.3.4].) By the equivalence $\mathtt{RZ}_{\mathbb{X}}^{\infty} \sim \varprojlim \mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}$ we mean that:

- The natural map on the topological space $|\mathtt{RZ}_{\mathbb{X}}^{\infty}| \to \varprojlim |\mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'}|$ is a homeomorphism.
- Using the notation introduced above the image of $\varinjlim A_{\mathsf{K}',\xi}$ in A_ξ is dense for each ξ.

Let $\mathsf{K}^{(i)} \subset G(\mathbb{Z}_p)$ and $\mathsf{K}'^{(i)} \subset \mathrm{GL}_{\mathbb{Z}_p}(\Lambda)(\mathbb{Z}_p)$ respectively denote the kernel of reduction modulo p^i. We define $\mathtt{RZ}_{G,b}^{\infty}$ to be the "projective limit" of $\mathtt{RZ}_{\mathbb{X}}^{\infty} \times_{\mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'(i)}} \mathtt{RZ}_{G,b}^{\mathsf{K}^{(i)}}$; more concretely, we let $\mathtt{RZ}_{G,b}^{\infty}$ be the closed adic subspace of $\mathtt{RZ}_{\infty}$ cut out by the equations defining $\mathtt{RZ}_{\mathbb{X}}^{\infty} \times_{\mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'(i)}} \mathtt{RZ}_{G,b}^{\mathsf{K}^{(i)}}$ for all i.

Note that the natural projection $\mathtt{RZ}_{\mathbb{X}}^{\infty} \twoheadrightarrow \mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'^{(i)}}$ restricts to $\mathtt{RZ}_{G,b}^{\infty} \twoheadrightarrow \mathtt{RZ}_{G,b}^{\mathsf{K}^{(i)}}$, which factors as

$$\mathtt{RZ}_{G,b}^{\infty} \hookrightarrow \mathtt{RZ}_{\mathbb{X}}^{\infty} \times_{\mathtt{RZ}_{\mathbb{X}}^{\mathsf{K}'(i)}} \mathtt{RZ}_{G,b}^{\mathsf{K}^{(i)}} \twoheadrightarrow \mathtt{RZ}_{G,b}^{\mathsf{K}^{(i)}}.$$

Therefore, a morphism $\mathrm{Spa}(A, A^+) \to \mathtt{RZ}_{G,b}^{\infty}$ gives rise to an isomorphism $\Lambda \xrightarrow{\sim} T(X_{G,b})_{(A,A^+)}$ of lisse $\mathbb{Z}_p$-sheaves on $\mathrm{Spa}(A, A^+)$, which matches (s_α) and $(t_{\alpha,\text{ét}})$, where $X_{G,b}$ is the universal p-divisible group over $\mathtt{RZ}_{G,b}$.

[37] Indeed, taking projective limits of adic spaces is problematic as explained in [42, §2.4].

Proposition 7.6.1. *The adic space* $\mathtt{RZ}^{\infty}_{G,b}$ *is a preperfectoid, and we have* $\mathtt{RZ}^{\infty}_{G,b} \sim \varprojlim_{\mathsf{K}} \mathtt{RZ}^{\mathsf{K}}_{G,b}$.

Proof. Since any closed subspace of a preperfectoid space is a preperfectoid space ([42, Proposition 2.3.11]) it remains to show $\mathtt{RZ}^{\infty}_{G,b} \sim \varprojlim_{\mathsf{K}} \mathtt{RZ}^{\mathsf{K}}_{G,b}$.

On the underlying topological space we clearly have a natural homeomorphism.

$$|\mathtt{RZ}^{\infty}_{G,b}| \xrightarrow{\sim} \varprojlim_{\mathsf{K}} |\mathtt{RZ}^{\mathsf{K}}_{G,b}|.$$

To verify the other condition, let $\{\mathrm{Spa}(A_\xi, A^+_\xi)\}$ be an affinoid open covering of $\mathtt{RZ}^{\infty}_{\mathbb{X}}$ whose image in $\mathtt{RZ}^{\mathsf{K}'(i)}_{\mathbb{X}}$ is an affinoid open cover $\{\mathrm{Spa}(A_{i,\xi}, A^+_{i,\xi})\}$. Let $\mathrm{Spa}(B_\xi, B^+_\xi) \subset \mathtt{RZ}^{\infty}_{G,b}$ be the pull-back of $\mathrm{Spa}(A_\xi, A^+_\xi)$, and we similarly define $\mathrm{Spa}(B_{i,\xi}, B^+_{i,\xi}) \subset \mathtt{RZ}^{\mathsf{K}^{(i)}}_{G,b}$. Then we have the following commutative diagram

$$\begin{array}{ccc} \varinjlim A_{i,\xi} & \longrightarrow & A_\xi \\ \downarrow & & \twoheaddownarrow \\ \varinjlim B_{i,\xi} & \longrightarrow & B_\xi \end{array},$$

where the right vertical arrow is a quotient map and the upper horizontal arrow has a dense image. It thus follows that the lower horizontal arrow also has a dense image. This shows $\mathtt{RZ}^{\infty}_{G,b} \sim \varprojlim_{\mathsf{K}} \mathtt{RZ}^{\mathsf{K}}_{G,b}$. ☐

Remark 7.6.2. As remarked in the introduction of [42], it should be possible to obtain an "infinite-level Rapoport-Zink space" $\mathtt{RZ}^{\infty}_{G,b}$ for (G, b) (or at least, an adic space equivalent to $\mathtt{RZ}^{\infty}_{G,b}$) directly without going through finite levels, and obtain an "explicit description" of $\mathtt{RZ}^{\infty}_{G,b}$ using the theory of vector bundles on Fargues-Fontaine curves (in the spirit of [42, Theorem D]). Such a construction should work for more general class of "local Shimura data" $(G, [b], \{\mu^{-1}\})$.

8. Digression on crystalline comparison for p-divisible groups

The goal of this section is to prove Theorem 7.1.6, for which we need to recall the basic constructions and crystalline comparison theory for p-divisible groups. We will use the notation and setting as in §7.1. and we additionally assume that $\mathfrak{X} = \mathrm{Spf}\, R$ is a connected formal scheme which is formally smooth and formally of finite type over W, $\widehat{\Omega}_{R/W}$ is free over R, and one can take an R-basis du_i such that $u_i \in R^\times$ for all i. The choice of R is more general than [7] (where various natural properties of (relative) period rings are proved), but one can rather easily deduce the properties of crystalline period rings that are relevant for us.[38]

In this section we allow $p = 2$. Although all the results hold when κ is a perfect field (instead of an algebraically closed field) with little modification in the proofs, we continue to assume that κ is algebraically closed for the notational simplicity.

8.1. Crystalline period rings.

Choose a separable closure $\mathbb{E}$ of $\mathrm{Frac}(R)$, and define $\overline{R}$ to be the union of normal R-subalgebras $R' \subset \mathbb{E}$ such that $R'[\frac{1}{p}]$ is finite étale over $R[\frac{1}{p}]$. Set $\widehat{\overline{R}} := \varprojlim_n \overline{R}/(p^n)$. (When R is a finite extension of W, we have $\overline{R} = \mathscr{O}_{\overline{K}_0}$.) We let $\bar{\eta}$ denote the geometric generic point of any of $\mathrm{Spec}\, R[\frac{1}{p}]$, $\mathrm{Spec}\, \overline{R}[\frac{1}{p}]$, and $\mathrm{Spec}\, \widehat{\overline{R}}[\frac{1}{p}]$.

[38] The properties of $B_{\mathrm{cris}}(R)$ that will be used can be rather easily deduced by the same proof as in [7]. More subtle properties which require refined almost étaleness, such as $R[\frac{1}{p}]$-flatness and the π_1-invariance, will not be used in this paper, although they are obtained in [25, §5] by slightly extending refined almost étaleness and repeating the proof of [7].

Let us briefly discuss the relation between the étale fundamental group of $\operatorname{Spec} R[\frac{1}{p}]$ and the algebraic fundamental group of $(\operatorname{Spf} R)^{\mathrm{rig}}$. For any finite étale $R[\frac{1}{p}]$-algebra A, let R_A be the normalisation of R in A. Since R is excellent[39], R_A is finite over R. By construction, $(\operatorname{Spf} R_A)^{\mathrm{rig}}$ is finite étale over $(\operatorname{Spf} R)^{\mathrm{rig}}$, so we obtain a functor $\operatorname{Spec} A \rightsquigarrow (\operatorname{Spf} R_A)^{\mathrm{rig}}$ from finite étale covers of $\operatorname{Spec} R[\frac{1}{p}]$ to finite étale covers of $(\operatorname{Spf} R)^{\mathrm{rig}}$. This induces a natural map of profinite groups

$$\pi_1^{\text{fét}}((\operatorname{Spf} R)^{\mathrm{rig}}, \bar{x}) \to \pi_1^{\text{ét}}(\operatorname{Spec} R[1/p], \bar{x}), \tag{8.1.1}$$

for any geometric closed point $\bar{x}$ of $\operatorname{Spec} R[1/p]$, which can also be viewed as a "geometric point" of $(\operatorname{Spf} R)^{\mathrm{rig}}$.

Remark 8.1.2. In the case we care about (such as $T(X)_{\bar{x}}$ for a p-divisible group X over R), the action of $\pi_1^{\text{fét}}((\operatorname{Spf} R)^{\mathrm{rig}}, \bar{x})$ factors through $\pi_1^{\text{ét}}(\operatorname{Spec} R[1/p], \bar{x})$.

We set[40]

$$\overline{R}^\flat := \varprojlim_{x \mapsto x^p} \overline{R}/(p),$$

which is a perfect $\mathscr{O}^\flat_{\overline{K}_0}$-algebra equipped with a natural action of $\pi_1^{\text{ét}}(\operatorname{Spec} R[\frac{1}{p}], \bar{\eta})$.

For any $(x_n)_{n\in\mathbb{Z}_{\geqslant 0}} \in \overline{R}^\flat$, define

$$x^{(n)} := \lim_{m\to\infty} (\tilde{x}_{m+n})^{p^m}$$

for any lift $\tilde{x}_{m+n} \in \widehat{\overline{R}}$ of $x_{m+n} \in \overline{R}/(p)$. Note that $x^{(n)}$ is well-defined and independent of the choices involved. Consider the following W-algebra map:

$$\theta : W(\overline{R}^\flat) \to \widehat{\overline{R}}, \quad \theta(a_0, a_1, \cdots) := \sum_{n=0}^{\infty} p^n a_n^{(n)}. \tag{8.1.3}$$

which is over the classical map $W(\mathscr{O}^\flat_{\overline{K}_0}) \twoheadrightarrow \mathscr{O}_{\widehat{\overline{K}}_0}$. The kernel of θ (8.1.3) is a principal ideal generated by an explicit element $p - [p^\flat]$, where $p^\flat = (a_n)$ with $a_n =$"$p^{1/n} \bmod p$". This claim follows from Lemma 8.1.4 below, since $p - [p^\flat]$ also generates the kernel of $W(\mathscr{O}^\flat_{\overline{K}_0}) \twoheadrightarrow \mathscr{O}_{\widehat{\overline{K}}_0}$.

We consider the R-linear extension $\theta_R : R \otimes_W W(\overline{R}^\flat) \twoheadrightarrow \widehat{\overline{R}}$ of θ, and define $A_{\mathrm{cris}}(R)$ to be the p-adic completion of the PD envelop of $R \otimes_W W(\overline{R}^\flat)$ with respect to $\ker(\theta_R)$. (Note that the notation is incompatible with $A_{\mathrm{cris}}(R)$ for f-semiperfect ring R introduced in §5.3. In this section, $A_{\mathrm{cris}}(R)$ as in §5.3 will not appear.) We let $\operatorname{Fil}^1 A_{\mathrm{cris}}(R)$ denote the kernel of $A_{\mathrm{cris}}(R) \twoheadrightarrow \widehat{\overline{R}}$, which is an PD ideal.

By choosing a lift of Frobenius $\sigma : R \to R$ (which exists by the formal smoothness of R), one defines a lift of Frobenius σ on $R \otimes_W W(\overline{R}^\flat)$, which extends to $A_{\mathrm{cris}}(R)$. The universal continuous connection $d : R \to \widehat{\Omega}_{R/W}$ extends (by usual divided power calculus) to a p-adically continuous connection $\nabla : A_{\mathrm{cris}}(R) \to A_{\mathrm{cris}}(R) \otimes_R \widehat{\Omega}_{R/W}$. Finally $A_{\mathrm{cris}}(R)$ has a natural $\pi_1^{\text{ét}}(\operatorname{Spec} R[\frac{1}{p}], \bar{\eta})$-action, which extends the natural action on $\overline{R}^\flat$ and fixes R.

Lemma 8.1.4. *The natural map*

$$(R \widehat{\otimes}_W W(\overline{R}^\flat)) \widehat{\otimes}_{W(\mathscr{O}^\flat_{\overline{K}_0})} A_{\mathrm{cris}}(W) \to A_{\mathrm{cris}}(R)$$

[39]Note that R is a quotient of some completion of a polynomial algebra over W by [11, Lemma 1.3.3], and such a ring is known to be excellent (*cf.* [43, Theorem 9]).

[40]Perhaps, $(\widehat{\overline{R}})^\flat$ would be a more precise notation as $(\widehat{\overline{R}}[\frac{1}{p}], \widehat{\overline{R}})$ is a perfectoid affinoid $\widehat{\overline{W}}$-algebra, but the notation $\overline{R}^\flat$ would cause no confusion.

is an isomoprhism, where $\widehat{\otimes}$ denote the p-adically completed tensor product.

Proof. We want to show that the above map is an isomorphism modulo p^m for each m. Since $A_{\mathrm{cris}}(R)/p^m$ is the PD envelop of $(\theta_R \bmod p^m)$ over W/p^m, it suffices to show that $R\otimes_W W_m(\overline{R}^\flat)$ is flat over $W_m(\mathscr{O}^\flat_{\overline{K}_0})$ for each m by [5, Proposition 3.21]. By local flatness criterion, it suffices to show that $\overline{R}^\flat$ is flat over $\mathscr{O}^\flat_{\overline{K}_0}$. (Note that R is flat over W.) Since $\mathscr{O}^\flat_{\overline{K}_0}$ is a valuation ring (of rank 1), $\mathscr{O}^\flat_{\overline{K}_0}$-flatness is equivalent to torsion-freeness, but clearly $\overline{R}^\flat$ has no nonzero $\mathscr{O}^\flat_{\overline{K}_0}$-torsion. □

Lemma 8.1.4 allows us to deduce explicit descriptions of $A_{\mathrm{cris}}(R)$ from $A_{\mathrm{cris}}(W)$, which is well-known; *cf.* [19, §5].

Since $A_{\mathrm{cris}}(R)$ is an $A_{\mathrm{cris}}(W)$-algebra, the element $t \in A_{\mathrm{cris}}(W)$, which is "Fontaine's p-adic analogue of $2\pi i$", can be viewed as an element of $A_{\mathrm{cris}}(R)$. We define

$$B^+_{\mathrm{cris}}(R) := A_{\mathrm{cris}}(R)[\tfrac{1}{p}], \quad B_{\mathrm{cris}}(R) := B^+_{\mathrm{cris}}(R)[\tfrac{1}{t}] = A_{\mathrm{cris}}(R)[\tfrac{1}{t}].$$

The Frobenius endomorphism σ and the connection ∇ extends to $B^+_{\mathrm{cris}}(R)$ and $B_{\mathrm{cris}}(R)$. We define the filtration $\mathrm{Fil}^r B^+_{\mathrm{cris}}(R)$ (for $r \geqslant 0$) to be the ideal generated by the rth divided power ideal of $A_{\mathrm{cris}}(R)$, and set

$$\mathrm{Fil}^r B_{\mathrm{cris}}(R) := \sum_{i\geqslant -r} t^{-i}\,\mathrm{Fil}^{i+r} B^+_{\mathrm{cris}}(R),$$

for any $r \in \mathbb{Z}$.

Lemma 8.1.5. *We have*

$$\mathbb{Z}_p = A_{\mathrm{cris}}(R)^{\sigma=1;\nabla=0};$$
$$\mathbb{Q}_p = (\mathrm{Fil}^0 B_{\mathrm{cris}}(R))^{\sigma=1;\nabla=0}.$$

Idea of the proof. One may repeat the proof of [7, Corollaire 6.2.19]. Indeed, the main ingredient of the proof is an explicit description of $A_{\mathrm{cris}}(R)$ in terms of "t-adic expansions" [7, Proposition 6.2.13], which can be deduced, via Lemma 8.1.4, from the classical result on $A_{\mathrm{cris}}(W)$ in [19, §5.2.7]. □

8.2. **Crystalline comparison for p-divisible groups.** Now, let X be a p-divisible group over R, and let $\bar\eta : R \to \mathbb{E}$ denote the geometric generic point, where $\mathbb{E}$ is the separable closure of $\mathrm{Frac}\,R$ that contains $\overline{R}$. We can consider $T(X)$ as a lisse $\mathbb{Z}_p$-sheaf on either $\mathrm{Spec}\,R[\frac{1}{p}]$ or $(\mathrm{Spf}\,R)^{\mathrm{rig}}$. (This will not lead to any serious confusion as observed in Remark 8.1.2.)

Then we have

$$T(X)_{\bar\eta} \cong \mathrm{Hom}_{\widehat{\overline{R}}}(\mathbb{Q}_p/\mathbb{Z}_p, X_{\widehat{\overline{R}}}), \tag{8.2.1}$$

which defines a natural map

$$\rho_X : T(X)_{\bar\eta} \to \mathrm{Hom}(\mathbb{D}(X_{\widehat{\overline{R}}})(A_{\mathrm{cris}}(R)), A_{\mathrm{cris}}(R)) \tag{8.2.2}$$

by sending $f \in T(X)_{\bar\eta}$ to the pull-back morphism $f^* : \mathbb{D}(X_{\widehat{\overline{R}}}) \to \mathbf{1} = \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$ evaluated at $A_{\mathrm{cris}}(R)$. First, note that $\mathbb{D}(X)(A_{\mathrm{cris}}(R))$ is naturally isomorphic to $A_{\mathrm{cris}}(R)\otimes_R \mathbb{D}(X)(R)$, and this identification respects the Frobenius endomorphism and the connections. So for any $f \in T(X)_{\bar\eta}$, the morphism $\rho_X(f) : \mathbb{D}(X)(R) \to A_{\mathrm{cris}}(R)$ respects both the Frobenius action F and the connections ∇, and $\rho_X(f)$ maps the Hodge filtration $\mathrm{Fil}^1_X \subset \mathbb{D}(X)(R)$ into $\mathrm{Fil}^1 A_{\mathrm{cris}}(R)$. Furthermore, ρ_X is equivariant under the natural $\pi_1^{\text{ét}}(\mathrm{Spec}\,R[\frac{1}{p}], \bar\eta)$-action, where it acts on $A_{\mathrm{cris}}(R)$ by the usual action.

To summarise, the following map can be obtained by $B_{\mathrm{cris}}(R)$-linearly extending ρ_X and dualising it:

$$B_{\mathrm{cris}}(R)\otimes_R \mathbb{D}(X)(R)\to B_{\mathrm{cris}}(R)\otimes_{\mathbb{Q}_p} V(X)^*_{\bar\eta}. \tag{8.2.3}$$

Furthermore, this map respects the naturally defined Frobenius-actions, connections, filtrations, and $\pi_1^{\text{ét}}(\operatorname{Spec} R[\frac{1}{p}],\bar\eta)$-action. (Here, we declare that $V(X)^*_{\bar\eta}$ is horizontal, is fixed by the Frobenius action, and lies in the 0th filtration, and $\mathbb{D}(X)(R)$ carries the trivial $\pi_1^{\text{ét}}(\operatorname{Spec} R[\frac{1}{p}],\bar\eta)$-action.)

Let $x: R\to \mathscr{O}_K$ be a map where $\mathscr{O}_K$ is a complete discrete valuation W-algebra with residue field κ, and choose a "geometric point" $\bar x: R\to\mathscr{O}_K\hookrightarrow \mathscr{O}_{\widehat{\overline{K}}}$. We can extend $\bar x$ to $\widehat{\overline{R}}\to\mathscr{O}_{\widehat{\overline{K}}}$, also denoted by $\bar x$.

We can repeat the construction of (8.2.3) for the p-divisible group X_x over $\mathscr{O}_K$, although $\mathscr{O}_K$ is not necessarily formally smooth over W (i.e., absolutely unramified). Recall that we have a natural isomorphism of isocrystals $\mathbb{D}(X_x)[\frac{1}{p}]\cong \mathbb{D}((X_{x,\kappa})_{\mathscr{O}_K/p})[\frac{1}{p}]$ induced by

$$\begin{aligned}\mathbb{D}(X_x)\cong\mathbb{D}(X_{x,\mathscr{O}_K/p}) &\xleftarrow{\mathrm{Frob}^r_{\mathscr{O}_K/p}} \mathbb{D}(\sigma^{r*}(X_{x,\mathscr{O}_K/p}))\\ &\cong \mathbb{D}((\sigma^{r*}X_{x,\kappa})_{\mathscr{O}_K/p})\xrightarrow{\mathrm{Frob}^r_\kappa}\mathbb{D}((X_{x,\kappa})_{\mathscr{O}_K/p}),\end{aligned}$$

lifting the identity map on $\mathbb{D}(X_{x,\kappa})[\frac{1}{p}]$, where r is chosen so that the maximal ideal of $\mathscr{O}_K/p$ is killed by p^rth power, and $\mathrm{Frob}^r_{\mathscr{O}_K/p}$ and Frob^r_κ respectively denote the rth iterated relative Frobenius morphisms for $X_{x,\mathscr{O}_K/p}$ and $X_{x,\kappa}$. With this choice of r, we have $\sigma^{r*}(X_{x,\mathscr{O}_K/p})\cong(\sigma^{r*}X_{x,\kappa})_{\mathscr{O}_K/p}$. The resulting isomorphism $\mathbb{D}(X_x)[\frac{1}{p}]\cong \mathbb{D}((X_{x,\kappa})_{\mathscr{O}_K/p})[\frac{1}{p}]$ is independent of the choice of r.

From this we get a natural isomorphism:

$$\mathbb{D}(X_x)(A_{\mathrm{cris}}(W))[\tfrac{1}{p}]\cong B^+_{\mathrm{cris}}(W)\otimes_W\mathbb{D}(X_{x,\kappa})(W). \tag{8.2.4}$$

We also have a natural $\mathrm{Gal}(\overline{K}/K)$-isomorphism

$$T(X_x)_{\bar x}\cong \mathrm{Hom}_{\widehat{\overline{K}}}(\mathbb{Q}_p/\mathbb{Z}_p, X_{\bar x}). \tag{8.2.5}$$

Now, by repeating the construction of the map (8.2.3) we obtain

$$B_{\mathrm{cris}}(W)\otimes_W\mathbb{D}(X_{x,\kappa})(W)\to B_{\mathrm{cris}}(W)\otimes_{\mathbb{Q}_p}V(X_x)^*_{\bar x}, \tag{8.2.6}$$

which respects the naturally defined Frobenius action, connection, filtration, and $\mathrm{Gal}(\overline{K}/K)$-action.

Theorem 8.2.7. *The maps (8.2.3) and (8.2.6) are isomorphisms.*

More general version of this theorem is proved in [25, Theorem 5.3].

Idea of the proof. By Theorem 7 in [17, §6], it follows that (8.2.6) is an isomorphism. To prove that (8.2.3) is an isomorphism, we repeat the proof of [17, Theorem 7] to show that the following map

$$A_{\mathrm{cris}}(R)\otimes_R\mathbb{D}(X)(R)\to A_{\mathrm{cris}}(R)\otimes_{\mathbb{Q}_p}V(X)^*_{\bar\eta},$$

induced by ρ_X (8.2.2), is injective with cokernel killed by t. One first handles the case when $X=\mu_{p^\infty}$ either by considering the PD completion of $A_{\mathrm{cris}}(R)$ as originally done by Faltings[41] or by some explicit computation with the Artin-Hasse exponential map as in [42, §4.2]. Now one deduces the general case from this by some Cartier duality argument, as explained in [17, §6]. □

[41] See the footnote in the proof of [25, Theorem 6.3] for slightly more details.

Let us now show that the (relative) crystalline comparison isomorphism (8.2.3) interpolates the crystalline comparison isomorphisms at classical points (8.2.6). For x as before, we set $\bar{x} : R \xrightarrow{x} \mathscr{O}_K \hookrightarrow \mathscr{O}_{\widehat{\overline{K}}}$, and choose an extension $\bar{x} : \widehat{\overline{R}} \to \widehat{\overline{K}}$. (Indeed, we can lift the geometric point $R[\frac{1}{p}] \to \overline{K}$ to $\overline{R}[\frac{1}{p}] \to \overline{K}$, and $\overline{R}$ maps to $\mathscr{O}_{\overline{K}}$. We then take the p-adic completion.)

By (8.2.1) and (8.2.5), we get an isomorphism

$$T(X)_{\bar{\eta}} \xrightarrow{\sim} T(X)_{\bar{x}} \cong T(X_x)_{\bar{x}}, \tag{8.2.8}$$

sending $\mathbb{Q}_p/\mathbb{Z}_p \to X_{\widehat{\overline{R}}}$ to its fibre at $\bar{x} : \widehat{\overline{R}} \to \mathscr{O}_{\widehat{\overline{K}}}$.

Note that $\bar{x}$ induces a map $\bar{x}^\flat : \overline{R}^\flat \to \mathscr{O}^\flat_{\widehat{\overline{K}}}$. Choose $x_0 : R \to W$ such that x_0 and x induce the same κ-point of R (which is possible as R is formally smooth over W). Then the map $x_0 \otimes W(\bar{x}^\flat) : R \otimes_W W(\overline{R}^\flat) \to W(\mathscr{O}^\flat_{\widehat{\overline{K}}})$ extends to $A_{\mathrm{cris}}(R) \to A_{\mathrm{cris}}(W)$, respecting all the extra structure possibly except σ; indeed, $x_0 : R \to W$ may not respect σ.

Lemma 8.2.9. *The following diagram commutes*

$$\begin{tikzcd} B_{\mathrm{cris}}(R) \otimes_R \mathbb{D}(X)(R) \arrow[r, "\sim", "(8.2.3)"'] \arrow[d] & B_{\mathrm{cris}}(R) \otimes_{\mathbb{Q}_p} V(X)^*_{\bar{\eta}} \arrow[d, "(8.2.8)"] \\ B_{\mathrm{cris}}(W) \otimes_W \mathbb{D}(X_{x,\kappa})(W) \arrow[r, "\sim", "(8.2.6)"'] & B_{\mathrm{cris}}(W) \otimes_{\mathbb{Q}_p} V(X)^*_{\bar{x}} \end{tikzcd},$$

where the left vertical arrow is induced from

$$\begin{aligned} &B^+_{\mathrm{cris}}(W) \otimes_{A_{\mathrm{cris}}(R)} \mathbb{D}(X)(A_{\mathrm{cris}}(R)) \\ &\qquad \cong \mathbb{D}(X_x)(A_{\mathrm{cris}}(W))[1/p] \cong B^+_{\mathrm{cris}}(W) \otimes_W \mathbb{D}(X_{x,\kappa})(W). \end{aligned}$$

Here, the second isomorphism is (8.2.4).

Proof. Clear from the construction. □

We extend the isomorphism (8.2.3) to the following isomorphism:

$$B_{\mathrm{cris}}(R)\otimes_R\mathbb{D}(X)(R)^{\otimes} \xrightarrow{\sim} B_{\mathrm{cris}}(R)\otimes_{\mathbb{Q}_p}(V(X)^*_{\bar{\eta}})^{\otimes} = B_{\mathrm{cris}}(R)\otimes_{\mathbb{Q}_p}V(X)^{\otimes}_{\bar{\eta}}, \tag{8.2.10}$$

respecting all the extra structures. Now, given $t : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ as in Theorem 7.1.6, the element

$$t(A_{\mathrm{cris}}(R)) = 1 \otimes t(R) \in A_{\mathrm{cris}}(R) \otimes_R \mathbb{D}(X)(R)^{\otimes} \subset B_{\mathrm{cris}}(R) \otimes_R \mathbb{D}(X)(R)^{\otimes}$$

is fixed by the Frobenius and $\pi_1^{\text{ét}}(\operatorname{Spec} R[\frac{1}{p}], \bar{\eta})$-action, is killed by the connection, and lies in the 0th filtration. By the isomorphism (8.2.10) and Lemma 8.1.5, the above element $1 \otimes t(R)$ corresponds to an element $t_{\text{ét},\bar{\eta}} \in V(X)^{\otimes}_{\bar{\eta}}$ fixed by the $\pi_1^{\text{ét}}(\operatorname{Spec} R[\frac{1}{p}], \bar{\eta})$-action. Therefore, by the usual dictionary there exists a unique map of lisse $\mathbb{Q}_p$-sheaves

$$t_{\text{ét}} : \mathbf{1} \to V(X)^{\otimes} \tag{8.2.11}$$

such that it induces the map $1 \mapsto t_{\text{ét},\bar{\eta}}$ on the fibre at $\bar{\eta}$. By Lemma 8.2.9, $t_{\text{ét},\bar{x}}$ interpolates the étale Tate tensors associated to the fibre of t at classical points.

We want to show that $t_{\text{ét}}$ is "integral"; i.e., it restricts to $t_{\text{ét}} : \mathbf{1} \to T(X)^{\otimes}$. For this, it suffices to show that $t_{\text{ét},\bar{x}} \in T(X)^{\otimes}_{\bar{x}}$ for some geometric point $\bar{x}$. (Note that R is assumed to be a domain.) By formal smoothness, we may choose $\bar{x}$ that lies over

$x : R \to W$. In the next section, we verify the integrality claim using the theory of Kisin modules.[42]

8.3. **Review of Kisin theory.** For simplicity[43], we assume that $\mathscr{O}_K = W$. We follow the treatment of §1.2 and §1.4 in [29]. Let $\mathfrak{S} := W[[u]]$ and define $\sigma : \mathfrak{S} \to \mathfrak{S}$ by extending the Witt vectors Frobenius by $\sigma(u) = u^p$.

Definition 8.3.1. By *Kisin module* we mean a finitely generated free $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a σ-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}[\frac{1}{p-u}]$ whose linearisation induces an isomorphism $1 \otimes \varphi : \sigma^*\mathfrak{M}[\frac{1}{p-u}] \to \mathfrak{M}[\frac{1}{p-u}]$.

For $i \in \mathbb{Z}$ we define $\mathrm{Fil}^i(\sigma^*\mathfrak{M}[\frac{1}{p}]) := (1 \otimes \varphi)^{-1}((p-u)^i\mathfrak{M}[\frac{1}{p}])$. There is a good notion of subquotients, direct sums, $\otimes$-products, and duals.

We choose $p^\flat \in \mathscr{O}^\flat_{\overline{K}_0}$ and define $\mathfrak{S} \to W(\mathscr{O}^\flat_{\overline{K}_0})(\subset A_{\mathrm{cris}}(W))$ by sending u to $[p^\flat]$. The following can be extracted from the main results of [28]:

Theorem 8.3.2. *There exists a covariant rank-preserving fully faithful exact functor $\mathfrak{M} : L \mapsto \mathfrak{M}(L)$ from the category of $\mathrm{Gal}(\overline{K}_0/K_0)$-stable $\mathbb{Z}_p$-lattices of some crystalline representations to the category of Kisin modules, respecting $\otimes$-products and duals. Furthermore, the functor $\mathfrak{M}$ satisfies the following additional properties:*

(1) *We have natural φ-equivariant isomorphisms*

$$\begin{aligned} B_{\mathrm{cris}}(W) \otimes_{\mathbb{Z}_p} L &\cong B_{\mathrm{cris}}(W) \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}(L) \\ &\cong B_{\mathrm{cris}}(W) \otimes_{\sigma,W} \mathfrak{M}(L)/u\mathfrak{M}(L), \end{aligned}$$

which identifies $D_{\mathrm{cris}}(L[\frac{1}{p}]) \cong \sigma^*(\mathfrak{M}(L)/u\mathfrak{M}(L))[\frac{1}{p}]$.

(2) *We have a natural filtered isomorphism*

$$D_{\mathrm{dR}}(L[1/p]) \cong (\sigma^*\mathfrak{M}(L)[\tfrac{1}{p}])/(u-p),$$

where on the target we take the image filtration of $\mathrm{Fil}^\bullet(\sigma^*\mathfrak{M}(L)[\frac{1}{p}])$.

(3) *For two $\mathbb{Z}_p$-lattice crystalline $\mathrm{Gal}(\overline{K}_0/K_0)$-representations L and L', let $\mathfrak{f} : \mathfrak{M}(L) \to \mathfrak{M}(L')$ be an φ-equivariant map. Then there exists at most one $\mathrm{Gal}(\overline{K}_0/K_0)$-equivariant map $f : L \to L'$ with $\mathfrak{M}(f) = \mathfrak{f}$, and such f exists if and only if the map*

$$B_{\mathrm{cris}}(W) \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}(L) \xrightarrow{1\otimes\mathfrak{f}} B_{\mathrm{cris}}(W) \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}(L')$$

is $\mathrm{Gal}(\overline{K}_0/K_0)$-equivariant, in which case $f[\frac{1}{p}]$ is obtained from the φ-invariance of the 0th filtration part of the isomorphism above.

Proof. The theorem follows from the statement and the proof of [28, Proposition 2.1.5]. Seel also [29, Theorem 1.2.1], where (1) and (2) are deduced from [28]. □

We continue to assume that $\mathscr{O}_K = W$, so we have $D_{\mathrm{cris}}(L[\frac{1}{p}]) = D_{\mathrm{dR}}(L[\frac{1}{p}])$ as K-modules.

We clearly have that $\mathfrak{M}(\mathbf{1}) = (\mathfrak{S}, \sigma)$ (where $\mathbf{1}$ denotes $\mathbb{Z}_p$ equipped with the trivial $\mathrm{Gal}(\overline{K}_0/K_0)$-action). If there is no risk of confusion, we let $\mathbf{1}$ also denote the Kisin module $(\mathfrak{S}, \sigma)$.

[42] The integral refinement of (8.2.10) a la Fontaine-Laffaille does not work in general unless t factors through a factor of $\mathbb{D}(X_x)^\otimes$ with the gradings concentrated in $[a, a+p-2]$.

[43] The rest of the discussion can be modified for $\mathscr{O}_K$ that are (finitely) ramified.

Corollary 8.3.3. *For $\mathfrak{M} := \mathfrak{M}(L)$, the isomorphisms in Theorem 8.3.2(1) induce*

$$L[1/p]^{\mathrm{Gal}(\overline{K}_0/K_0)} \cong \sigma^*\mathfrak{M}[1/p]^{\varphi=1} \cong \mathrm{Fil}^0 D_{\mathrm{dR}}(L[1/p]) \cap D_{\mathrm{cris}}(L[1/p])^{\varphi=1},$$

which restrict to

$$L^{\mathrm{Gal}(\overline{K}_0/K_0)} \cong (\sigma^*\mathfrak{M})^{\varphi=1} \cong \mathrm{Fil}^0 D_{\mathrm{dR}}(L[1/p]) \cap (W \otimes_{\sigma,\mathfrak{S}} \mathfrak{M})^{\varphi=1}.$$

Proof. The first isomorphism is obtained by taking $\mathrm{Gal}(\overline{K}_0/K_0)$- and φ- invariance to the 0th filtration part of the isomorphisms in Theorem 8.3.2(1). Indeed, we have $(\sigma^*\mathfrak{M}[\frac{1}{p}])^{\varphi=1} \subset \mathrm{Fil}^0(\sigma^*\mathfrak{M}[\frac{1}{p}])$ by definition of $\mathrm{Fil}^0(\sigma^*\mathfrak{M}[\frac{1}{p}])$.

Let us show the last integrality assertion. Note that we have $1\otimes\varphi : (\sigma^*\mathfrak{M})^{\varphi=1} \xrightarrow{\sim} \mathfrak{M}^{\varphi=1}$. Now Theorem 8.3.2(3) shows that the following map

$$L^{\mathrm{Gal}(\overline{K}_0/K_0)} = \mathrm{Hom}_{\mathrm{Gal}(\overline{K}_0/K_0)}(\mathbf{1}, L) \to \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathbf{1}, \mathfrak{M}(L)) = \mathfrak{M}^{\varphi=1},$$

sending $f : \mathbf{1} \to L$ to $\mathfrak{M}(f) : \mathbf{1} \to \mathfrak{M}$, is an isomorphism, and it induces the isomorphism $L[\frac{1}{p}]^{\mathrm{Gal}(\overline{K}_0/K_0)} \xrightarrow{\sim} \sigma^*\mathfrak{M}[1/p]^{\varphi=1} \xrightarrow[1\otimes\varphi]{\sim} \mathfrak{M}[\frac{1}{p}]^{\varphi=1}$ that we have just proved.

To show that the mod u reduction $(\sigma^*\mathfrak{M})^{\varphi=1} \to \mathrm{Fil}^0 D_{\mathrm{dR}}(L[1/p]) \cap (W \otimes_{\sigma,\mathfrak{S}} \mathfrak{M})^{\varphi=1}$ is an isomorphism, note that the natural inclusion $L^{\mathrm{Gal}(\overline{K}_0/K_0)} \to L$ corresponds to the following inclusion of Kisin modules

$$\mathfrak{S} \otimes_{\mathbb{Z}_p} L^{\mathrm{Gal}(\overline{K}_0/K_0)} \cong \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathfrak{M}^{\varphi=1} \hookrightarrow \mathfrak{M}.$$

The image of this map is a direct factor as a $\mathfrak{S}$-module since $L^{\mathrm{Gal}(\overline{K}_0/K_0)} \to L$ is a direct factor as a $\mathbb{Z}_p$-module. Then, it is not difficult to show the $\mathbb{Q}_p$-linear isomorphism $\sigma^*\mathfrak{M}[1/p]^{\varphi=1} \cong \mathrm{Fil}^0 D_{\mathrm{dR}}(L[1/p]) \cap D_{\mathrm{cris}}(L[1/p])^{\varphi=1}$ matches the given $\mathbb{Z}_p$-lattices, as desired. □

Theorem 8.3.4. *For any p-divisible group Y over W, the isomorphism $D_{\mathrm{cris}}(V(Y)^*) \cong \mathbb{D}(Y)(W)[\frac{1}{p}]$ restricts to an isomorphism of F-crystals*

$$W \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}(T(Y)^*) \cong \mathbb{D}(Y)(W).$$

If we invert p, then the Hodge filtration on the right hand side induces the the image filtration of $\mathrm{Fil}^\bullet\, \sigma^*\mathfrak{M}(T(Y)^*)[\frac{1}{p}]$ on the left hand side.

Proof. If $p > 2$ or $X^\vee$ is connected, then this is a result of Kisin (*cf.* [29, Theorem 1.4.2]. The remaining case when $p = 2$ follows from [27, Proposition 4.2(1)]. □

Proof of Theorem 7.1.6. Let us first show Theorem 7.1.6 when X is a p-divisible group over R, where R is as in the beginning of §8. For $t : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ as in Theorem 7.1.6, let $t_{\text{ét}} : \mathbf{1} \to V(X)^{\otimes}$ denote its étale realisation as constructed in (8.2.11). We choose $x : R \to W$ and a "geometric point" $\bar{x}$ supported at x. Set $\mathfrak{M}_x := \mathfrak{M}(T(X_x)^*_{\bar{x}})$ and $\mathbf{M}_x := \mathbb{D}(X_x)(W)$. By Theorem 8.3.4 we have a natural F-equivariant isomorphism

$$W \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}_x \cong \mathbf{M}_x. \tag{8.3.5}$$

Let $t_{\mathfrak{S},\bar{x}} \in (\mathfrak{M}_x^{\otimes}[\frac{1}{p}])^{\varphi=1}$ be the tensor corresponding to $t_{\text{ét},\bar{x}} \in (V(X)^{\otimes}_{\bar{x}})^{\mathrm{Gal}(\overline{K}_0/K_0)}$ by Corollary 8.3.3.

We want to show that $t_{\text{ét}}$ is integral, for which it suffices to show that $t_{\mathfrak{S},\bar{x}} \in (\mathfrak{M}_x^{\otimes})^{\varphi=1}$ by Corollary 8.3.3. Recall that $t_{\text{ét},\bar{x}}$ is constructed so that it corresponds to the morphism $t_x : \mathbf{1} \to \mathbb{D}(X_x)^{\otimes}$ by the crystalline comparison isomorphism. It now follows from Theorem 8.3.2(1) and (8.3.5) that the following natural isomorphism

$$K_0 \otimes_{\sigma,\mathfrak{S}[1/p]} \mathfrak{M}_x^{\otimes}[\tfrac{1}{p}] \cong \mathbf{M}_x^{\otimes}[\tfrac{1}{p}]\big(\cong D_{\mathrm{cris}}(V(X)^*_{\bar{x}})^{\otimes}\big)$$

matches $1 \otimes t_{\mathfrak{S},\bar{x}}$ with $t_x(W)$. But since $t_x(W) \in \mathrm{Fil}^0\,\mathbf{M}_x^{\otimes}$ (not just in $\mathbf{M}_x^{\otimes}$), we obtain $t_{\mathfrak{S},\bar{x}} \in (\mathfrak{M}_x^{\otimes})^{\varphi=1}$ from Corollary 8.3.3 and (8.3.5). This shows Theorem 7.1.6 when $\mathfrak{X} = \mathrm{Spf}\,R$.

To prove Theorem 7.1.6 in general, note that that $\mathfrak{X}$ admits a Zariski open covering $\{\mathfrak{U}_\xi\}$ where each $\mathfrak{U}_\xi = \mathrm{Spf}\,R_\xi$ satisfies the assumption as in the beginning of §8. We have just proved that there exists a morphism $t_{\text{ét}}|_{\mathfrak{U}_\xi^{\mathrm{rig}}} : \mathbf{1} \to T(X_{\mathfrak{U}_\xi})^{\otimes}$ for each ξ that satisfies the condition in the theorem, and these morphisms should coincide at each overlap by uniqueness. So the locally defined tensors $\{t_{\text{ét}}|_{\mathfrak{U}_\xi^{\mathrm{rig}}}\}$ glue to give a tensor $t_{\text{ét}}$ on $\mathscr{X}$, which concludes the proof. □

REFERENCES

[1] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189. MR 0399094 (53 #2945)

[2] P. Berthelot and A. Ogus, *F-isocrystals and de Rham cohomology. I*, Invent. Math. **72** (1983), no. 2, 159–199. MR 700767 (85e:14025)

[3] Pierre Berthelot, *Cohomologie rigide et cohomologie rigide à support propre. première partie*, Prépublication IRMAR 96-03 (1996).

[4] Pierre Berthelot, Lawrence Breen, and William Messing, *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982. MR 667344 (85k:14023)

[5] Pierre Berthelot and Arthur Ogus, *Notes on crystalline cohomology*, Princeton University Press, Princeton, N.J., 1978. MR MR0491705 (58 #10908)

[6] Siegfried Bosch and Werner Lütkebohmert, *Formal and rigid geometry. II. Flattening techniques*, Math. Ann. **296** (1993), no. 3, 403–429. MR 1225983 (94e:11070)

[7] Olivier Brinon, *Représentations p-adiques cristallines et de de Rham dans le cas relatif*, Mém. Soc. Math. Fr. (N.S.) (2008), no. 112, vi+159. MR 2484979 (2010a:14034)

[8] Miaofen Chen, Mark Kisin, and Eva Viehmann, *Connected components of affine Deligne-Lusztig varieties in mixed characteristic*, Preprint, `arXiv:math/1307.3845` (2013), 60 pages.

[9] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *Fibrés quadratiques et composantes connexes réelles,*, Math. Ann. **244** (1979), 105–134.

[10] Brian Conrad and A. J. de Jong, *Approximation of versal deformations*, J. Algebra **255** (2002), no. 2, 489–515. MR 1935511 (2004a:14003)

[11] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Inst. Hautes Études Sci. Publ. Math. **82** (1995), 5–96. MR 1383213 (97f:14047)

[12] ———, *Étale fundamental groups of non-Archimedean analytic spaces*, Compositio Math. **97** (1995), no. 1-2, 89–118, Special issue in honour of Frans Oort. MR 1355119 (97c:32047)

[13] ———, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93. MR 1423020 (98e:14011)

[14] A. J. de Jong and W. Messing, *Crystalline Dieudonné theory over excellent schemes*, Bull. Soc. Math. France **127** (1999), no. 2, 333–348. MR 1708635 (2001b:14075)

[15] Pierre Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289. MR 546620 (81i:10032)

[16] Vladimir G. Drinfel′d, *Coverings of p-adic symmetric domains*, Funkcional. Anal. i Priložen. **10** (1976), no. 2, 29–40. MR MR0422290 (54 #10281)

[17] Gerd Faltings, *Integral crystalline cohomology over very ramified valuation rings*, J. Amer. Math. Soc. **12** (1999), no. 1, 117–144. MR MR1618483 (99e:14022)

[18] Laurent Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, Astérisque (2004), no. 291, 1–199, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales. MR 2074714 (2005g:11110b)

[19] Jean-Marc Fontaine, *Le corps des périodes p-adiques*, Astérisque (1994), no. 223, 59–111, With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988). MR MR1293971 (95k:11086)

[20] Alexander Grothendieck, *Eléments de géométrie algébrique, I–IV*, Inst. Hautes Études Sci. Publ. Math. (1960-1967), no. 4, 8, 11, 17, 20, 24, 28, and 32.

[21] Alexandre Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Les Presses de l'Université de Montréal, Montreal, Que., 1974, Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970). MR 0417192 (54 #5250)

[22] Benjamin Howard and George Pappas, *Rapoport-Zink spaces for spinor groups*, Preprint, arXiv:math/1509.03914 (2015), 75 pages.
[23] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. MR 1734903 (2001c:14046)
[24] Luc Illusie, *Déformations de groupes de Barsotti-Tate (d'après A. Grothendieck)*, Astérisque (1985), no. 127, 151–198, Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84). MR 801922
[25] Wansu Kim, *The relative Breuil-Kisin classification of p-divisible groups and finite flat group schemes*, Int. Math. Res. Not. **Advance Online Access** (2014), 81 pages.
[26] ———, *Rapoport-Zink uniformisation of Hodge-type Shimura varieties*, Preprint (2015), 28 pages.
[27] Wausu Kim, *The classification of p-divisible groups over 2-adic discrete valuation rings*, Math. Res. Lett. **19** (2012), no. 01, 121–141.
[28] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 459–496. MR MR2263197 (2007j:11163)
[29] ———, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.
[30] Donald Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, Vol. 203, Springer-Verlag, Berlin, 1971. MR 0302647 (46 #1791)
[31] Robert E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220. MR 809866 (87i:14040)
[32] Gérard Laumon and Laurent Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000. MR 1771927 (2001f:14006)
[33] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid. MR 879273 (88h:13001)
[34] B. Mazur and William Messing, *Universal extensions and one dimensional crystalline cohomology*, Lecture Notes in Mathematics, Vol. 370, Springer-Verlag, Berlin, 1974. MR 0374150 (51 #10350)
[35] William Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, Vol. 264, Springer-Verlag, Berlin, 1972. MR 0347836 (50 #337)
[36] J. S. Milne, *Introduction to Shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 265–378. MR 2192012 (2006m:11087)
[37] Ben Moonen, *Models of Shimura varieties in mixed characteristics*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 267–350. MR 1696489 (2000e:11077)
[38] Michael Rapoport and Eva Viehmann, *Towards a theory of local Shimura varieties*, Preprint, arXiv:math/1401.2849 (2014), 53 pages.
[39] Michael Rapoport and Thomas Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR 1393439 (97f:14023)
[40] Peter Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Preprint, arXiv:1205.3463 (2012).
[41] ———, *Perfectoid spaces: A survey*, Preprint, arXiv:1303.5948 (2013).
[42] Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, Preprint, arXiv:1211.6357 (2012), 74 pages.
[43] Paolo Valabrega, *A few theorems on completion of excellent rings*, Nagoya Math. J. **61** (1976), 127–133. MR 0407007 (53 #10790)
[44] Adrian Vasiu, *Good reductions of Shimura varieties of Hodge type in arbitrary unramified mixed characteristic, part I*, Preprint, arXiv:0707.1668 (2007), 53 pages.
[45] ———, *A motivic conjecture of Milne*, to appear in J. Reine Angew. Math. (2012), 60 pages.
[46] Torsten Wedhorn, *On Tannakian duality over valuation rings*, J. Algebra **282** (2004), no. 2, 575–609. MR 2101076 (2005j:18007)
[47] Jean-Pierre Wintenberger, *Un scindage de la filtration de Hodge pour certaines variétés algébriques sur les corps locaux*, Ann. of Math. (2) **119** (1984), no. 3, 511–548. MR 744862 (86k:14015)
[48] Xinwen Zhu, *Affine grassmannians and the geometric Satake in mixed characteristic*, Preprint, arXiv:math/1407.8519 (2014).

WANSU KIM, DEPARTMENT OF MATHEMATICS, KING'S COLLEGE LONDON, STRAND, LONDON, WC2R 2LS, UNITED KINGDOM
E-mail address: wansu.kim@kcl.ac.uk