RAPOPORT-ZINK UNIFORMISATION OF HODGE-TYPE SHIMURA VARIETIES

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ABSTRACT. We show that the integral canonical models of Hodge-type Shimura varieties at odd good reduction primes admits "p-adic uniformisation" by Rapoport-Zink spaces of Hodge type constructed in [15].

CONTENTS

1. Introduction 1
2. Notation and preliminaries 3
3. Review on Shimura varieties of Hodge type 8
4. Rapoport-Zink uniformisation via formal schemes 15
5. Rapoport-Zink uniformisation via rigid geometry 24
References 26

1. INTRODUCTION

Shimura varieties have many interesting structures and symmetries which encode arithmetic information. It is now a standard folklore conjecture that the cohomology of Shimura varieties should realise the global Langlands correspondence. It is natural to look for a purely local analogue of Shimura varieties, whose cohomology should realise the local Langlands correspondence, and ask how the local-global compatibility is encoded geometrically. For example, Carayol [5] showed that the (height-2) Lubin-Tate tower plays the role of "local Shimura varieties" and the identification of the Lubin-Tate tower with the completion of the modular tower at a supersingular point (by Serre-Tate deformation theory) encodes the local-global compatibility.

Many interesting examples of Shumura varieties can be understood as moduli spaces of certain polarised abelian varieties equipped with the action of some semi-simple algebra and level structure. Such Shimura varieties are called *of PEL type*, and examples include modular curves, Siegel modular varieties, and unitary Shimura varieties. The purely local analogue of PEL Shimura varieties was constructed by Rapoport and Zink [30], which are now called Rapoport-Zink spaces of EL or PEL type. In the good reduction case, Rapoport-Zink spaces are moduli spaces of p-divisible groups with some action of semi-simple algebra (and possibly with polarisation), up to rigidification (by quasi-isogeny). Furthermore, they showed the relationship between certain Rapoport-Zink spaces of (P)EL type and PEL Shimura varieties in a way that is analogous to the complex analytic uniformisation of Shimura varieties and generalises some known examples (of modular and Shimura curves via Lubin-Tate and Drinfeld towers); *cf.* [30, Ch.6]. We call this result the *Rapoport-Zink uniformisation* of PEL Shimura varieties.

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There is a more general class of Shimura varieties which, over $\mathbb{C}$, parametrise abelian varieties with certain Hodge cycles. They are called *Shimura varieties of Hodge type*. An example naturally comes up in relation to the construction of an abelian variety associated to a polarised complex K3 surfaces (due to Kuga and Satake). Although such moduli spaces are essentially defined only in characteristic 0 (as Hodge cycles are defined using singular cohomology with $\mathbb{Q}$-coefficients), recent developments in integral p-adic Hodge theory allow us to study certain "natural" integral models of such Shimura varieties at odd good reduction primes. See Kisin [17] and Vasiu [31, 32, 33] for the statement and the proof.

In the author's previous work [15], the local analogue of Hodge-type Shimura varieties for $p > 2$ (called *Rapoport-Zink spaces of Hodge type*) was constructed under a certain unramifiedness assumption, generalising the construction of Rapoport-Zink spaces of (P)EL type; loosely speaking, Rapoport-Zink spaces of Hodge type can be thought of as moduli spaces of p-divisible groups with Tate tensors (instead of endomorphisms and polarisation) up to rigidification by quasi-isogeny. In this paper, we prove the Hodge-type generalisation of the Rapoport-Zink uniformisation for odd good reduction prime.

To simplify the statement, we recall the theorem in the *basic* case, although we obtain the Rapoport-Zink uniformisation in the non-basic case as well. Let $(G, \mathfrak{H})$ be a Shimura datum of Hodge type (with G connected), and assume that G admits a reductive $\mathbb{Z}_{(p)}$-model for $p > 2$, also denoted as G. Let $E := E(G, \mathfrak{H})$ denote the reflex field, and we choose a prime $\mathfrak{p}$ over p, which is necessarily unramified. Then the aforementioned result of Vasiu and Kisin produces an "integral canonical model" $\mathscr{S}_{\mathsf{K}}$ of $\mathrm{Sh}_{\mathsf{K}}(G, \mathfrak{H})$, where $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ with $\mathsf{K}_p = G(\mathbb{Z}_p)$ and $\mathsf{K}^p \subset G(\mathbb{A}_{\mathrm{f}}^p)$ is a "small enough" open compact subgroup.

Let $W := W(\overline{\mathbb{F}}_p)$ and $K_0 := \mathrm{Frac}\, W$, viewed as a $\mathscr{O}_{E,\mathfrak{p}}$-algebra. Let $b \in G(K_0)$ be a basic element (in the sense of Definition 2.6.6) which come from an $\overline{\mathbb{F}}_p$-point of $\mathscr{S}_{\mathsf{K}}$, and let $\mathtt{RZ}_{G,b}$ denote the Rapoport-Zink space of Hodge type [15] associated to (G, b); see §4.1 for the details. Let $\mathscr{S}_{\mathsf{K},[b]}$ denote the (basic) Newton stratum indexed by the σ-conjugacy class $[b]$ of b (*cf.* Theorem 4.10.1).

Theorem 1 (*cf.* Theorems 4.11 and 5.4)**.** *There exists an isomorphism of formal schemes over W*

$$\Theta^{\phi} : I^{\phi}(\mathbb{Q})\backslash\mathtt{RZ}_{G,b} \times G(\mathbb{A}_{\mathrm{f}}^p)/\mathsf{K}^p \to (\mathscr{S}_{\mathsf{K},W})_{/\mathscr{S}_{\mathsf{K},[b]}},$$

where $I^{\phi}(\mathbb{Q})$ is the group of self quasi-isogenies of abelian varieties with tensors coming from a closed point of $\mathscr{S}_{\mathsf{K},[b]}$. The target of the isomorphism is the completion of $\mathscr{S}_{\mathsf{K},W}$ at the basic Newton stratum $\mathscr{S}_{\mathsf{K},[b]}$. Furthermore, the isomorphism Θ^{ϕ} naturally descends over $\mathscr{O}_{E,\mathfrak{p}}$, and on the rigid analytic generic fibres the isomorphism extends to a $G(\mathbb{A}_{\mathrm{f}})$-equivariant isomorphism of towers on the both sides.

When b is not basic, we still obtain some variant of the theorem where the target of the isomorphism is replaced by a more complicated formal scheme (*cf.* Theorem 4.7).

Let us make a remark on the proof. Unlike the PEL case, $\mathscr{S}_{\mathsf{K}}$ does not have a good moduli interpretation and this causes number of additional difficulties.

First, it is not trivial to construct the morphism $\mathtt{RZ}_{G,b} \to \widehat{\mathscr{S}_{\mathsf{K},W}}$, where the target is the p-adic completion of $\mathscr{S}_{\mathsf{K},W}$. To overcome this problem, we use a deformation-theoretic trick, exploiting that the completions of $\widehat{\mathscr{S}_{\mathsf{K},W}}$ at closed points are well-understood by construction, and the work of Chen, Kisin and Viehmann [6] which allows us to control the connected components of $\mathtt{RZ}_{G,b}$. See Proposition 4.3 and subsequent remarks for more details.

At this stage, one can repeat the proof of Theorems 6.21 and 6.23 in [30] to obtain some preliminary version of Rapoport-Zink uniformisation (Theorem 4.7). In order to promote it to Theorem 1, we need to show that the basic Newton stratum $\mathscr{S}_{\mathsf{K},[b]}$ contains only one isogeny class (Proposition 4.10.3), and the source of the isomorphism Θ^ϕ descends over $\mathscr{O}_{E,\mathfrak{p}}$ (Corollary 4.9.3). To deal with these problems, we use Kisin's result of CM lifting of mod p points of $\mathscr{S}_{\mathsf{K}}$ [18, §2], and its corollaries. In particular, the aforementioned Kisin's theorem shows that the quasi-isogeny group $I^\phi(\mathbb{Q})$ is the group of $\mathbb{Q}$-points of the inner form I^ϕ of G which is trivial outside p and ∞, $I^\phi_{\mathbb{Q}_p} = J_b$ (2.6.4), and $I^\phi(\mathbb{R})$ is compact modulo centre.

Recently, Ben Howard and George Pappas [12] gave another construction (using global techniques) of Hodge-type Rapoport-Zink spaces that come from global Hodge-type Shimura data, in such a way that the Rapoport-Zink uniformisation holds by construction. Indeed, their construction relies on the existence of integral canonical models of Hodge-type Shimura varieties and the Rapoport-Zink uniformisation for Siegel modular varieties, and the Hodge-type Rapoport-Zink uniformisation is obtained by pulling back the Siegel case of Rapoport-Zink uniformisation. Note that some of the ideas in this paper are used in [12] for the construction of Hodge-type Rapoport-Zink spaces. Our approach is to construct Hodge-type Rapoport-Zink spaces by purely local means in [15], and separately obtain the link with the global theory (i.e., Rapoport-Zink uniformisation).

We have excluded the case of $p = 2$ because Rapoport-Zink spaces of Hodge type at $p = 2$. (Note that the 2-adic integral canonical models were constructed in [16].) But once we have Rapoport-Zink spaces of Hodge type when $p = 2$ (which is the author's work in progress) then the Rapoport-Zink uniformisation can be extended.

The Rapoport-Zink uniformisation is more interesting in the bad reduction case, but we do not consider this case as the construction of Rapoport-Zink spaces in [15] has not been generalised in the bad reduction case. On the other hand, the recent work of Kisin and Pappas on integral models of Hodge-type Shimura varieties with parahoric level structure [19] suggests that the uniformisation result can be generalised to some bad reduction cases.

In §2 and §3 we review basic notions and set up the notation – §2 is for general notions, and §3 is for Shimura varieties and Rapoport-Zink spaces of Hodge type. In §4 we obtain the Rapoport-Zink uniformisation at the hyperspecial maximal level at p, and in §5 we extend the uniformisation to rigid analytic towers.

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2. Notation and preliminaries

2.1. For any ring R, an R-module M, and an R-algebra R', we write $M_{R'} := R' \otimes_R M$. Similarly, if R is a noetherian adic ring and $\mathfrak{X}$ is a formal scheme over $\operatorname{Spf} R$, then for any continuous morphism of adic rings $R \to R'$ we write $\mathfrak{X}_{R'} := \mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spf} R'$.

2.2. For definitions in category theory, see [35] and references therein. Let $\mathcal{C}$ be a pseudo-abelian[1] symmetric tensor category such that arbitrary (infinite) direct sum exists. Let $\mathbf{1}$ denote the identity object for $\otimes$-product in $\mathcal{C}$ (which exists by the axioms of tensor categories).

[1]Pseudo-abelian categories are defined in the same way as abelian categories, except that we only require the existence of kernel for idempotent morphisms instead of requiring the existence of kernel and cokernel for any morphism. In practice, the pseudo-abelian categories that we will encounter are the category of filtered or graded objects in some abelian category.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ which is stable under direct sums, tensor products, and direct factors. Assume furthermore that $\mathcal{D}$ is rigid; i.e., every object of $\mathcal{D}$ has a dual. (For example, $\mathcal{C}$ can be the category of R-modules filtered by direct factors, and $\mathcal{D}$ can be the full subcategory of finitely generated projective R-modules.) Then for any object $M \in \mathcal{D}$, we let

$$M^{\otimes} \in \mathcal{C}$$

denote the direct sum of any (finite) combination of tensor products, symmetric products, alternating products, and duals of M. Note that we naturally have

$$M^{\otimes} = (M^*)^{\otimes}.$$

2.3. Let S be a (not necessarily connected) scheme, and $\bar{x}$ a geometric point of S. Then $\pi_1^{\text{ét}}(S, \bar{x})$ denotes the étale fundamental group of the connected component of S containing $\bar{x}$.

2.4. **Abelian Schemes.** For any abelian scheme $f : \mathcal{A} \to S$ (where S is any scheme), we define

$$\mathcal{V}_{\mathrm{dR}}(\mathcal{A})\big(= \mathcal{V}_{\mathrm{dR}}(\mathcal{A}/S)\big) := \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/S)^*; \tag{2.4.1a}$$

$$\mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A})\big(= \mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A}/S)\big) := \prod_{\ell\neq\infty}{}' (R^1 f_{\text{ét}*}\mathbb{Q}_\ell)^*, \tag{2.4.1b}$$

where $\prod'$ is the restricted product with respect to $\{(R^1 f_{\text{ét}*}\mathbb{Z}_\ell)^*\}$. Note that (if S is connected then) for any geometric point $\bar{x}$ of S the fibre $\mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A})_{\bar{x}}$ has a natural continuous action of $\pi_1^{\text{ét}}(S, \bar{x})$.

For any prime p, we can define the p-component $\mathcal{V}_{\mathbb{Q}_p}(\mathcal{A}) := (R^1 f_{\text{ét}*}\mathbb{Q}_p)^*$ and the prime-to-p component $\mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A})$ with

$$\mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A}) = \mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A}) \times \mathcal{V}_{\mathbb{Q}_p}(\mathcal{A}). \tag{2.4.1c}$$

With suitably chosen $\mathcal{C}$ and $\mathcal{D}$, we can form $\mathcal{V}_{\mathrm{dR}}(\mathcal{A})^{\otimes}$, $\mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A})^{\otimes}$, etc., as in §2.2.

Definition 2.4.2. Let $\mathcal{A}$ be an abelian scheme over some scheme S. A *de Rham tensor* on $\mathcal{A}$ is a filtered $\mathcal{O}_S$-morphism $t_{\mathrm{dR}} : \mathbf{1} \to \mathcal{V}_{\mathrm{dR}}(\mathcal{A})^{\otimes}$, where $\mathbf{1}$ is $\mathcal{O}_S$ equipped with the filtration whose grading is concentrated in 0. We will often abuse the notation and denote by $t_{\mathrm{dR}} \in \Gamma(S, \mathcal{V}_{\mathrm{dR}}(\mathcal{A})^{\otimes})$ the image of $1 \in \Gamma(S, \mathcal{O}_S)$ by $t_{\mathrm{dR}} : \mathbf{1} \to \mathcal{V}_{\mathrm{dR}}(\mathcal{A})$.

An *étale tensor* on $\mathcal{A}$ is an $\mathbb{A}_{\mathrm{f}}$-linear morphism $t_{\text{ét}} : \mathbf{1} \to \mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A})^{\otimes}$ of étale sheaves on S, where $\mathbf{1}$ is the constant $\mathbb{A}_{\mathrm{f}}$-local system of rank 1. We similarly define a *prime-to-p étale tensor* $t^p_{\text{ét}} : \mathbf{1} \to \mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A})^{\otimes}$ and a *p-adic étale tensor* $t_{\text{ét},p} : \mathbf{1} \to \mathcal{V}_{\mathbb{Q}_p}(\mathcal{A})^{\otimes}$.

If S is a smooth variety over $\mathbb{C}$, then we can also define the following $\mathbb{Q}$-local system

$$\mathcal{V}(\mathcal{A})\big(= \mathcal{V}(\mathcal{A}/S)\big) := (R^1 f^{\mathrm{an}}_* \mathbb{Q})^*, \tag{2.4.3}$$

and we have natural isomorphisms $\mathbb{A}_{\mathrm{f}} \otimes_{\mathbb{Q}} \mathcal{V}(\mathcal{A}) \cong \mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A})$ and $\mathcal{O}_S \otimes_{\mathbb{Q}} \mathcal{V}(\mathcal{A}) \cong \mathcal{V}_{\mathrm{dR}}(\mathcal{A})$. By classical Hodge theory, we obtain a variation of $\mathbb{Q}$-Hodge structures.

With suitably chosen $\mathcal{C}$ and $\mathcal{D}$, we can form $\mathcal{V}(\mathcal{A})^{\otimes}$ as in §2.2. Given a $\mathbb{Q}$-linear morphism of locally constant sheaves $t_B : \mathbf{1} \to \mathcal{V}(\mathcal{A})^{\otimes}$ (where $\mathbf{1}$ is the constant sheaf $\mathbb{Q}$ on S), we define the *étale* and *de Rham components* $t_{\text{ét}}$ and t_{dR} of t_B as follows:

$$t_{\text{ét}} : \mathbf{1} \xrightarrow{\mathbb{A}_{\mathrm{f}}\otimes t_B} \mathbb{A}_{\mathrm{f}} \otimes_{\mathbb{Q}} \mathcal{V}(\mathcal{A})^{\otimes} \xrightarrow{\sim} \mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A})^{\otimes} \tag{2.4.4a}$$

$$t_{\mathrm{dR}} : \mathbf{1} \xrightarrow{\mathcal{O}_S\otimes t_B} \mathcal{O}_S \otimes_{\mathbb{Q}} \mathcal{V}(\mathcal{A})^{\otimes} \xrightarrow{\sim} \mathcal{V}_{\mathrm{dR}}(\mathcal{A})^{\otimes}. \tag{2.4.4b}$$

Clearly $t_{\text{ét}}$ is an étale tensor, and we are often interested in t such that t_{dR} is a de Rham tensor. Such t_B can be thought of as a "family of Hodge tensors".

2.5. **Group theory preliminaries.** Throughout this section, let R be either a field of characteristic zero or a discrete valuation ring of mixed characteristic. In practice, R will be one of $\mathbb{Q}$, $\mathbb{Z}_{(p)}$, and $\mathbb{Z}_p$. Let G be a reductive group over R; i.e., an affine smooth group scheme over R such that all the fibres are reductive groups. Let M be a free R-module of finite rank, and we fix a closed immersion of group schemes $G \hookrightarrow \mathrm{GL}_R(M)$. Let $M^{\otimes}$ be as defined in §2.2, where $\mathcal{C}$ is the category of R-modules and $\mathcal{D}$ is the category of locally free R-modules of finite rank.

Proposition 2.5.1. *In the above setting, here exists a finitely many elements $s_\alpha \in M^{\otimes}$ such that G is the pointwise stabiliser of (s_α); i.e., for any R-algebra R', we have*

$$G(R') = \{g \in \mathrm{GL}_R(M)(R');\ g(s_\alpha) = s_\alpha\ \forall \alpha\}.$$

Proof. The case when R is a field is proved in [9, Proposition 3.1], and the case of discrete valuation rings is proved in [17, Proposition 1.3.2]. □

Example 2.5.2. If G is a "classical group" then one can often explicitly write down (s_α) that define G in the sense of Proposition 2.5.1. For example, for a perfect alternating form $\psi : M \otimes M \to R$ on a projective R-module M, we can find a tensor $s_\psi \in M^{\otimes}$ whose pointwise stabiliser is $\mathrm{GSp}(M, \psi)$, which is explained in [15, Example 2.1.4].

Definition 2.5.3. Let $\mathfrak{X}$ be an R-scheme[2]. For a cocharacter $\mu : \mathbb{G}_m \to \mathrm{GL}_R(M)_{\mathfrak{X}}$, we say that a grading $\mathrm{gr}^\bullet(\mathcal{O}_{\mathfrak{X}} \otimes_R M)$ is *induced from* μ if the $\mathbb{G}_m$-action on $\mathcal{O}_{\mathfrak{X}} \otimes_R M$ via μ leaves each grading stable, and the resulting $\mathbb{G}_m$-action on $\mathrm{gr}^a(\mathcal{O}_{\mathfrak{X}} \otimes_R M)$ is given by

$$\mathbb{G}_m \xrightarrow{z \mapsto z^{-a}} \mathbb{G}_m \xrightarrow{z \mapsto z\,\mathrm{id}} \mathrm{GL}(\mathrm{gr}^a(\mathcal{O}_{\mathfrak{X}} \otimes_R M)).$$

We additionally fix finitely many $(s_\alpha) \subset M^{\otimes}$ defining $G \subset \mathrm{GL}_R(M)$. Let $\mathscr{E}$ be a vector bundle on $\mathfrak{X}$. Then we can form $\mathscr{E}^{\otimes}$ in the categogy of quasi-coherent sheaves. For (finitely many) global sections $(t_\alpha) \subset \Gamma(\mathfrak{X}, \mathscr{E}^{\otimes})$, we define the following scheme over $\mathfrak{X}$

$$P_{\mathfrak{X}} := \mathrm{isom}_{\mathcal{O}_{\mathfrak{X}}}\big([\mathscr{E}, (t_\alpha)], [\mathcal{O}_{\mathfrak{X}} \otimes_R M, (1 \otimes s_\alpha)]\big) \subset \mathrm{isom}_{\mathcal{O}_{\mathfrak{X}}}\big(\mathscr{E}, \mathcal{O}_{\mathfrak{X}} \otimes_R M\big), \tag{2.5.4}$$

which classifies isomorphisms of vector bundles over $\mathfrak{X}$ which match (t_α) and $(1 \otimes s_\alpha)$. There is a natural left $G_{\mathfrak{X}}$-action on $P_{\mathfrak{X}}$ through its natural action on $\mathcal{O}_{\mathfrak{X}} \otimes_R M$. Note that $P_{\mathfrak{X}}$ is a trivial G-torsor if and only if there exists an isomorphism $\varsigma : \mathscr{E} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}} \otimes_R M$ which matches (t_α) and $(1 \otimes s_\alpha)$. Indeed, such ς defines a section $\mathfrak{X} \to P_{\mathfrak{X}}$ and any other sections are translates by the G-action.

For a cocharacter $\mu : \mathbb{G}_m \to G_{\mathfrak{X}}$ and $g \in \Gamma(\mathfrak{X}, G)$, we write ${}^g\mu := g\mu g^{-1}$ and let $\{\mu\} := \{{}^g\mu :\ g \in G(W)\}$ denote the $G(W)$-conjugacy class of μ. The following terminology of "$\{\mu\}$-filtrations" was introduced in [15, Definition 2.2.3].

Definition 2.5.5. Let $\mathscr{E}$ be a vector bundle over $\mathfrak{X}$, with $(t_\alpha) \subset \Gamma(\mathfrak{X}, \mathscr{E}^{\otimes})$.

First, assume that $P_{\mathfrak{X}}$, defined in (2.5.4), is a *trivial* G-torsor. Then a filtration $\mathrm{Fil}^\bullet \mathscr{E}$ of $\mathscr{E}$ is called a *$\{\mu\}$-filtration (with respect to (t_α))* if there exists an isomorphism $\varsigma : \mathscr{E} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{X}} \otimes_R M$, matching (t_α) and $(1 \otimes s_\alpha)$, that takes $\mathrm{Fil}^\bullet \mathscr{E}$ to a filtration of $\mathcal{O}_{\mathfrak{X}} \otimes_R M$ induced by ${}^g\mu$ for some $\mu \in \{\mu\}$ and $g \in \Gamma(\mathfrak{X}, G)$.

When $P_{\mathfrak{X}}$ is a G-torsor, a filtration $\mathrm{Fil}^\bullet \mathscr{E}$ of $\mathscr{E}$ is called a *$\{\mu\}$-filtration (with respect to (t_α))* if it is étale-locally a $\{\mu\}$-filtration; in other words, there exists an étale covering $f : \mathfrak{Y} \to \mathfrak{X}$ such that $P_{\mathfrak{Y}}$ is a trivial G-torsor and $(\mathrm{Fil}^\bullet \mathscr{E})_{\mathfrak{Y}}$ is a $\{\mu\}$-filtration with respect to $(f^* t_\alpha)$.

[2]It is often convenient and natural to allow $\mathfrak{X}$ to be an analytic space or a formal scheme. But it will be quite obvious how to adapt the subsequent discussion to these cases.

Note that $\Gamma(\mathfrak{X}, G)$ naturally acts on the set of $\{\mu\}$-filtrations. In practice (i.e., when $\mathscr{E}$ comes from a suitable cohomology sheaf for an abelian scheme), it is too much to expect that $P_{\mathfrak{X}}$ is a trivial G-torsor – for example, $\mathscr{E}$ may not necessarily be a free $\mathcal{O}_{\mathfrak{X}}$-module. But it is certainly reasonable to impose that $P_{\mathfrak{X}}$ is a G-torsor; i.e., that $(\mathscr{E}, (t_\alpha))$ étale-locally looks like $(M, (s_\alpha))$.

When $G = \mathrm{GL}_R(M)$, then a filtration $\mathrm{Fil}^\bullet \mathscr{E}$ of $\mathscr{E}$ is a $\{\mu\}$-filtration for some cocharacter μ if and only if associated grading $\mathrm{gr}^\bullet \mathscr{E}$ is of constant rank, and the conjugacy class of μ is uniquely determined by the rank of each grading.

Let us fix $G \subset \mathrm{GL}_R(M)$, (s_α), and $\{\mu\}$ as before, and consider a vector bundle $\mathscr{E}$ on $\mathfrak{X}$ and $(t_\alpha) \subset \Gamma(\mathfrak{X}, \mathscr{E}^{\otimes})$. Let $\mathtt{Fl}^{\mathscr{E},(t_\alpha)}_{G,\{\mu\}}$ denote the functor on schemes on $\mathfrak{X}$, which associates to $\mathfrak{Y} \xrightarrow{f} \mathfrak{X}$ the set of $\{\mu\}$-filtration of $f^*\mathscr{E}$ with respect to (f^*t_α). We write $\mathtt{Fl}^{\mathscr{E}}_{\{\mu\}} := \mathtt{Fl}^{\mathscr{E},\emptyset}_{\mathrm{GL}(M),\{\mu\}}$, and we use the same letter to denote the the scheme representing the functor, which is relative projective and smooth over $\mathfrak{X}$.

Lemma 2.5.6. *The natural inclusion $\mathtt{Fl}^{\mathscr{E},(t_\alpha)}_{G,\{\mu\}} \hookrightarrow \mathtt{Fl}^{\mathscr{E}}_{\{\mu\}}$ can be represented by a closed immersion of schemes over $\mathfrak{X}$. Furthermore, if $P_{\mathfrak{X}}$ (as in (2.5.4)) is a G-torsor, then $\mathtt{Fl}^{\mathscr{E},(t_\alpha)}_{G,\{\mu\}}$ is representable by a (non-empty) connected scheme which is relatively projective smooth over $\mathfrak{X}$.*

Proof. This follows from étale descent of closed immersions; *cf.* the proof of [15, Lemma 2.2.6]. □

2.6. **Review on G-isocrystals.**

Definition 2.6.1. Let $\mathbf{D}$ be a pro-torus with character group $X^*(\mathbf{D}) = \mathbb{Q}$; i.e., $\mathbf{D} = \varprojlim \mathbb{G}_m$ where the transition maps is the Nth power maps ordered by divisibility.

For any morphism $\nu : \mathbf{D} \to \mathrm{GL}(n)_{K_0}$ we obtain a $\mathbb{Q}$-grading of K_0^n by the weight decomposition. More explicitly, choose an integer r such that $r\nu$ factors through $\mathbb{G}_m$. Then the (d/r)th grading of K_0^n is the subspace where the action of $(r\nu)(z)$ for $z \in \mathbb{G}_m(K_0)$ coincides with the scalar multiplication of z^d.

Proposition 2.6.2 (Kottwitz [21])**.** *Let G be a connected reductive group over $\mathbb{Q}_p$, and $K_0 = W(\overline{\mathbb{F}}_p)_{\mathbb{Q}}$. Then, for each $b \in G(K_0)$, there exists a unique homomorphism*

$$\nu_b : \mathbf{D} \to G_{K_0}$$

such that for any representation $\rho : G_{K_0} \to \mathrm{GL}(n)_{K_0}$ the $\mathbb{Q}$-grading associated to $\rho \circ \nu_b$ is the slope decomposition for $(K_0^n, b\sigma)$. The $G(K_0)$-conjugacy class of ν_b only depends on the σ-conjugacy class of b in $G(K_0)$.

Furthermore, any σ-conjugacy class of $G(K_0)$ contains an element $b \in G(K_0)$ which satisfy the following "decency equation" for some $r \in \mathbb{Z}$:

$$(b\sigma)^r = (r\nu_b)(p)\sigma^r, \tag{2.6.3}$$

where the equality takes place in $\langle\sigma\rangle \ltimes G(K_0)$.

It follows (from the uniqueness assertion) that for any $g, b \in G(K_0)$ we have $\nu_{gb\sigma(g)^{-1}} = g\nu_b g^{-1}$.

Consider the following group valued functor $J_b = J_{G,b}$ defined as follows:

$$J_b(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} K_0) |\ gb\sigma(g)^{-1} = b\} \tag{2.6.4}$$

for any $\mathbb{Q}_p$-algebra R. Note that for any $g, b \in G(K_0)$ we have $J_{gb\sigma(g)^{-1}}(R) = gJ_b(R)g^{-1}$ as a subgroup of $G(R \otimes_{\mathbb{Q}_p} K_0)$; in particular, J_b essentially depends only on the σ-conjugacy class of b in $G(K_0)$.

Proposition 2.6.5. *Assume that $b \in G(K_0)$ satisfies the decency equation (2.6.3) for $r \in \mathbb{Z}$. Then we have $(r\nu_b)(p) \in G(\mathbb{Q}_{p^r}) \cap J_b(\mathbb{Q}_p)$, where the intersection takes place* *in $G(K_0)$, and J_b is representable by an inner form of the centraliser $G_{(r\nu_b)(p)}$, which is a Levi subgroup of G. (In particular, $(r\nu_b)(p)$ lies in the centre of $J_b(\mathbb{Q}_p)$.)*

Proof. See [30, Corollaries 1.9, 1.14] for the proof. $\square$

Definition 2.6.6. We say that $b \in G(K_0)$ is *basic* if the following equivalent conditions are satisfied:

(1) The slope morphism ν_b factors through the centre of G.
(2) The group J_b is an inner form of G.

(The equivalence is clear from Proposition 2.6.5.) The definition only depends on the σ-conjugacy class of b in $G(K_0)$.

2.7. **Review of Dieudonné crystals.** Let $\mathfrak{X}$ be a formal scheme over $\operatorname{Spf}\mathbb{Z}_p$, and consider the crystalline site $(\mathfrak{X}/\mathbb{Z}_p)$. By an *isocrystal* over $\mathfrak{X}$, we mean an object in the isogeny category of crystals of quasi-coherent $\mathcal{O}_{(\mathfrak{X}/\mathbb{Z}_p)}$-modules. For any crystal of quasi-coherent $\mathcal{O}_{(\mathfrak{X}/\mathbb{Z}_p)}$-modules $\mathbb{D}$, we let $\mathbb{D}[\frac{1}{p}]$ denote the associated isocrystal.

For a p-divisible group X over $\mathfrak{X}$, we have a contravariant Dieudonné crystal[3] $\mathbb{D}(X)$ equipped with a filtration $(\operatorname{Lie}X)^* \cong \operatorname{Fil}^1_X \subset \mathbb{D}(X)_{\mathfrak{X}}$ by a subvector bundle, where $\mathbb{D}(X)_{\mathfrak{X}}$ is the pull-back of $\mathbb{D}(X)$ to the Zariski site of $\mathfrak{X}$. We call Fil^1_X the *Hodge filtration* for X. If $\mathfrak{X} = \operatorname{Spf}R$, then we can regard the Hodge filtration as a filtration on the R-sections $\operatorname{Fil}^1_X \subset \mathbb{D}(X)(R)$. From the relative Frobenius morphism $F : X_{\overline{\mathfrak{X}}} \to \sigma^* X_{\overline{\mathfrak{X}}}$, we obtain the Frobenius morphism $F : \sigma^*\mathbb{D}(X) \to \mathbb{D}(X)$. On tensor products of $\mathbb{D}(X)$'s, we naturally extend the Frobenius structure and filtration.

If $X = \mathcal{A}[p^\infty]$ for some abelian scheme $f : \mathcal{A} \to \mathfrak{X}$, then we have $\mathbb{D}(X) \cong R^1 f_{\mathrm{CRIS},*}\mathcal{O}_{(\mathcal{A}/\mathbb{Z}_p)}$, where the Frobenius morphism F on $\mathbb{D}(X)$ matches with the crystalline Frobenius on the right hand side. Furthermore, restricting the isomorphism to the Zariski site, we obtain a filtered isomorphism between the vector bundle $\mathbb{D}(X)_{\mathfrak{X}}$ and the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathfrak{X}) = \mathcal{V}(\mathcal{A})^*$ (both equipped with the Hodge filtration).

Let $\mathbf{1} := \mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$ and $\mathbf{1}(-1) := \mathbb{D}(\mu_{p^\infty})$. We set

- $\mathbf{1}(-c) := \mathbf{1}(-1)^{\otimes c}$ if $c > 0$;
- $\mathbf{1}(-c) := (\mathbf{1}(-1)^*)^{\otimes |c|}$ if $c < 0$;
- $\mathbf{1}(0) := \mathbf{1}$.

We will often use the same notation $\mathbf{1}(-c)$ for the isocrystal associated to it. Note that the underlying crystal of $\mathbf{1}(-c)$ is the structure sheaf $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ with $F = p^c\,\mathrm{id}$ (identifying $\sigma^*\mathbf{1}(-c)$ with $\mathcal{O}_{\mathfrak{X}/\mathbb{Z}_p}$ as well)[4]. The Hodge filtration on $\mathbf{1}(-c)_{\mathfrak{X}}$ is concentrated at degree c.

We now define $\mathbb{D}(X)^{\otimes}$ as in §2.2 by letting $\mathcal{C}$ the category of crystals of quasi-coherent $\mathcal{O}_{(\mathfrak{X}/\mathbb{Z}_p)}$-modules and $\mathcal{D} \subset \mathcal{C}$ be the full subcategory of finitely generated locally free objects. Then the Hodge filtration on $\mathbb{D}(X)_{\mathfrak{X}}$ induces a natural filtration on $\mathbb{D}(X)^{\otimes}_{\mathfrak{X}}$, and the Frobenius morphism on $\mathbb{D}(X)$ induces an isomorphism of isocrystals $F : \sigma^*\mathbb{D}(X)^{\otimes}[\frac{1}{p}] \xrightarrow{\sim} \mathbb{D}(X)^{\otimes}[\frac{1}{p}]$.

Definition 2.7.1. Let X be a p-divisible group over a formal scheme $\mathfrak{X}$ over $\operatorname{Spf}\mathbb{Z}_p$. For a morphism of crystals $t : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$, we let $t_{\mathrm{dR}} : \mathbf{1} \to \mathbb{D}(X)^{\otimes}_{\mathfrak{X}}$ denote the pull-back of t to the Zariski site. By abuse of notation, we also denote by $t_{\mathrm{dR}} \in \Gamma(\mathfrak{X}, \mathbb{D}(X)^{\otimes}_{\mathfrak{X}})$ the image of $1 \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ by t_{dR}.

Definition 2.7.2. Let X be a p-divisible group over a formal scheme $\mathfrak{X}$ over $\operatorname{Spf}\mathbb{Z}_p$. A *crystalline Tate tensor* on X is a morphism of crystals $t : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$, which satisfies the following properties:

[3] See [25], [24], or [1] for the construction of $\mathbb{D}(X)$ and the extra structure.

[4] Note that such F is only defined up to isogeny if $c < 0$

(1) The map on isocrystals $\mathbf{1} \to \mathbb{D}(X)^{\otimes}[\frac{1}{p}]$, induced t, is F-equivariant.
(2) The map t_{dR} is a de Rham tensor; i.e., the section $t_{\mathrm{dR}} \in \Gamma(\mathfrak{X}, \mathbb{D}(X)^{\otimes}_{\mathfrak{X}})$ lies in the 0th filtration with respect to the filtration induced by $\mathrm{Fil}^1_X \subset \mathbb{D}(X)_{\mathfrak{X}}$.

Let $\mathfrak{X}$ be a formal scheme over $\mathrm{Spf}\,\mathbb{Z}_p$, and $f : \mathcal{A} \to \mathfrak{X}$ be an abelian scheme. Then a *crystalline Tate tensor* on A is a morphism of crystals

$$t : \mathbf{1} \to (R^1 f_{\mathrm{CRIS},*}\mathcal{O}_{\mathcal{A}/\mathbb{Z}_p})^{\otimes} = \mathbb{D}(\mathcal{A}[p^\infty])^{\otimes}, \tag{2.7.3}$$

which is a crystalline Tate tensor on $\mathcal{A}[p^\infty]$.

Example 2.7.4. Given an endomorphism of p-divisible groups $f : X \to X$ we obtain a morphism of crystals $\mathbb{D}(f) : \mathbb{D}(X) \to \mathbb{D}(X)$, which gives rise to the following crystalline Tate tensor:

$$t_f : \mathbf{1} \to \mathbb{D}(X) \otimes \mathbb{D}(X)^* \subset \mathbb{D}(X)^{\otimes}.$$

To a principal polarisation $\lambda : X \xrightarrow{\sim} X^\vee$ one can associate a crystalline Tate tensor $t_\lambda : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ by the same recipe as in [15, Example 2.1.4].

3. Review on Shimura varieties of Hodge type

We review basic results on Shimura varieties of Hodge type and their integral models in the good reduction case. Our notation is a global analogue of the notation introduced in [15, §2]. In §4 we recall the main results of [15].

3.1. **Review of Shimura varieties of Hodge type in characteristic** 0**.** Consider a $2g$-dimensional $\mathbb{Q}$-vector space V, equipped with a nondegenerate alternating bilinear form (i.e., a symplectic form) $\psi : V \times V \to \mathbb{Q}$. Consider the symplectic similitude group $\mathrm{GSp}(V,\psi)$ which is a connected reductive group. One can find an $\mathbb{R}$-basis of $V_{\mathbb{R}}$ so that the matrix representation of $V_{\mathbb{R}}$ is $J := \begin{pmatrix} & \mathrm{id}_g \\ -\mathrm{id}_g & \end{pmatrix}$, which identifies $\mathrm{GSp}(V,\psi)_{\mathbb{R}}$ with $\mathrm{GSp}_{2g/\mathbb{R}}$ defined by $(\mathbb{R}^{2g}, J)$. Let $\boldsymbol{\mathcal{S}}^{\pm}$ be the set of $\mathrm{GSp}_{2g}(\mathbb{R})$-conjugates of the cocharacter $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \to \mathrm{GSp}_{2g/\mathbb{R}}$ which induces the following on the $\mathbb{R}$-points:

$$\mathbb{C}^\times \to \mathrm{GSp}_{2g}(\mathbb{R}); \quad a+bi \mapsto \begin{pmatrix} a\,\mathrm{id}_g & b\,\mathrm{id}_g \\ -b\,\mathrm{id}_g & a\,\mathrm{id}_g \end{pmatrix}.$$

Then $(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$ is a Shimura datum, often referred to as a *Siegel Shimura datum*. Its reflex field is $\mathbb{Q}$.

Definition 3.1.1. A Shimura datum $(G,\mathfrak{H})$ is called *of Hodge type* if there is an embedding of Shimura data

$$(G,\mathfrak{H}) \hookrightarrow (\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$$

for some rational symplectic vector space (V,ψ).

Clearly, $(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$ is of Hodge type. More generally, PEL-type Shimura data (*cf.* [8, §4], [22, §4]) are of Hodge type.

3.1.2. Let $(G,\mathfrak{H})$ be a Shimura datum of Hodge type. To simplify the notation, let $E := E(G,\mathfrak{H})$ denote the reflex field and we write $\mathrm{Sh}_{\mathsf{K}} := \mathrm{Sh}_{\mathsf{K}}(G,\mathfrak{H})$ to denote the canonical model over E. We fix an embedding $(G,\mathfrak{H}) \hookrightarrow (\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$, and let $\mathsf{K} \subset G(\mathbb{A}_{\mathrm{f}})$ and $\mathsf{K}' \subset \mathrm{GSp}(V,\psi)(\mathbb{A}_{\mathrm{f}})$ be "small enough" open compact subgroups with $\mathsf{K} \subset \mathsf{K}'$ such that the natural map $\mathrm{Sh}_{\mathsf{K}}(G,\mathfrak{H}) \to \mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})_{E(G,\mathfrak{H})}$ is a closed immersion. (Indeed, up to replacing $\mathsf{K} \subset G(\mathbb{A}_{\mathrm{f}})$ with some finite-index open subgroup it is always possible to find K' as above; *cf.* [8, Proposition 1.15].)

Recall that $\mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$ can be interpreted as a moduli space of polarised complex abelian varieties with level structure, so we have a universal abelian scheme $\mathcal{A}_{\mathsf{K}',\mathbb{Q}} \to \mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$ defined up to isogeny. By restriction, we obtain an abelian scheme $f : \mathcal{A}_{\mathsf{K},E} \to \mathrm{Sh}_{\mathsf{K}}(G,\mathfrak{H})$. Pulling back by $E \hookrightarrow \mathbb{C}$, we can explicitly write down a "universal abelian scheme (up to isogeny)" $f : \mathcal{A}_{\mathsf{K},\mathbb{C}} \to \mathrm{Sh}_{\mathsf{K}}(G,\mathfrak{H})_{\mathbb{C}}$ in terms of the associated variation of $\mathbb{Q}$-Hodge structures. First, the first Betti homology can be obtained as follows:

(3.1.3) $$\mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}}) = G(\mathbb{Q})\backslash(V \times \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}}))/\mathsf{K}, \qquad \textit{cf. } (2.4.3),$$

where $G(\mathbb{Q})$ acts diagonally and K acts only on $G(\mathbb{A}_{\mathrm{f}})$. To define the Hodge filtration, consider the following filtration $\mathrm{Fil}^\bullet_{\mathfrak{H}}$ of $V \times \mathfrak{H}$ whose fibre at $h \in \mathfrak{H}$ is given by grading induced from the cocharacter $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$ (in the sense of Definition 2.5.3), where μ_h is as below:

(3.1.4) $$\mu_h : \mathbb{G}_m \xrightarrow{z\mapsto(z,1)} \mathbb{G}_m \times \mathbb{G}_m \cong \mathbb{S}_{\mathbb{C}} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}}.$$

Then $\mathrm{Fil}^\bullet_{\mathfrak{H}}$ descends to a holomorphic filtration $\mathrm{Fil}^\bullet_{\mathsf{K}}$ of $\mathcal{O}_{\mathrm{Sh}_{\mathsf{K},\mathbb{C}}} \otimes_{\mathbb{Q}} \mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})$. This define a variation of $\mathbb{Q}$-Hodge structures that defines $\mathcal{A}_{\mathsf{K},\mathbb{C}}$.

Lemma 3.1.5. *Let $s \in V^{\otimes}$ be an element fixed by G. Then the morphism $\mathbf{1} \to V^{\otimes}$ defined by $1 \mapsto s$ induce a morphism $t_B^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})^{\otimes}$ of "variations of $\mathbb{Q}$-Hodge structures"[5]. Furthermore, such t_B^{univ} is compatible under the natural projection maps of the tower $\{\mathrm{Sh}_{\mathsf{K}}(G,\mathfrak{H})_{\mathbb{C}}\}_{\mathsf{K}\subset G(\mathbb{A}_{\mathrm{f}})}$, and are invariant under the natural $G(\mathbb{A}_{\mathrm{f}})$-action.*

Proof. If s is fixed by G, then the global section $s \in \Gamma(\mathfrak{H}, V^{\otimes} \times \mathfrak{H})$ induce a global section s of $\mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})$. Therefore we obtain a $\mathbb{Q}$-linear morphism of locally constant sheaves $t_B^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})^{\otimes}$. To show that the image of this map is in the 0th filtration, it suffices to show the claim over $\mathfrak{H}$, but by definition of the filtration $\mathrm{Fil}^\bullet_{\mathfrak{H}}$ the global section s has to lie in the 0th filtration (as s is fixed by G). The last assertion (on the compatibility with the tower and the Hecke $G(\mathbb{A}_{\mathrm{f}})$-action) is clear. □

Lemma 3.1.6. *Let $s \in V^{\otimes}$ be an element fixed by G, and $t_B^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})^{\otimes}$ be the morphism constructed from s by the recipe in Lemma 3.1.5. Then the de Rham component $t_{\mathrm{dR}}^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},\mathbb{C}})^{\otimes}$ of t_B^{univ}* (cf. *(2.4.4b)) descends to a de Rham tensor $t_{\mathrm{dR}}^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},E})^{\otimes}$, and the étale component $t_{\text{ét}}^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K},\mathbb{C}})^{\otimes}$ of t_B^{univ}* (cf. *(2.4.4a)) descends to an étale tensor $t_{\text{ét}}^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K},E})^{\otimes}$.*

Proof. This lemma is essentially proved in Lemma 2.2.1 and Corollary 2.2.2 in [17], by choosing a finitely many tensors $(s_\alpha) \subset V^{\otimes}$ such that their pointwise stabiliser is G and one of s_α is s. We now explain how to deduce the lemma from *loc. cit.*

The existence of the de Rham tensor $t_{\mathrm{dR}}^{\mathrm{univ}}$ on $\mathcal{A}_{\mathsf{K},E}$ is proved in [17, Corollary 2.2.2]. Let us now prove the assertion on the étale components. Let η be a generic point of Sh_{K}, and $\bar\eta$ be a geometric point supported at η. By [17, Lemma 2.2.1], the fibre $t_{\text{ét},\bar\eta}^{\mathrm{univ}}$ is invariant under the action of $\mathrm{Gal}(\bar\eta/\eta)$.

Let $\overline{\{\eta\}} \subset \mathrm{Sh}_{\mathsf{K}}$ be the connected component of Sh_{K} containing η. Since $\pi_1^{\text{ét}}(\overline{\{\eta\}}, \bar\eta)$ is a quotient of $\mathrm{Gal}(\bar\eta/\eta)$ by normality, it follows $t_{\text{ét},\eta}^{\mathrm{univ}}$ extends over $\overline{\{\eta\}}$. Hence we obtain $t_{\text{ét}}^{\mathrm{univ}}$ over Sh_{K} by repeating this process for each of the generic points. □

3.2. **"Universal" abelian schemes over Hodge-type Shimura varieties (in characteristic 0).** We fix finitely many elements $(s_\alpha) \subset V^{\otimes}$ whose pointwise stabiliser is $G \subset \mathrm{GL}_{\mathbb{Q}}(V)$; *cf.* Proposition 2.5.1. Consider $t_{\alpha,B}^{\mathrm{univ}} : \mathbf{1} \to \mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})^{\otimes}$ associated to (s_α) by Lemma 3.1.5, which produce $t_{\alpha,\mathrm{dR}}^{\mathrm{univ}}$ and $t_{\alpha,\text{ét}}^{\mathrm{univ}}$ defined over Sh_{K} by Lemma 3.1.6. We now list the properties and extra structures possessed by $\mathcal{A}_{\mathsf{K},E}$.

[5]To be precise, there is a finite-rank direct factor in $\mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})^{\otimes}$ which is a variation of $\mathbb{Q}$-Hodge structures, such that t_B^{univ} factors through and induces a morphism of variations of $\mathbb{Q}$-Hodge structures. The subscript in t_B^{univ} stands for Betti.

3.2.1 (G-torsor). Consider the following closed subscheme of the isom scheme over Sh_{K}

$$P_{\mathsf{K},E} \subset \mathrm{isom}_{\mathrm{Sh}_{\mathsf{K}}}\left(\mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},E}), \mathcal{O}_{\mathrm{Sh}_{\mathsf{K}}} \otimes_{\mathbb{Q}} V\right),$$

defined by the condition of matching $(t^{\mathrm{univ}}_{\alpha,\mathrm{dR}}) \subset \Gamma(\mathrm{Sh}_{\mathsf{K}}, \mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},E})^{\otimes})$ with $(s_\alpha) \subset V^{\otimes}$ for each α. Then $P_{\mathsf{K},E}$ is a G-torsor. Indeed, it suffices to show that $P_{\mathsf{K},\mathbb{C}}$ is a G-torsor over $\mathrm{Sh}_{\mathsf{K},\mathbb{C}}$. On the other hand, $P_{\mathsf{K},\mathbb{C}}$ splits under the complex analytic topology (which can be seen from the natural isomorphism $\mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},\mathbb{C}}) \cong \mathcal{O}_{\mathrm{Sh}_{\mathsf{K},\mathbb{C}}} \otimes_{\mathbb{Q}} \mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})$ and the explicit construction of $\mathcal{V}(\mathcal{A}_{\mathsf{K},\mathbb{C}})$), so $P_{\mathsf{K},\mathbb{C}} \to \mathrm{Sh}_{\mathsf{K},\mathbb{C}}$ is flat with non-empty fibre everywhere and the natural G-action is simple and transitive.

3.2.2 (The Hodge filtration is a $\{\mu\}$-filtration.). Let $\mu : \mathbb{G}_m \to G_E$ be the cocharacter such that $\mu_{\mathbb{C}}$ belongs to a conjugacy class associated to some $h \in \mathfrak{H}$ by (3.1.4). (Such a μ exists by definition of $E = E(G, \mathfrak{H})$.) Then the Hodge filtration $\mathrm{Fil}^0(\mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},E}))$ is a $\{\mu\}$-filtration with respect to $(t^{\mathrm{univ}}_{\alpha,\mathrm{dR}})$ in the sense of Definition 2.5.5. Indeed, since Sh_{K} is reduced and of finite type, it suffices (by Lemma 2.5.6) to show that at each closed point $x \in \mathrm{Sh}_{\mathsf{K}}(\mathbb{C})$ the fibre

$$\mathrm{Fil}^0(\mathcal{V}_{\mathsf{K},x}) \subset \mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},E})_x \cong \mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},x}/\mathbb{C})^*$$

defines a point in $\mathtt{Fl}^{\mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K},E}),(t^{\mathrm{univ}}_{\alpha,\mathrm{dR}})}_{G,\{\mu\}}$ over $x \in \mathrm{Sh}_{\mathsf{K}}(\mathbb{C})$. Indeed, this is clear from the definition, as the Hodge filtration at x is given by the cocharacter μ_h associated to some $h \in \mathfrak{H}$.

3.2.3 (Level Structure). For an open compact subgroup $\mathsf{K} \subset G(\mathbb{A}_{\mathrm{f}})$, we will define a universal global section

$$\eta_{\mathsf{K}} \in \Gamma\left(\mathrm{Sh}_{\mathsf{K}}, \mathrm{isom}\left[(V_{\mathbb{A}_{\mathrm{f}}}, (1\otimes s_\alpha)), (\mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K},E}), (t^{\mathrm{univ}}_{\alpha,\text{ét}}))\right]/\mathsf{K}\right),$$

where $V_{\mathbb{A}_{\mathrm{f}}} := \mathbb{A}_{\mathrm{f}} \otimes_{\mathbb{Q}} V$. Note that η_K only depends on the isogeny class of $(\mathcal{A}_{\mathsf{K},E}, (t^{\mathrm{univ}}_{\alpha,\text{ét}}))$; i.e., $\mathcal{A}_{\mathsf{K},E}$ up to isogeny respecting $(t^{\mathrm{univ}}_{\alpha,\text{ét}})$.

For a geometric point $x \in \mathrm{Sh}_{\mathsf{K}}(\mathbb{C})$, let $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K}}, x)$ and $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K},\mathbb{C}}, x)$ to denote the étale fundamental group of the component containing x. Then defining η_{K} is equivalent to giving, for a point $x \in \mathrm{Sh}_{\mathsf{K}}(\mathbb{C})$ on each connected component, an isomorphism

$$\eta_x : V_{\mathbb{A}_{\mathrm{f}}} \xrightarrow{\sim} \mathcal{V}_{\mathbb{A}_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K},E})_x,$$

matching $(1\otimes s_\alpha)$ and $(t^{\mathrm{univ}}_{\alpha,,\text{ét},x})$, such that the right coset $\eta_x\mathsf{K}$ is stable under the action of $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K}}, x)$.

Note that the pull-back of $\mathcal{V}_{\mathsf{K}}$ to $\mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}})$ is a trivial local system. We first define $\widetilde{\eta} : V_{\mathbb{A}_{\mathrm{f}}} \times \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}}) \to V_{\mathbb{A}_{\mathrm{f}}} \times \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}})$ by $(v, h, g) \mapsto (gv, h, g)$. Given a point $x \in \mathrm{Sh}_{\mathsf{K}}(\mathbb{C})$, we pick a lift $(h, g) \in \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}})$ of x and set

$$\eta_x := \widetilde{\eta}|_{V_{\mathbb{A}_{\mathrm{f}}} \times \{(h,g)\}} : V_{\mathbb{A}_{\mathrm{f}}} \to V_{\mathbb{A}_{\mathrm{f}}}.$$

We now show that the right coset $\eta_x\mathsf{K}$ only depends on x, not on the choice of lift (h, g), where $g' \in \mathsf{K}$ acts as $\eta_x \mapsto \eta_x \circ g'$.

Firstly, for any $g' \in \mathsf{K}$ we obtain another lift $(h, gg') \in \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}})$ of x. Then we have

$$\widetilde{\eta}(v, h, gg'\mathsf{K}) = (gg'v, h, gg'\mathsf{K}) = \widetilde{\eta}(g'v, h, g\mathsf{K}) \in V_{\mathbb{A}_{\mathrm{f}}} \times \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}})/\mathsf{K};$$

i.e., we have $\widetilde{\eta}_{(h,gg'\mathsf{K})} = \eta_{(h,g\mathsf{K})} \circ g'$.

Secondly, for any $\gamma \in G(\mathbb{Q})$ we obtain another lift $({}^\gamma h, \gamma g) \in \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}})$ of x. Then we have

$$\widetilde{\eta}(v, {}^\gamma h, \gamma g\mathsf{K}) = (\gamma gv, {}^\gamma h, \gamma g\mathsf{K}) = (\gamma, \gamma, \gamma) \circ \widetilde{\eta}(v, h, g\mathsf{K}).$$

Now, recall that $\mathcal{V}_{\mathsf{K}} = G(\mathbb{Q})\backslash(V \times \mathfrak{H} \times G(\mathbb{A}_{\mathrm{f}}))/\mathsf{K}$ where $G(\mathbb{Q})$ acts diagonally. Therefore, we obtain the same map η_x if we replace (h, g) with $({}^\gamma h, \gamma g)$.

This shows that the right coset $\eta_x\mathsf{K}$ is stable under the action of $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K},\mathbb{C}}, x)$. We now show that $\eta_x\mathsf{K}$ is stable under the action of $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K}}, x)$. Clearly, we may replace K with a finite-index open normal subgroup, so we may assume that there exists a "small enough" open compact subgroup $\mathsf{K}' \subset \mathrm{GSp}(V,\psi)(\mathbb{A}_\mathrm{f})$ containing K such that $\mathrm{Sh}_{\mathsf{K}} \to \mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})_E$ is a closed immersion. Then $\eta_x\mathsf{K}'$ defines a universal level structure on $\mathcal{A}_{\mathsf{K}',\mathbb{C}}$, so it "descents" to a level structure on $\mathcal{A}_{\mathsf{K}',E}$ (by the universal property of $\mathcal{A}_{\mathsf{K}',E}$). In particular, $\eta_x\mathsf{K}'$ is stable under the action of $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K}}, x)$. But since $\mathrm{Sh}_{\mathsf{K}},(\mathbb{C}) \to \mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})(\mathbb{C})$ is injective, $\eta_x\mathsf{K}$ is the only right K-coset contained in $\eta_x\mathsf{K}'$ whose elements match $(1\otimes s_\alpha)$ and $(t^{\mathrm{univ}}_{\alpha,\text{ét},x})$. (Indeed, if there were any other K-coset $\eta_y\mathsf{K} \subset \eta_x\mathsf{K}'$ with this property, then $\eta_x\mathsf{K}$ and $\eta_y\mathsf{K}$ define $\mathbb{C}$-points of Sh_{K} which map to the same point in $\mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$.) Since $(t^{\mathrm{univ}}_{\alpha,\text{ét},x})$ are invariant under the action of $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K}}, x)$ by Lemma 3.1.6, it also follows that $\eta_x\mathsf{K}$ is stable under the action of $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K}}, x)$.

3.2.4 (Hecke action). For any $\mathsf{K} \subset G(\mathbb{A}_\mathrm{f})$, the right translation by $g \in G(\mathbb{A}_\mathrm{f})$ on $\mathfrak{H} \times G(\mathbb{A}_\mathrm{f})$ descends to an isomorphism

$$[g] : \mathrm{Sh}_{g\mathsf{K}g^{-1},\mathbb{C}} \xrightarrow{\sim} \mathrm{Sh}_{\mathsf{K},\mathbb{C}}\,.$$

By the standard rigidity result (*cf.* [27, Theorem 13.6]), this map is defined over the reflex field

$$[g] : \mathrm{Sh}_{g\mathsf{K}g^{-1}} \xrightarrow{\sim} \mathrm{Sh}_{\mathsf{K}}\,.$$

We can describe the pull-back by $[g]$ of the universal abelian scheme and the level structure $(\mathcal{A}_{\mathsf{K}}, \eta_{\mathsf{K}})$ as follows. The isogeny class of $[g]^*(\mathcal{A}_{\mathsf{K}}, (t^{\mathrm{univ}}_{\alpha,\text{ét}}))$ coincides with $(\mathcal{A}_{g\mathsf{K}g^{-1}}, (t^{\mathrm{univ}}_{\alpha,\text{ét}}))$, and $[g]^*\eta_{\mathsf{K}}$ corresponds to $(\eta'_x g)\mathsf{K}$ for any $x \in \mathrm{Sh}_{g\mathsf{K}g^{-1}}(\mathbb{C})$ where $\eta'_x : V_{\mathbb{A}_\mathrm{f}} \xrightarrow{\sim} \mathcal{V}_{\mathbb{A}_\mathrm{f}}(\mathcal{A}_{g\mathsf{K}g^{-1}})_x$ is a representative of the fibre of $\eta_{g\mathsf{K}g^{-1}}$ at x. (These claims can be explicitly verified over $\mathbb{C}$.)

3.3. **Integral canonical models.** In this section, we review the basic properties of integral canonical models of Hodge-type Shimura varieties in the good reduction case, constructed independently by Kisin [17] and Vasiu [31, 32, 33]. We refer to the aforementioned references for the full details including the definition of integral canonical model.

3.3.1 (Good Reduction Hypothesis). From now on, we fix a prime p. Let $(G,\mathfrak{H})$ be a Hodge-type Shimura datum, and assume that G admits a reductive $\mathbb{Z}_{(p)}$-model $G_{\mathbb{Z}_{(p)}}$. Then we can choose the following extra data:

(1) We choose an embedding of Shimura data $(G,\mathfrak{H}) \hookrightarrow (\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$, and a ψ-stable $\mathbb{Z}_{(p)}$-lattice $\Lambda_{\mathbb{Z}_{(p)}} \subset V$ such that the closed immersion $G \hookrightarrow \mathrm{GSp}(V,\psi) \hookrightarrow \mathrm{GL}(V)$ over $\mathbb{Q}$ extends to a closed immersion $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(\Lambda_{\mathbb{Z}_{(p)}})$ of reductive groups over $\mathbb{Z}_{(p)}$. If $p > 2$ then for any embedding $(G,\mathfrak{H}) \hookrightarrow (\mathrm{GSp}(V,\psi), \boldsymbol{\mathcal{S}}^{\pm})$ there exists a lattice $\Lambda_{\mathbb{Z}_{(p)}}$ with the above property by [17, Proposition 2.3.1].[6]

(2) We choose finitely many elements $(s_\alpha) \subset \Lambda^{\otimes}_{\mathbb{Z}_{(p)}}$ such that the pointwise stabiliser of (s_α) in $\mathrm{GL}(\Lambda_{\mathbb{Z}_{(p)}})$ is $G_{\mathbb{Z}_{(p)}}$, which is possible by Proposition 2.5.1.

We do not require ψ to be a perfect alternating form on $\Lambda_{\mathbb{Z}_{(p)}}$, although by Zarhin's trick it is possible to arrange (V,ψ) so that ψ induces a perfect alternating form on some choice of $\Lambda_{\mathbb{Z}_{(p)}}$.

The following lemma is proved in [26, Corollary 4.7]:

[6]Indeed, [17, Proposition 2.3.1] asserts that for any $G(\mathbb{Z}_p)$-stable $\mathbb{Z}_p$-lattice $\Lambda \subset \mathbb{Q}_p \otimes_{\mathbb{Q}} V$, $\Lambda_{\mathbb{Z}_{(p)}} := \Lambda \cap V$ satisfies the desired property. Note that [17, Proposition 2.3.1] also proves the claim when $p = 2$ and $G_{\overline{\mathbb{Q}}}$ has no normal subgroup isomorphic to SO_{2n+1}.

Lemma 3.3.2. *Let $(G,\mathfrak{H})$ be any Shimura datum. Assume that G is unramified at p; i.e., there exists a reductive $\mathbb{Z}_{(p)}$-model of G. Then the reflex field $E(G,\mathfrak{H})$ is unramified at any prime $\mathfrak{p}$ over p.*

Recall that we fix a reductive $\mathbb{Z}_{(p)}$-model of G, also denoted by G, which is a closed subgroup of $\mathrm{GL}(\Lambda_{\mathbb{Z}_{(p)}})$. Set $\mathsf{K}_p := G(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$, which is a hyperspecial maximal compact subgroup. Choose an open compact subgroup $\mathsf{K}^p \subset G(\mathbb{A}^p_{\mathrm{f}})$ so that the product $\mathsf{K}_p\mathsf{K}^p \subset G(\mathbb{A}_{\mathrm{f}})$ is "small enough". From now on, we always assume that $\mathsf{K} := \mathsf{K}_p\mathsf{K}^p$ with *the* hyperspecial maximal compact subgroup K_p, in which case we expect that Sh_{K} should admit a smooth integral model over $\mathscr{O}_{E,(\mathfrak{p})}$ for any prime $\mathfrak{p}$ of $E := E(G,\mathfrak{H})$ over p. Here, $\mathscr{O}_{E,(\mathfrak{p})}$ is the (uncompleted) localisation of $\mathscr{O}_E$ at $\mathfrak{p}$, which is an unramified extension of $\mathbb{Z}_{(p)}$ by Lemma 3.3.2.

Let us recall the main result and basic properties on integral canonical models:

Theorem 3.3.3 (Vasiu, Kisin)**.** *Assume that $p > 2$.*[7] *Then for any small enough K^p there exists an integral canonical $\mathscr{O}_{E,(\mathfrak{p})}$-model $\mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}$ of $\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$ in the following sense: The $G(\mathbb{A}^p_{\mathrm{f}})$-equivariant tower $\{\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}\}_{\mathsf{K}^p}$ extends to a $G(\mathbb{A}^p_{\mathrm{f}})$-equivariant tower $\{\mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}\}_{\mathsf{K}^p}$ of smooth $\mathscr{O}_{E,(\mathfrak{p})}$-schemes with finite étale transition maps, and the tower satisfies the (uniquely characterising) extension property formulated by Milne (*cf. [28, §3]*).*

Furthermore, the following additional properties hold:

1. *Choose a ψ-stable $\mathbb{Z}$-lattice $\Lambda_{\mathbb{Z}} \subset \Lambda_{\mathbb{Z}_{(p)}}$. Then the universal abelian scheme $\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p,E} \to \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$, corresponding to the choice of $\mathbb{Z}$-lattice $\Lambda_{\mathbb{Z}}$, extends to an abelian scheme $\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p} \to \mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}$.*
2. *The de Rham tensors $(t^{\mathrm{univ}}_{\alpha,\mathrm{dR}})$ on $\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p,E}$, associated to $(s_\alpha) \subset \Lambda^{\otimes}_{\mathbb{Z}_{(p)}}$ by Lemma 3.1.6, extends over the integral canonical model $t^{\mathrm{univ}}_{\alpha,\mathrm{dR}} : \mathbf{1} \to \mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p})^{\otimes}$. Furthermore, the formation of $t^{\mathrm{univ}}_{\alpha,\mathrm{dR}}$ respects the natural projections and the natural $G(\mathbb{A}^p)$-action on the tower $\{\mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}\}_{\mathsf{K}^p}$.*

Proof. Vasiu ([31], [32, 33]) and Kisin [17] constructed an integral canonical model $\mathscr{S}_{\mathsf{K}} = \mathscr{S}_{\mathsf{K}}(G,\mathfrak{H})$ of Sh_{K} by normalising Sh_{K} in a certain integral model of $\mathrm{Sh}_{\mathsf{K}'}(\mathrm{GSp}(V,\psi),\boldsymbol{\mathcal{S}}^{\pm})$ (for a suitable choice of $\mathsf{K}' \subset \mathrm{GSp}(V,\psi)(\mathbb{A}_{\mathrm{f}})$) constructed from a Mumford moduli scheme. (The main content is to verify that this construction yields an integral canonical model.) The existence of $\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p} \to \mathscr{S}_{\mathsf{K}_p\mathsf{K}^p}$ follows from the construction. For (2), see [17, Corollary 2.3.9] □

Remark 3.3.4. The discussion on Hecke action in §3.2.4 can be extended to the prime-to-p Hecke action (i.e., the $G(\mathbb{A}^p_{\mathrm{f}})$-action) on the integral canonical models, which we explain now. Let $\mathsf{K} := \mathsf{K}_p\mathsf{K}^p$ be as before, and pick a geometric point x of Sh_{K}, viewed also as a geometric point of $\mathscr{S}_{\mathsf{K}}$. As $\mathscr{S}_{\mathsf{K}}$ is normal, the open immersion $\mathrm{Sh}_{\mathsf{K}} \hookrightarrow \mathscr{S}_{\mathsf{K}}$ induces a (surjective) quotient morphism $\pi_1^{\text{ét}}(\mathrm{Sh}_{\mathsf{K}},x) \to \pi_1^{\text{ét}}(\mathscr{S}_{\mathsf{K}},x)$.

Note that the lisse sheaf $\mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K},E})$ on Sh_{K} extends to a lisse sheaf $\mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K}})$ on $\mathscr{S}_{\mathsf{K}}$. By considering the monodromy action at geometric points, it now follows that the prime-to-p étale tensors $t^{\mathrm{univ},p}_{\alpha,\text{ét}} : \mathbf{1} \to \mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K},E})^{\otimes}$ on the generic fibre extend to the integral canonical model:

$$t^{\mathrm{univ},p}_{\alpha,\text{ét}} : \mathbf{1} \to \mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K}})^{\otimes}.$$

Furthermore, the prime-to-p part of the level structure η_{K^p} (i.e., the image of η_K in $\mathrm{isom}(V_{\mathbb{A}^p_{\mathrm{f}}}, \mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K},E}))$ extends to the integral canonical model:

$$\eta_{\mathsf{K}^p} \in \Gamma(\mathscr{S}_{\mathsf{K}}, \mathrm{isom}\left[(V_{\mathbb{A}^p_{\mathrm{f}}},(s_\alpha)),(\mathcal{V}_{\mathbb{A}^p_{\mathrm{f}}}(\mathcal{A}_{\mathsf{K}}),(t^{\mathrm{univ},p}_{\alpha,\text{ét}}))\right]/\mathsf{K}^p), \tag{3.3.5}$$

[7] Note that the construction of integral canonical models is claimed by Vasiu [32, 33] for any p.

which only depends on the *prime-to-p* isogeny class of $(\mathcal{A}_{\mathsf{K}}, (t^{\mathrm{univ},p}_{\alpha,\text{ét}}))$. Now for any $g \in G(\mathbb{A}^p_{\mathrm{f}})$, Theorem 3.3.3 gives an isomorphism $[g] : \mathscr{S}_{g\mathsf{K}g^{-1}} \xrightarrow{\sim} \mathscr{S}_{\mathsf{K}}$ extending the map on the generic fibre described in §3.2.4. Then, we have $[g]^*(\mathcal{A}_{\mathsf{K}}, (t^{\mathrm{univ},p}_{\alpha,\text{ét}})) = (\mathcal{A}_{g\mathsf{K}g^{-1}}, (t^{\mathrm{univ},p}_{\alpha,\text{ét}}))$ as prime-to-p isogeny classes, and the pull-back $[g]^*\eta_{\mathsf{K}^p}$ over the integral canonical model has exactly the same description as in §3.2.4.

Let us now move on to the p-adic part of the (co)homology of $\mathcal{A}_{\mathsf{K}}$ and tensors thereof. Since the p-adic lisse sheaf $\mathcal{V}_{\mathbb{Q}_p}(\mathcal{A}_{\mathsf{K},E})$ on Sh_{K} does not extend over $\mathscr{S}_{\mathsf{K}}$, we cannot extend the p-adic étale tensor $t^{\mathrm{univ}}_{\alpha,\text{ét},p} : \mathbf{1} \to \mathcal{V}_{\mathbb{Q}_p}(\mathcal{A}_{\mathsf{K},E})^{\otimes}$ to the integral canonical model. Instead, one would expect that $t^{\mathrm{univ}}_{\alpha,\text{ét},p}$ should give rise to a natural crystalline Tate tensor by p-adic Hodge theory.

Let $\widehat{\mathscr{S}_{\mathsf{K}}}$ denote the p-adic completion of $\mathscr{S}_{\mathsf{K}}$ (i.e., the formal completion of $\mathscr{S}_{\mathsf{K}}$ at the special fibre), and $\widehat{f} : \widehat{\mathcal{A}}_{\mathsf{K}} \to \widehat{\mathscr{S}_{\mathsf{K}}}$ the p-adic completion of $f : \mathcal{A}_{\mathsf{K}} \to \mathscr{S}_{\mathsf{K}}$. Then we have a natural isomorphism $\mathrm{H}^1_{\mathrm{dR}}(\widehat{\mathcal{A}}_{\mathsf{K}}/\widehat{\mathscr{S}_{\mathsf{K}}}) \cong (R^1\widehat{f}_{\mathrm{CRIS},*}\mathcal{O}_{\widehat{\mathcal{A}}_{\mathsf{K}}/\mathbb{Z}_p})_{\widehat{\mathscr{S}_{\mathsf{K}}}}$, where the right hand side is the pull-back of the crystal to the Zariski site. In particular, the de Rham tensor $(t^{\mathrm{univ}}_{\alpha,\mathrm{dR}})$ on $\mathcal{A}_{\mathsf{K}}$ induce an $\mathcal{O}_{\widehat{\mathscr{S}_{\mathsf{K}}}/\mathbb{Z}_p}$-linear morphisms of crystals:

$$t^{\mathrm{univ}}_{\alpha} : \mathbf{1} \to (R^1\widehat{f}_{\mathrm{CRIS},*}\mathcal{O}_{\widehat{\mathcal{A}}_{\mathsf{K}}/\mathbb{Z}_p})^{\otimes} = \mathbb{D}(\widehat{\mathcal{A}}_{\mathsf{K}}[p^\infty])^{\otimes}. \tag{3.3.6}$$

By construction, $t^{\mathrm{univ}}_{\alpha,\mathrm{dR}}$ coincides with the de Rham tensor associated to $t^{\mathrm{univ}}_{\alpha}$ by Definition 2.7.1.

Proposition 3.3.7. *The morphisms $(t^{\mathrm{univ}}_{\alpha})$ in (3.3.6) are crystalline Tate tensors on $\widehat{\mathcal{A}}_{\mathsf{K}}$ in the sense of Definition 2.7.2. Furthermore, the p-adic comparison isomorphism matches $(t^{\mathrm{univ}}_{\alpha})$ with $(t^{\mathrm{univ}}_{\alpha,\text{ét},p})$.*

Proof. This is essentially a corollary of relative crystalline comparison for p-divisible groups and the theorem of Blasius and Wintenberger; *cf.* [3, Theorem 0.3].

Consider the p-adic étale tensor

$$t^{\mathrm{univ}}_{\alpha,\text{ét},p} : \mathbf{1} \to \mathcal{V}_{\mathbb{Q}_p}(\mathcal{A}_{\mathsf{K},E})^{\otimes}.$$

Then the relative comparison isomorphism provides an F-equivariant morphism of isocrystals

$$t_\alpha : \mathbf{1} \to (\mathbb{D}(\mathcal{A}_{\mathsf{K}}[p^\infty])^*)^{\otimes}[1/p] = \mathbb{D}(\mathcal{A}_{\mathsf{K}}[p^\infty])^{\otimes}[1/p].$$

Indeed, the argument [10, §6] can be generalised to prove this; see [**?**, Theorem 5.3] for the precise statement, which globalise to show the claim.

It remains to show that $t_\alpha = t^{\mathrm{univ}}_{\alpha}$, which can be extracted from the construction of $\mathscr{S}_{\mathsf{K}}$ (*cf.* the proof of Proposition 2.3.5 and Corollary 2.3.9 in [17]). One can also extract a direct argument from *loc. cit.* as follows. By smoothness of $\mathscr{S}_{\mathsf{K}}$, both t_α and $t^{\mathrm{univ}}_{\alpha}$ are determined by the induced sections on $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K}}/\widehat{\mathscr{S}}^{\mathrm{rig}}_{\mathsf{K}})^{\otimes}$, so the claim $t_\alpha = t^{\mathrm{univ}}_{\alpha}$ can be verified on the fibres at a Zariski dense set of points of $\widehat{\mathscr{S}}^{\mathrm{rig}}_{\mathsf{K}}$. Indeed, the (classical) points of $\widehat{\mathscr{S}}^{\mathrm{rig}}_{\mathsf{K}}$ which come from $\overline{\mathbb{Q}}$-points of $\mathscr{S}_{\mathsf{K}}$ is Zariski dense in $\widehat{\mathscr{S}}^{\mathrm{rig}}_{\mathsf{K}}$, and the fibres of t_α and $t^{\mathrm{univ}}_{\alpha}$ (at $\overline{\mathbb{Q}}$-points) coincide by the theorem of Blasius and Wintenberger [3, Theorem 0.3]. □

We fix an embedding $\kappa(\mathfrak{p}) \hookrightarrow \overline{\mathbb{F}}_p$, and set $W := W(\overline{\mathbb{F}}_p)$ and $K_0 = \mathrm{Frac}\, W$. Let σ denote the Witt vectors Frobenius endomorphism on W and K_0.

For $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p$ with $\mathsf{K}_p = G(\mathbb{Z}_p)$, we consider $\tilde{x} : \mathrm{Spec}\, W \to \mathscr{S}_{\mathsf{K}}$, and let x denote the $\overline{\mathbb{F}}_p$-point induced by $\tilde{x}$. Let $\mathcal{A}_{\mathsf{K},\tilde{x}}$ and $\mathcal{A}_{\mathsf{K},x}$ respectively denote the pull-back of $\mathcal{A}_{\mathsf{K}}$.

The following result was originally conjectured by Milne and was proved by Vasiu and Kisin (independently) in the course of constructing $\mathscr{S}_{\mathsf{K}}$ (i.e., proving Theorem 3.3.3):

Proposition 3.3.8. *There is a W-linear isomorphism*

$$W \otimes_{\mathbb{Z}_{(p)}} \Lambda^*_{\mathbb{Z}_{(p)}} \cong \mathbb{D}(\mathcal{A}_{\mathsf{K},\tilde{x}}[p^\infty])(W)$$

matching $(1 \otimes s_\alpha)$ and $(t^{\mathrm{univ}}_{\alpha,\mathrm{dR},\tilde{x}})$. In particular, the pointwise stabiliser of $(t^{\mathrm{univ}}_{\alpha,\mathrm{dR},\tilde{x}})$ in $\mathrm{GL}(\mathbb{D}(\mathcal{A}_{\mathsf{K},x}[p^\infty])(W)$ is isomorphic to G_W.

Proof. We first show that there exists an isomorphism

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} \Lambda_{\mathbb{Z}_{(p)}} \cong T_p(\mathcal{A}_{\mathsf{K},\tilde{x}})$$

which matches $(1 \otimes s_\alpha)$ and $(t^{\mathrm{univ}}_{\alpha,\text{ét},p})$. Indeed, by fixing an embedding $\tau : W \hookrightarrow \mathbb{C}$ we obtain an isomorphism

$$T_p(\mathcal{A}_{\mathsf{K},\tilde{x}}) \cong \mathrm{H}_1(\tau^*\mathcal{A}_{\mathsf{K},\tilde{x}}, \mathbb{Z}_p)$$

matching $(t^{\mathrm{univ}}_{\alpha,\text{ét},p})$ and the "Betti tensors" $(t^{\mathrm{univ}}_{\alpha,B})$ constructed in Lemma 3.1.5. Now by construction, there exists an isomorphism $\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} \Lambda_{\mathbb{Z}_{(p)}} \cong \mathrm{H}_1(\tau^*\mathcal{A}_{\mathsf{K},\tilde{x}}, \mathbb{Z}_p)$ matching the tensors.

Now it remains to show the existence of an isomorphism

$$W \otimes_{\mathbb{Z}_p} T_p(\mathcal{A}_{\mathsf{K},\tilde{x}})^* \cong \mathbb{D}(\mathcal{A}_{\mathsf{K},x}[p^\infty])(W) \tag{3.3.9}$$

matching $(1 \otimes t^{\mathrm{univ}}_{\alpha,\text{ét},p})$ and $(t^{\mathrm{univ}}_{\alpha})$. Since these étale and crystalline tensors are related by the p-adic comparison isomorphism by Proposition 3.3.7, the existence of such an isomorphism was proved by Vasiu and Kisin in the course of constructing integral canonical models; *cf.* [17, Proposition 1.3.4], [34]. ☐

We now extend the G-torsor $P_{\mathsf{K},E}$ over Sh_{K} (§3.2.1) to the integral canonical model $\mathscr{S}_{\mathsf{K}}$. Consider the following closed subscheme of the isom scheme over $\mathscr{S}_{\mathsf{K}}$

$$P_{\mathsf{K}} \subset \mathrm{isom}_{\mathscr{S}_{\mathsf{K}}}\big(\mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K}}), \mathcal{O}_{\mathscr{S}_{\mathsf{K}}} \otimes_{\mathbb{Z}_{(p)}} \Lambda_{\mathbb{Z}_{(p)}}\big),$$

defined by the condition of matching $(t^{\mathrm{univ}}_{\alpha,\mathrm{dR}}) \subset \Gamma(\mathscr{S}_{\mathsf{K}}, \mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K}})^{\otimes})$ with $(s_\alpha) \subset \Lambda^{\otimes}_{\mathbb{Z}_{(p)}}$ for each α. Then we have $P_{\mathsf{K},E} = P_{\mathsf{K}} \times_{\mathrm{Spec}\,\mathscr{O}_{E,(\mathfrak{p})}} \mathrm{Spec}\,E$, which is a G-torsor over Sh_{K}.

Lemma 3.3.10. *The scheme P_{K} above is a G-torsor over $\mathscr{S}_{\mathsf{K}}$.*

Proof. It follows from the construction of $\mathscr{S}_{\mathsf{K}}$ (*cf.* [18, Proposition 1.3.9(1)]) and Proposition 3.3.8 that P_{K} pulls back to a G-torsor over the completion of $\mathscr{S}_{\mathsf{K}}$ at any $\overline{\mathbb{F}}_p$-point. This proves the claim. ☐

Let us now consider the Hodge filtration for $\mathcal{A}_{\mathsf{K}}$. We first need the following lemma:

Lemma 3.3.11. *Let $(G, \mathfrak{H})$ be any Shimura datum such that G is unramified at p (i.e., G admits a reductive $\mathbb{Z}_{(p)}$-model). Write $E := E(G, \mathfrak{H})$, and choose a prime $\mathfrak{p} \subset \mathscr{O}_E$ over p. Then there exists a cocharacter $\mu : \mathbb{G}_m \to G_{\mathscr{O}_{E,(\mathfrak{p})}}$ such that $\mu_{\mathbb{C}} = \mu_h$ for some $h \in \mathfrak{H}$ (by the recipe given in (3.1.4)).*

Proof. By unramifiedness, $G_{\mathbb{Q}_p}$ is quasi-split. Then by [20, Lemma 1.1.3(a)], there exists a cocharacter $\mu' : \mathbb{G}_m \to G_E$ over E with $\mu'_{\mathbb{C}} = \mu_h$ for some $h \in \mathfrak{H}$. Let $S \subset G_E$ be a maximal E-split torus containing the image of μ'. Since any maximal E-split tori are $G(E)$-conjugate to each other (*cf.* [4, Theorem 20.9(ii)]), there exists $g \in G(E)$ such that ${}^gS := gSg^{-1}$ is the generic fibre of a maximal split torus in $G_{\mathscr{O}_{E,(\mathfrak{p})}}$. We set $\mu := {}^g\mu'$, which extends over $\mathscr{O}_{E,(\mathfrak{p})}$. ☐

Corollary 3.3.12. *The Hodge filtration for $\mathcal{A}_{\mathsf{K}} \to \mathscr{S}_{\mathsf{K}}$ is a $\{\mu\}$-filtration with respect to $(t_{\alpha,\mathrm{dR}})$, where $\{\mu\}$ is the $G(W)$-conjugacy class of μ as in Lemma 3.3.11.*

Proof. By Lemmas 3.3.10 and 3.3.11, $\{\mu\}$-filtrations in $\mathcal{V}_{\mathrm{dR}}(\mathcal{A}_{\mathsf{K}})^*$ form a smooth closed subscheme of a certain grassmannian over $\mathscr{S}_{\mathsf{K}}$ (*cf.* Lemma 2.5.6). Since the Hodge filtration for $\mathcal{A}_{\mathsf{K},E} \to \mathrm{Sh}_{\mathsf{K}}$ is a $\{\mu\}$-filtration (*cf.* §3.2.2), to prove the corollary it suffices to show that the Hodge filtration of $\mathcal{A}_{\mathsf{K}}$ becomes a $\{\mu\}$-filtration after pulling back by any morphism $\tilde{x} : \mathrm{Spec}\, R \to \mathscr{S}_{\mathsf{K}}$, where R is a p-adic discrete valuation ring. But this follows from the valuative criterion for properness (applied to the grassmannian of $\{\mu\}$-filtrations over $\mathscr{S}_{\mathsf{K}}$). $\square$

3.3.13. We fix $x \in \mathscr{S}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$ and $\tilde{x} \in \mathscr{S}_{\mathsf{K}}(W)$ as before, and write $\mathbb{X} := \mathcal{A}_{\mathsf{K},x}[p^\infty]$ and $\widetilde{\mathbb{X}} := \mathcal{A}_{\mathsf{K},\tilde{x}}[p^\infty]$. We choose a W-isomorphism $\mathbb{D}(\mathbb{X})(W) \cong W \otimes_{\mathbb{Z}_{(p)}} \Lambda^*_{\mathbb{Z}_{(p)}}$ matching $(t^{\mathrm{univ}}_{\alpha,\tilde{x}})$ and $(1 \otimes s_\alpha)$, as in Proposition 3.3.8. Then we obtain $b \in \mathrm{GL}(K_0 \otimes_{\mathbb{Z}_{(p)}} \Lambda_{\mathbb{Z}_{(p)}})$ such that $F = b(\sigma \otimes \mathrm{id})$. Since each of $(t^{\mathrm{univ}}_{\alpha,\tilde{x}})$ is fixed by F, it follows that b fixes each of $(1 \otimes s_\alpha)$; i.e., we have $b \in G(K_0)$. By Corollary 3.3.12, the Hodge filtration $\mathrm{Fil}^1_{\widetilde{\mathbb{X}}}$ is induced by ${}^g\mu$ for some $g \in G(W)$ where μ is a cocharacter as in Lemma 3.3.11.

Lemma 3.3.14. *In the above setting, we have $b \in G(W)(p^{-1})^{\sigma^*\mu}G(W)$.*

Proof. This lemma follows from [15, Lemma 2.5.7(2)], which can be applied thanks to Corollary 3.3.12 and Proposition 3.3.8. $\square$

4. Rapoport-Zink uniformisation via formal schemes

In this section, we relate Rapoport-Zink spaces of Hodge type constructed in [15] with a certain completion of $\mathscr{S}_{\mathsf{K}}$ (*cf.* Theorem 4.7), generalising (the unramified case of) Rapoport-Zink uniformisation of PEL Shimura varieties (*cf.* [30, Theorem 6.23]).

Using Kisin's theorem on quasi-isogeny groups of abelian varieties with tensors (which we recall in Theorem 4.8), we refine the uniformisation; namely, we descend the uniformisation over $\mathscr{O}_{E,\mathfrak{p}}$ (§4.9)[8], and simplify the statement in the "basic case" (Theorem 4.11)[9].

From now on, we *always assume that* $p > 2$ without mentioning it.

4.1. **Review of Rapoport-Zink spaces of Hodge type.** We recall the definitions and main results in [15]. We work in the setting of §3.3.13.

Let Nilp_W be the category of W-algebras where p is nilpotent. For $b \in G(K_0)$ and X_0 as in §3.3.13, we define a functor $\mathtt{RZ}_b : \mathrm{Nilp}_W \to (\mathbf{Sets})$ so that $\mathtt{RZ}_b(R)$ is the set of isomorphism classes of pairs (X, ι) where X is a p-divisible group over R and $\iota : \mathbb{X}_{R/p} \dashrightarrow X_{R/p}$ is a quasi-isogeny (i.e., an invertible global section of $\mathrm{Hom}(\mathbb{X}_{R/p}, X_{R/p})[\frac{1}{p}]$). Note that $\mathtt{RZ}_b$ only depends, up to isomorphism, on the σ-conjugacy class of b in $\mathrm{GL}(K_0 \otimes \Lambda_{\mathbb{Z}_{(p)}})$. By [30, Theorem 2.16], $\mathtt{RZ}_b$ is representable by a formal scheme which is locally formally of finite type and formally smooth over W. We will also let $\mathtt{RZ}_b$ also denote the representing formal schemes.

Remark 4.1.1. For any p-divisible group X over $R \in \mathfrak{AR}_W$ which lifts $\mathbb{X}$, there exists a unique quasi-isogeny $\mathbb{X}_{R/p} \dashrightarrow X_{R/p}$ lifting the identification $\mathbb{X} \xrightarrow{\sim} X_{\overline{\mathbb{F}}_p}$. This identifies the universal deformation space of $\mathbb{X}$ with the completion of $\mathtt{RZ}_b$ at the point $(\mathbb{X}, \mathrm{id}) \in \mathtt{RZ}_b(\overline{\mathbb{F}}_p)$; *cf.* [30, Proposition 3.33]

Let $s_{\alpha,\mathbb{D}} (:= t^{\mathrm{univ}}_{\alpha,x}) : \mathbf{1} \to \mathbb{D}(\mathbb{X})^\otimes$ be the crystalline Tate tensors induced from $(t^{\mathrm{univ}}_\alpha)$ on $\widehat{\mathcal{A}}_{\mathsf{K}}$; i.e., we have $s_{\alpha,\mathbb{D}}(W) = 1 \otimes s_\alpha$ under the identification as in Propositiion 3.3.8, where $(s_\alpha) \subset \Lambda^\otimes$ define G. Then, for any $(X, \iota) \in \mathtt{RZ}_b(R)$ with

[8]See [30, Proposition 6.16] for the PEL case.

[9]See [30, Theorem 6.30] for the PEL case.

$R \in \mathrm{Nilp}_W$, we have a Frobenius-equivariant morphism of isocrystals $s_{\alpha,\mathbb{D}} : \mathbf{1} \to \mathbb{D}(X)^{\otimes}[\frac{1}{p}]$ which uniquely lifts

$$s_{\alpha,\mathbb{D}} : \mathbf{1} \xrightarrow{(s_{\alpha,\mathbb{D}})_{R/p}} \mathbb{D}(X_{0,R/p})^{\otimes}[\tfrac{1}{p}] \xrightarrow{\mathbb{D}(\iota)^{-1}} \mathbb{D}(X_{R/p})^{\otimes}[\tfrac{1}{p}]; \tag{4.1.2}$$

alternatively, one may uniquely lift ι to $\tilde{\iota} : \widetilde{\mathbb{X}}_R \dashrightarrow X$ and obtain $s_{\alpha,\mathbb{D}}$ from the tensor on $\widetilde{\mathbb{X}}$.

In general, there may not exist any morphism of (integral) crystals giving rise to $s_{\alpha,\mathbb{D}}$. On the other hand, there is a natural closed formal subscheme of $\mathtt{RZ}_b$ over which $(s_{\alpha,\mathbb{D}})$ is induced from crystalline Tate tensors.

Theorem 4.1.3 ([15, Theorem 4.9.1]). *Assume that $p > 2$. Then there exists a closed formal subscheme $\mathtt{RZ}_{G,b} \subset \mathtt{RZ}_b$, which is formally smooth over W, with the following universal property: Let R be a formally smooth formally finitely generated algebra over either W or W/p^m, and consider a morphism $f : \mathrm{Spf}\, R \to \mathtt{RZ}_b$. Let X be a p-divisible group over $\mathrm{Spec}\, R$ which pulls back to $f^*X_{\mathtt{RZ}_b}$ over $\mathrm{Spf}\, R$. Then f factors through $\mathtt{RZ}_{G,b}$ if and only if there exists a crystalline Tate tensors $t_\alpha : \mathbf{1} \to \mathbb{D}(X)^{\otimes}$ for each α such that*

(1) *For some ideal of definition J of R containing p, the pull-back of t_α over R/J induces the map of isocrystals $s_{\alpha,\mathbb{D}} : \mathbf{1} \to \mathbb{D}(X_{R/J})^{\otimes}[\frac{1}{p}]$.*
(2) *Let $\widetilde{R}$ be a p-adic lift of R which is formally smooth over W. Then the $\widetilde{R}$-scheme*

$$P_{\widetilde{R}} := \mathrm{isom}_{\widetilde{R}}\left[(\mathbb{D}(X)(\widetilde{R}), (t_\alpha(\widetilde{R}))], [\widetilde{R} \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]\right),$$

defined as in (2.5.4), is a G-torsor.
(3) *The Hodge filtration $\mathrm{Fil}^1_X \subset \mathbb{D}(X)(R)$ is a $\{\mu\}$-filtration with respect to $(t_{\alpha,\mathrm{dR}}) \subset \mathbb{D}(X)(R)^{\otimes}$.*

The closed formal subscheme $\mathtt{RZ}_{G,b} \subset \mathtt{RZ}_b$ is independent of the choice of (s_α).

4.1.4. Let $f : \mathrm{Spf}\, R \to \mathtt{RZ}_{G,b}$ be as in Theorem 4.1.3 and we use the notation as above. Then (t_α) are uniquely determined by f (*cf.* [15, Lemma 4.6.4]). Therefore, applying the universal property to an open affine covering of $\mathtt{RZ}_{G,b}$ we obtain a "universal family" of crystalline Tate tensors

$$t_\alpha : \mathbf{1} \to \mathbb{D}((X_{\mathtt{RZ}_b})|_{\mathtt{RZ}_{G,b}})^{\otimes}.$$

Let us recall the description of $\mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$ and the formal completion $\widehat{\mathtt{RZ}}_{G,b,y}$ at $y \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$ from [15, §4.8].

If we fix an isomorphism $[\mathbb{D}(\mathbb{X})(W), (s_{\alpha,\mathbb{D}})] \cong [W \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]$, then $(X, \iota) \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$, the quasi-isogeny ι induces a map $g \in G(K_0)$ on $W \otimes_{\mathbb{Z}_p} \Lambda^*$. Since the choice of the isomorphism $[\mathbb{D}(\mathbb{X})(W), (s_{\alpha,\mathbb{D}})] \cong [W \otimes_{\mathbb{Z}_p} \Lambda^*, (1 \otimes s_\alpha)]$ admits a simply transitive $G(W)$-action, we get an injective map $\mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p) \hookrightarrow G(K_0)/G(W)$. And its image is given by the following affine Deligne-Lusztig set (*cf.* [15, Proposition 2.5.9]):

$$\mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \{g \in G(K_0) |\ g^{-1} b \sigma(g) \in G(W) b G(W)\}/G(W). \tag{4.1.5}$$

Now, given $y \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$, one can identify the formal completion $\widehat{\mathtt{RZ}}_{G,b,y}$ with the explicit deformation space with Tate tensors constructed by Faltings; *cf.* [15, §4.8]. Instead of recalling the precise description, let us record the following consequence which will be used later.

Let $\widehat{\mathscr{S}}_{\mathsf{K},x}$ denote the completion of $\mathscr{S}_{\mathsf{K}}$ at $x : \mathrm{Spec}\,\overline{\mathbb{F}}_p \to \mathscr{S}_{\mathsf{K}}$. We also view $x = (\mathbb{X}, \mathrm{id}) \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$. As observed in Remark 4.1.1, we have a morphism $\widehat{\mathscr{S}}_{\mathsf{K},x} \to \widehat{\mathtt{RZ}}_{b,x}$ given by rigidity of quasi-isogeny, which is a closed immersion of formal schemes by Serre-Tate deformation theory [13, Theorem 1.2.1]. Furthermore, this closed immersion factors through $\widehat{\mathtt{RZ}}_{G,b,x}$ by the universal property of $\mathtt{RZ}_{G,b}$ (Theorem 4.1.3). Indeed, the crystalline Tate tensors $(t_\alpha^{\mathrm{univ}})$ on $\widehat{\mathcal{A}}_{\mathsf{K}}$ induce the required (t_α), which satisfy (1) by taking J to be the maximal ideal and the remaining conditions by Corollary 3.3.12.

Proposition 4.1.6. *The morphism of formal schemes $\widehat{\mathscr{S}}_{\mathsf{K},x} \to \widehat{\mathtt{RZ}}_{G,b,x}$, defined above, is an isomorphism.*

Proof. Note that both completions as well as the deformations of $\mathbb{X}$ over them have the same explicit description, and the morphism we constructed match them; *cf.* [15, Theorem 4.9.1] and [18, Proposition 1.3.9(1)]. □

4.1.7. Assume that there is another embedding $(G,\mathfrak{H}) \hookrightarrow (\mathrm{GSp}(V',\psi'),\boldsymbol{\mathcal{S}}'^{\pm})$ of Shimura data and a $\mathbb{Z}_{(p)}$-lattice $\Lambda'_{\mathbb{Z}_{(p)}} \subset V'$ as in §3.3.1. This choice gives rise to another abelian scheme $\mathcal{A}'_{\mathsf{K}}$ over $\mathscr{S}_{\mathsf{K}}$. We set $\mathbb{X}' := \mathcal{A}'_{\mathsf{K},x}$ for $x \in \mathscr{S}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$ and let $\mathtt{RZ}'_{G,b} \subset \mathtt{RZ}'_b$ denote the moduli spaces constructed using $\mathbb{X}'$ instead. Then the functoriality part of [15, Theorem 4.9.1] shows that there exists a unique isomorphism $\mathtt{RZ}_{G,b} \cong \mathtt{RZ}'_{G,b}$ respecting the description of $\overline{\mathbb{F}}_p$-points and the completions thereof given in (4.1.5) and Proposition 4.1.6.

We recall some of the extra structures that $\mathtt{RZ}_{G,b}$ possesses.

4.1.8. By classical Dieudonné theory, $J_b(\mathbb{Q}_p)$ (2.6.4) can be identified with the group of quasi-isogenies of $\mathbb{X}$ that preserve the crystalline Tate tensors $(s_{\alpha,\mathbb{D}})$. It turns out to be a $\mathbb{Q}_p$-point of an inner form of a Levi subgroup of G; *cf.* [30, Corollary 1.14]. We define the left action of $J_b(\mathbb{Q}_p)$ on $\mathtt{RZ}_{G,b}$ as follows. For any $(X,\iota) \in \mathtt{RZ}_{G,b}(R)$ and $g \in J_b(\mathbb{Q}_p)$, we set

$$g(X,\iota) = (X,\iota\circ g^{-1}).$$

4.1.9. Put $d := [E_{\mathfrak{p}} : \mathbb{Q}_p]$, and let $q = p^d$ be the cardinality of the residue field of $E_{\mathfrak{p}}$. Let $\tau = \sigma^d \in \mathrm{Gal}(K_0/E_{\mathfrak{p}})$ denote the q-Frobenius element (i.e., the lift of the qth power map on $\overline{\mathbb{F}}_p$). For any formal scheme $\mathfrak{X}$ over $\mathrm{Spf}\, W$, we write $\mathfrak{X}^\tau := \mathfrak{X} \times_{\mathrm{Spf}\, W,\tau} \mathrm{Spf}\, W$. For any $R \in \mathrm{Nilp}_W$, note that $\mathfrak{X}^\tau(R) = \mathfrak{X}(R^\tau)$, where R^τ is R viewed as a W-algebra via τ. By *Weil descent datum* over $\mathscr{O}_{E,\mathfrak{p}}$ we mean an isomorphism $\Phi : \mathfrak{X} \to \mathfrak{X}^\tau$. Note that if there exists an $\mathscr{O}_{E,\mathfrak{p}}$-formal scheme $\mathfrak{X}_0$ with $(\mathfrak{X}_0)_W \cong \mathfrak{X}$, then $\mathfrak{X}$ has a Weil descent datum over $\mathscr{O}_{E,\mathfrak{p}}$. Such a Weil descent datum is called *effective*.

We define a Weil descent datum $\Phi : \mathtt{RZ}_b \xrightarrow{\sim} \mathtt{RZ}_b^\tau$ over $\mathscr{O}_{E,\mathfrak{p}}$, sending $(X,\iota) \in \mathtt{RZ}_{G,b}(R)$ to $(X^\Phi,\iota^\Phi) \in \mathtt{RZ}_{G,b}(R^\tau)$, where X^Φ is X viewed as a p-divisible group over R^τ, and ι^Φ is the following quasi-isogeny:

$$\iota^\Phi_{R^\tau/p} : \mathbb{X}_{R^\tau/p} = (\tau^*\mathbb{X})_{R/p} \overset{\mathrm{Frob}^{-d}}{\dashrightarrow} X_{R/p} \overset{\iota}{\dashrightarrow} X_{R/p} = X^\Phi_{R/p},$$

where $\mathrm{Frob}^d : \mathbb{X} \to \tau^*\mathbb{X}$ is the relative q-Frobenius (with $q = p^d$). One can check that Φ restricts to a Weil descent datum $\Phi : \mathtt{RZ}_{G,b} \xrightarrow{\sim} \mathtt{RZ}_{G,b}^\tau$ over $\mathscr{O}_{E,\mathfrak{p}}$ (by looking at $\overline{\mathbb{F}}_p$-points and the formal completions thereof; *cf.* [15, Definition 7.3.5]).

Clearly the $J_b(\mathbb{Q}_p)$ action commutes with the Weil descent datum Φ. Note that Φ is *not* an effective Weil descent datum for $\mathtt{RZ}_{G,b}$.

4.1.10. Over the rigid analytic generic fibre of $\mathtt{RZ}_{G,b}$ we can naturally define a tower of étale coverings with Galois group $G(\mathbb{Z}_p)$ equipped with a natural $G(\mathbb{Q}_p)$-action. The $J_b(\mathbb{Q}_p)$-action and the Weil descent datum naturally lifts to each layer of the tower in a compatible way. We will give a brief review when we use it (§5.2), and see [15, §7.4] for the details.

4.2. **Isogeny classes of mod p points.** We continue to work in the setting of §3.3.13. Let $\iota : \mathcal{A} \dashrightarrow \mathcal{A}'$ be a quasi-isogeny of abelian schemes over $R \in \mathrm{Nilp}_W$; i.e., an invertible global section of $\mathrm{Hom}(\mathcal{A}, \mathcal{A}') \otimes_{\mathbb{Z}} \mathbb{Q}$. Then ι induces the following isomorphisms:

$$\mathbb{D}(\mathcal{A}[p^\infty])[1/p] \xleftarrow{\sim} \mathbb{D}(\mathcal{A}'[p^\infty])[1/p]; \tag{4.2.1a}$$

$$\mathcal{V}_{\mathbb{A}_f^p}(\mathcal{A}) \xrightarrow{\sim} \mathcal{V}_{\mathbb{A}_f^p}(\mathcal{A}'). \tag{4.2.1b}$$

Definition 4.2.2. We define an equivalence relations $x \sim x'$ for $x, x' \in \mathscr{S}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$ if there exists a quasi-isogeny $\iota : \mathcal{A}_{\mathsf{K},x} \dashrightarrow \mathcal{A}_{\mathsf{K},x'}$ such that the isomorphisms (4.2.1) induced by ι matches $(t_{\alpha,x}^{\mathrm{univ}})$ with $(t_{\alpha,x'}^{\mathrm{univ}})$, and $(t_{\alpha,\text{ét},x}^{\mathrm{univ},p})$ with $(t_{\alpha,\text{ét},x'}^{\mathrm{univ},p})$. An equivalence class ϕ containing $x \in \mathscr{S}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$ is called an *isogeny class* of x.

Let $(X, \iota) \in \mathtt{RZ}_{G,b}(R)$ for $R \in \mathrm{Nilp}_W$, and for the choice of the W-lift $\widetilde{\mathbb{X}}$ as in §3.3.13 let $\tilde{\iota} : \widetilde{\mathbb{X}}_R \dashrightarrow X$ denote the unique lift of ι. Assume that $p^n\tilde{\iota} : \widetilde{\mathbb{X}}_R \to X$ is an isogeny, and let $\mathcal{A} := (\mathcal{A}_{\mathsf{K},\tilde{x}})_R / \ker(p^n\tilde{\iota})$ be an abelian scheme over R. Note that $\mathcal{A}[p^\infty] = X$ by construction, and we have a quasi-isogeny

$$\tilde{\iota} : (\mathcal{A}_{\mathsf{K},\tilde{x}})_R \overset{p^{-n}}{\dashrightarrow} (\mathcal{A}_{\mathsf{K},\tilde{x}})_R \twoheadrightarrow \mathcal{A}.$$

Note that $\tilde{\iota}$ induces crystalline Tate tensors $t_\alpha : \mathbf{1} \to \mathbb{D}(\mathcal{A}[p^\infty])^{\otimes}[\frac{1}{p}]$, and an isomorphism of $\mathbb{A}_f^p$-local systems

$$\mathcal{V}_{\mathbb{A}_f^p}(\tilde{\iota}) : \mathcal{V}_{\mathbb{A}_f^p}((\mathcal{A}_{\mathsf{K},\tilde{x}})_R) \xrightarrow{\sim} \mathcal{V}_{\mathbb{A}_f^p}(\mathcal{A}).$$

Via this isomorphism, $(t_{\alpha,\text{ét},\tilde{x}}^{\mathrm{univ}})$ induces prime-to-p étale tensors on $\mathcal{A}$ as follows

$$t_\alpha^p := \mathcal{V}_{\mathbb{A}_f^p}(\tilde{\iota}) \circ t_{\alpha,\text{ét},\tilde{x}}^{\mathrm{univ}},$$

and the prime-to-p level structure $\tilde{x}^*\eta_{\mathsf{K}^p}$ (3.3.5) induces

$$\eta^p = \mathcal{V}(\tilde{\iota}) \circ (\tilde{x}^*\eta_{\mathsf{K}^p}) \in \Gamma(\operatorname{Spec} R, \operatorname{isom}\left[(V_{\mathbb{A}_f^p}, (s_\alpha)), (\mathcal{V}_{\mathbb{A}_f^p}(\mathcal{A}), (t_\alpha^p))\right]/\mathsf{K}^p). \tag{4.2.3}$$

The next aim is to construct a morphism of formal schemes $\mathtt{RZ}_{G,b} \to \widehat{\mathscr{S}_{\mathsf{K}}}$ where the target is the p-adic completion of $\mathscr{S}_{\mathsf{K}}$.

Proposition 4.3. *In the above setting, there exists a unique map $f : \operatorname{Spec} R \to \widehat{\mathscr{S}_{\mathsf{K}}}$, depending on $(X, \iota) \in \mathtt{RZ}_{G,b}(R)$, such that*

$$f^*(\mathcal{A}_{\mathsf{K}}, (t_\alpha^{\mathrm{univ}}), \eta_{\mathsf{K}^p}) \cong (\mathcal{A}, (t_\alpha), \eta^p),$$

where η^p is defined in (4.2.3).

In particular, we obtain a morphism of formal schemes

$$\Theta^\phi : \mathtt{RZ}_{G,b} \to \widehat{\mathscr{S}_{\mathsf{K},W}},$$

which commutes with the Weil descent data over $\mathscr{O}_{E,\mathfrak{p}} = W(\kappa(\mathfrak{p}))$. Furthermore, Θ^ϕ is independent of the auxiliary choice of $\Lambda_{\mathbb{Z}_{(p)}}$ and (s_α).

Proof. The unique existence of $\Theta^\phi(\overline{\mathbb{F}}_p) : \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p) \to \mathscr{S}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$, as well as independence of choice, follows from [18, Proposition 1.4.4][10], which was proved using the main result of [6]. Considering the case when R is an artin local ring with residue field $\overline{\mathbb{F}}_p$, it follows that Θ^ϕ should induce the isomorphism $\widehat{\mathtt{RZ}}_{G,b,y} \xrightarrow{\sim} \widehat{\mathscr{S}}_{\mathsf{K},\Theta^\phi(y)}$ for any $y \in \mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$. Note that this isomorphism is independent of the choice of $\Lambda_{\mathbb{Z}_{(p)}}$ and (s_α). This shows Θ^ϕ is uniquely defined and independent of auxiliary choice if it is defined.

[10]To obtain the map map $\mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p) \to \mathscr{S}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$ from [18, Proposition 1.4.4], note that $\mathtt{RZ}_{G,b}(\overline{\mathbb{F}}_p)$ can be identified with a certain affine Deligne-Lusztig set by [15, (4.8.1)].

It remains to show the existence of Θ^{ϕ} using some suitable choice of $\Lambda_{\mathbb{Z}_{(p)}}$. Indeed, by Zarhin's trick we may assume that $G_{\mathbb{Z}_{(p)}} \subset \mathrm{GSp} := \mathrm{GSp}(\Lambda_{\mathbb{Z}_{(p)}}, \psi)$ where ψ is a *perfect* alternating form on $\Lambda_{\mathbb{Z}_{(p)}}$. Then we may choose an open compact subgroup $\mathsf{K}' = \mathsf{K}'_p\mathsf{K}'^p \subset \mathrm{GSp}(\mathbb{A}_\mathrm{f})$ such that $\mathsf{K}'_p = \mathrm{GSp}(\mathbb{Z}_p)$ and we have a natural closed immersion $\mathrm{Sh}_\mathsf{K} \hookrightarrow \mathrm{Sh}_{\mathsf{K}',E}$ (with the obvious notation); *cf.* [17, Lemma 2.1.2]. Since K'_p is hyperspecial maximal, we have an integral canonical model $\mathscr{S}_{\mathsf{K}'}$ of $\mathrm{Sh}_{\mathsf{K}'}$, and $\mathscr{S}_\mathsf{K}$ is the normalisation of Sh_K in $\mathscr{S}_{\mathsf{K}'}$.

Given $x \in \mathscr{S}_\mathsf{K}(\overline{\mathbb{F}}_p)$, we let $x \in \mathscr{S}_{\mathsf{K}'}(\overline{\mathbb{F}}_p)$ also denote its image. Then we also obtain $\mathtt{RZ}_{\mathrm{GSp},b} \subset \mathtt{RZ}_b$ by working with GSp instead of G, and clearly $\mathtt{RZ}_{G,b}$ is a closed formal subscheme of $\mathtt{RZ}_{\mathrm{GSp},b}$. Now, the desired map for GSp instead of G

$$\mathtt{RZ}_{\mathrm{GSp},b} \to \widehat{\mathscr{S}_{\mathsf{K}',W}}$$

was already constructed in [30, Theorem 6.21]. We want to show that the restriction $\mathtt{RZ}_{G,b} \to \widehat{\mathscr{S}_{\mathsf{K}',W}}$ factors through $\widehat{\mathscr{S}_{\mathsf{K},W}}$. For this, it suffices to verify the claim on the level of the $\overline{\mathbb{F}}_p$-points and the completions thereof. This can be seen from the fact that the inclusion $\mathtt{RZ}_{G,b} \hookrightarrow \mathtt{RZ}_{\mathrm{GSp},b}$ has the expected effect on the $\overline{\mathbb{F}}_p$-points and the completions thereof.

To show that Θ^{ϕ} commutes with the Weil descent data, note that it suffices to check this for $\Theta^{\phi}_{\overline{\mathbb{F}}_p}$, in which case the claim is more or less clear from the definition. *Cf.* the proof of [30, Theorem 6.21]. □

Remark 4.3.1. In some sense, the proof of [18, Proposition 1.4.4] essentially proves Proposition 4.3, except that $\mathtt{RZ}_{G,b}$ was not defined in [18] and some *ad hoc* notion for $\mathtt{RZ}_{G,b}(R)$ was used instead. So Proposition 4.3 can be proved by "repeating" the proof [18, Proposition 1.4.4] in the following way (taking [6] as the main input). By the argument in [18, §1.4.10], the map can be extended to $\mathtt{RZ}^{\circ}_{G,b} \to \widehat{\mathscr{S}_{\mathsf{K},W}}$, where $\mathtt{RZ}^{\circ}_{G,b}$ is the connected component containing $x = (\mathbb{X}, \mathrm{id})$. Now, it follows from the main result of [6] that the Hecke action at p transitively permutes the connected components of $\mathtt{RZ}_{G,b}$; *cf.* [18, Proposition 1.2.22].

Corollary 4.3.2. *The map $\Theta^{\phi} : \mathtt{RZ}_{G,b} \to \widehat{\mathscr{S}_\mathsf{K}}$ in Proposition 4.3 extends to*

$$\Theta^{\phi} : \mathtt{RZ}_{G,b} \times G(\mathbb{A}^p_\mathrm{f})/\mathsf{K}^p \to \widehat{\mathscr{S}_{\mathsf{K},W}}$$

so that on points over $R \in \mathrm{Nilp}_W$ we have $(X, \iota, g\mathsf{K}^p) \mapsto (\mathcal{A}, (t_\alpha), \eta^p g)$. This morphism commutes with the Weil descent data over $\mathscr{O}_{E,\mathfrak{p}}$.

Definition 4.4. Let $I^{\phi}(\mathbb{Q})$ be the group of quasi-isogenies $\mathcal{A}_{\mathsf{K},x} \dashrightarrow \mathcal{A}_{\mathsf{K},x}$ which preserves $(t^{\mathrm{univ}}_{\alpha,x})$ and $(t^{\mathrm{univ},p}_{\alpha,\text{ét},x})$. Note that $I^{\phi}(\mathbb{Q})$ only depends on ϕ, not on the individual x. We view $I^{\phi}(\mathbb{Q})$ naturally as a subgroup of $J_b(\mathbb{Q}_p)$ and $G(\mathbb{A}^p_\mathrm{f})$. We let $I^{\phi}(\mathbb{Q})$ act on $\mathtt{RZ}_{G,b} \times G(\mathbb{A}^p_\mathrm{f})/\mathsf{K}^p$ via left translation.

Remark 4.4.1. In the general Hodge-type (non-PEL) setting, it is a non-trivial theorem of Kisin that $I^{\phi}(\mathbb{Q})$ is the $\mathbb{Q}$-points of an inner form of some Levi subgroup of G with explicit description at each place of $\mathbb{Q}$.[11] We state this result in Theorem 4.8, and it will be used to prove the stronger statement of Rapoport-Zink uniformisation; *cf.* §4.9, §4.10.

Lemma 4.4.2. *The subgroup $I^{\phi}(\mathbb{Q}) \subset J_b(\mathbb{Q}_p) \times G(\mathbb{A}^p_\mathrm{f})$ is discrete.*

Proof. (Compare with the proof of Theorem 6.23 in [30, p.289].) Note that $J_b(\mathbb{Q}_p)$ has an open compact subgroup consisting of isomorphisms of $\mathbb{X}$; namely, $J_b(\mathbb{Q}_p) \cap \mathrm{GL}(W \otimes \Lambda)$. Let $U \subset J_b(\mathbb{Q}_p) \times G(\mathbb{A}^p_\mathrm{f})$ be an open subgroup such that the image in $G(\mathbb{A}^p_\mathrm{f})$ stabilises $\prod_{\ell \neq p} T_\ell(\mathcal{A}_{\mathsf{K},x})$ and the image in $J_b(\mathbb{Q}_p)$ is contained in the open

[11] This result can be proved much more easily in the PEL case.

compact subgroup of isomorphisms. This is always possible to arrange by replacing U with an open subgroup of finite index. Then $I^\phi(\mathbb{Q})\cap U$ is a finite group since it is a subgroup of the automorphism group of polarised abelian variety $(\mathcal{A}_{\mathsf{K},x},\lambda)$. □

Proposition 4.5. *Assume that K^p is "small enough". Then the quotient*

$$I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$$

is representable by a formal scheme which is locally formally of finite type and formally smooth over W, and the Weil descent datum Φ of $\mathtt{RZ}_{G,b}$ induces a Weil descent datum on this quotient.

The morphism $\Theta^\phi:\mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p\to\widehat{\mathscr{S}_{\mathsf{K}}}$, defined in Corollary 4.3.2, is invariant under the $I^\phi(\mathbb{Q})$-action and the induced morphism of formal schemes

$$\Theta^\phi: I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p\to\widehat{\mathscr{S}_{\mathsf{K},W}}$$

is a monomorphism of functors on Nilp_W.

Proof. Let us first show that the quotient $I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$ is representable by a formal algebraic space. Note that

$$I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p=\coprod_{\Gamma}\Gamma\backslash\mathtt{RZ}_{G,b} \tag{4.5.1}$$

where $\Gamma\subset J_b(\mathbb{Q}_p)$ runs over discrete subgroups of the form $I^\phi(\mathbb{Q})\cap g\mathsf{K}^pg^{-1}$ for $g\in G(\mathbb{A}^p_{\mathrm{f}})$. Such a group Γ is separated with respect to the profinite topology and discrete by Lemma 4.4.2. Also Γ is torsion-free if K^p is "small enough" (more precisely, if K^p fixes the n-torsion points of $\mathcal{A}_{\mathsf{K},x}$ for some $n\geqslant 3$; *cf.* the proof of Theorem 6.23 in [30, p.289–290].) Then, the Γ-action on $\mathtt{RZ}_{G,b}$ has no fixed point since the Γ-action on $\mathtt{RZ}_b$ has no fixed point by [30, Corollary 2.35]. We then show that $\Gamma\backslash\mathtt{RZ}_{G,b}$ is representable by a formal algebraic space by repeating the proof of [30, Proposition 2.37]. (Alternatively, one may apply [30, Proposition 2.37] to show that the quotient $\Gamma\backslash\mathtt{RZ}_b$ is representable by a formal algebraic space and observe that $\mathtt{RZ}_{G,b}$ is a Γ- stable closed formal subscheme of $\mathtt{RZ}_b$.)

It is clear that $\Theta^\phi:\mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p\to\widehat{\mathscr{S}_{\mathsf{K},W}}$ is invariant under the $I^\phi(\mathbb{Q})$-action. We now show that the induced map of formal algebraic spaces

$$\Theta^\phi: I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p\to\widehat{\mathscr{S}_{\mathsf{K},W}}$$

is a monomorphism of functors on Nilp_W. Indeed, the injectivity on $\overline{\mathbb{F}}_p$-points is clear from Proposition 4.3, and Θ^ϕ induces an isomorphism on the completions at any $\overline{\mathbb{F}}_p$-point (by Proposition 4.1.6). The claim now follows from descent and direct limit consideration.

Note that any algebraic space which is separated and locally quasi-finite over a scheme is a scheme (*cf.* [23, Théorème (A.2)]). This shows that any closed algebraic subspace of $I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$ is a scheme[12], which shows that $I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$ can be represented by a formal scheme.

The assertion on the Weil descent datum follows since $I^\phi(\mathbb{Q})$ act on $\mathtt{RZ}_{G,b}$ via $I^\phi(\mathbb{Q})\hookrightarrow J_b(\mathbb{Q}_p)$ whose action commutes with the Weil descent datum Φ of $\mathtt{RZ}_{G,b}$. This concludes the proof. □

We finish by identifying $I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$ as the completion of $\widehat{\mathscr{S}_{\mathsf{K},W}}$ at a (possibly infinite) chain of closed subschemes. We first recall the following definition:

[12]For example, the image of $\mathtt{RZ}_{G,b}(h)^{m,n}\times G(\mathbb{A}^p_{\mathrm{f}})$ in $I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$ is a scheme for each (m,n), where $\mathtt{RZ}_{G,b}(h)^{m,n}$ is introduced in [15, §6.1]

Definition 4.6. Let $\mathfrak{X}$ be a formal scheme and $\mathcal{Z} := \{Z_i\}_{i\in\mathbf{I}}$ where $Z_i \subset |\mathfrak{X}|$ is a closed subset such that for each $i \in \mathbf{I}$ there are only finitely many $j \in \mathbf{I}$ with $Z_i \cap Z_j \neq \emptyset$.

We define the *completion* $\mathfrak{X}_{/\mathcal{Z}}$ *of* $\mathfrak{X}$ *along* $\mathcal{Z}$ to be the following formal scheme. The underlying topological space is

$$|\mathfrak{X}_{/\mathcal{Z}}| := \bigcup_{i\in\mathbf{I}} Z_i$$

with the direct limit topology. For each $x \in |\mathfrak{X}_{/\mathcal{Z}}|$, we consider the open subset of $|\mathfrak{X}_{/\mathcal{Z}}|$:

$$\mathcal{Z}(x) := \left(\bigcup_{x\in Z_i} Z_i\right) \setminus \left(\bigcup_{x\notin Z_i} Z_i\right),$$

which is also a locally closed subset of $\mathfrak{X}$. We give a formal scheme structure on $\mathcal{Z}(x)$ as the completion of $\mathfrak{X}$ along $\mathcal{Z}(x)$. The formal scheme $\mathfrak{X}_{/\mathcal{Z}}$ is obtained by glueing these formal schemes on $\mathcal{Z}(x)$ as we vary $x \in |\mathfrak{X}_{/\mathcal{Z}}|$.

Note that if the index set $\mathbf{I}$ is finite (i.e., $Z := \bigcup_{i\in\mathbf{I}} Z_i$ is a Zariski-closed subset of $|\mathfrak{X}|$) then $\mathfrak{X}_{/\mathcal{Z}}$ is the completion of $\mathfrak{X}$ along Z.

Example 4.6.1. We give an example of $\mathcal{Z} = \mathscr{I}^{\phi}$ for $\mathfrak{X} = \widehat{\mathscr{S}_{\mathsf{K},W}}$. For an isogeny class ϕ, set $\mathscr{I}^{\phi} := \{Z_i\}_{i\in\mathbf{I}}$, where $\mathbf{I}$ be the set of $I^{\phi}(\mathbb{Q})$-orbits of irreducible components of $\mathtt{RZ}_{G,b} \times G(\mathbb{A}_{\mathrm{f}}^{p})/\mathsf{K}^{p}$, and $Z_i \subset |\widehat{\mathscr{S}_{\mathsf{K},W}}|$ for $i \in \mathbf{I}$ is the image by Θ^{ϕ} of the $I^{\phi}(\mathbb{Q})$-orbit of irreducible components corresponding to $i \in \mathbf{I}$. To see that Z_i is a closed subset, note that any irreducible component of $\mathtt{RZ}_{G,b}^{\mathrm{red}}$ is projective. One can check that any Z_i intersects with only finitely many Z_j's from (4.5.1). Therefore we can define $(\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{I}^{\phi}}$ as in Definition 4.6.

The following theorem is a Hodge-type generalisation of the unramified case of [30, Theorem 6.23].

Theorem 4.7. *The morphism Θ^{ϕ}, obtained in Proposition 4.5, induces an isomorphism of formal schemes respecting the natural Weil descent datum over $\mathscr{O}_{E,\mathfrak{p}}$:*

$$\Theta^{\phi} : I^{\phi}(\mathbb{Q})\backslash\mathtt{RZ}_{G,b} \times G(\mathbb{A}_{\mathrm{f}}^{p})/\mathsf{K}^{p} \xrightarrow{\sim} (\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{I}^{\phi}}.$$

Proof. Note that Θ^{ϕ} in the statement is a formally étale surjective monomorphism which induces a proper morphism on the underlying reduced schemes. Such a morphism between locally noetherian formal schemes is an isomorphism; see the proof of Theorem 6.23 in [30, p.290]. □

For the remainder of the section, we prove some refinements of Theorem 4.7; namely, we descend the isomorphism Θ^{ϕ} in Theorem 4.7 over $\mathscr{O}_{E,\mathfrak{p}}$ (not just over $W = W(\overline{\mathbb{F}}_p)$), and simplify the statement of Theorem 4.7 in the "basic" case. For this, we need the following theorem of Kisin (which is highly non-trivial in the non-PEL case):

Theorem 4.8 (Kisin)**.** *The group $I^{\phi}(\mathbb{Q})$ as in Definition 4.4 is the $\mathbb{Q}$-points of reductive $\mathbb{Q}$-group I^{ϕ}, which is an inner form of some Levi subgroup of G. More precisely, there exists an element $\gamma_0 \in G(\mathbb{Q})$ such that I^{ϕ} is an inner form of the centraliser $G_{\gamma_0} \subset G$ of γ_0.*

Furthermore, $I^{\phi}(\mathbb{R})$ is compact modulo centre, and we have $I^{\phi}_{\mathbb{Q}_p} = J_b$ and $I^{\phi}_{\mathbb{Q}_\ell} \cong (G_{\gamma_0})_{\mathbb{Q}_\ell}$ for $\ell \neq p, \infty$, which naturally recovers the embedding $I^{\phi}(\mathbb{Q}) \subset J_b(\mathbb{Q}_p) \times G(\mathbb{A}_{\mathrm{f}}^{p})$.

Proof. This is a direct consequence of Corollaries 2.3.1 and 2.3.5 in [18]. Note that the assertion on $I^{\phi}_{\mathbb{Q}_v}$ for all places v of $\mathbb{Q}$ can be deduced from the properties of γ_0 in [18, Corollary 2.3.1]. □

4.9. **Effectivity of Weil descent.** Although the Weil descent datum Φ on $\mathtt{RZ}_{G,b}$ is not effective, we will show that Φ induces an effective Weil descent datum on $I^{\phi}(\mathbb{Q})\backslash\mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$. In particular, by Theorem 4.7 $(\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{I}^{\phi}}$ descends over $\operatorname{Spf}\mathscr{O}_{E,\mathfrak{p}}$; *cf.* Corollary 4.9.3. In the PEL case, this result can be obtained from Theorem 3.49 and Proposition 6.16 in [30].

By Kottwitz' theorem (Proposition 2.6.2), we may assume that $b\in G(K_0)$ satisfies the equation $(b\sigma)^r=(r\nu_b)(p)\sigma^r$ by replacing b up to σ-conjugacy in $G(K_0)$. Viewing $(r\nu_b)(p)\in J_b(\mathbb{Q}_p)$ as a quasi-isogeny of X_0 (*cf.* Proposition 2.6.5), the height of $(r\nu_b)(p)$ is precisely $r\dim X_0$.[13] Therefore, we have an isomorphism

$$\langle(r\nu_b)(p)\rangle\backslash\mathtt{RZ}_{G,b}\cong\coprod_{h=0}^{r\dim X_0-1}\mathtt{RZ}_{G,b}(h),$$

where $\mathtt{RZ}_{G,b}(h)$ is a quasi-compact open and closed formal subscheme defined by requiring the height of the quasi-isogeny to be $h\in\mathbb{Z}$.

Since $(r\nu_b)(p)$ is in the centre of $J_b(\mathbb{Q}_p)$ (*cf.* Proposition 2.6.5), the natural left action of $J_b(\mathbb{Q}_p)$ on $\mathtt{RZ}_{G,b}$ descends to the quotient, and the Weil descent datum Φ on $\mathtt{RZ}_{G,b}$ induces a Weil descent datum on this quotient.

Proposition 4.9.1. *The Weil descent datum Φ on $\langle(r\nu_b)(p)\rangle\backslash\mathtt{RZ}_{G,b}$ is effective for any $r\in\mathbb{Z}$ such that $r\nu_b:\mathbf{D}\to G_{K_0}$ factors through $\mathbb{G}_m$ (via the natural projection $\mathbf{D}\twoheadrightarrow\mathbb{G}_m$.*

Proof. Note that the closed immersion $\mathtt{RZ}_{G,b}\hookrightarrow\mathtt{RZ}_b$ commutes with the Weil descent datum Φ over $\mathscr{O}_{E,\mathfrak{p}}$, so it suffices to prove the claim for $\mathtt{RZ}_b$ instead of $\mathtt{RZ}_{G,b}$. The case of $\mathtt{RZ}_b$ was already handled in [30, Theorem 3.49]. □

Next, we would like to approximate a suitable power of $(r\nu_b)(p)$ to a global element. Let us introduce some notation. Let $Z^{\phi}\subset I^{\phi}$ denote the centre. Note that $Z^{\phi}(\mathbb{Q}_p)$ is precisely the centre of $J_b(\mathbb{Q}_p)$, so we view $(r\nu_b)(p)$ as an element of $Z^{\phi}(\mathbb{Q}_p)$, which is contained in $Z^{\phi}(\mathbb{A}_{\mathrm{f}})$.

Set $U^{\phi,p}=Z^{\phi}(\mathbb{A}^p_{\mathrm{f}})\cap\mathsf{K}^p$ where the intersection is taken inside $G(\mathbb{A}^p_{\mathrm{f}})$, and choose an open compact subgroup $U^{\phi}_p\subset Z^{\phi}(\mathbb{Q}_p)$ so that it is contained in the open compact subgroup of $J_b(\mathbb{Q}_p)$ consisting of automorphisms of $\mathbb{X}$. Since $U^{\phi}:=U^{\phi}_pU^{\phi,p}$ is an open compact subgroup of $Z^{\phi}(\mathbb{A}_{\mathrm{f}})$, the following abelian group

$$Z^{\phi}(\mathbb{Q})\backslash Z^{\phi}(\mathbb{A}_{\mathrm{f}})/U^{\phi}$$

is finite. We may assume that $(r\nu_b)(p)\in Z^{\phi}(\mathbb{Q})\cdot U^{\phi}$ by replacing r with a suitable integer multiple of r. Therefore, we may (and do) choose $r\in\mathbb{Z}$, so that there exists $z\in Z^{\phi}(\mathbb{Q})$ with $z\equiv(r\nu_b)(p)\bmod U^{\phi}$.

We have just proved the following proposition, which generalises [30, Proposition 6.16]:

Proposition 4.9.2. *The map $\Theta^{\phi}:\mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p\to\widehat{\mathscr{S}_{\mathsf{K},W}}$ (*cf.* Corollary 4.3.2) factors through $\langle(r\nu_b)(p)\rangle\backslash\mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$, where r is chosen as above.*

The following Corollary is straightforward from Propositions 4.9.1 and 4.9.2:

Corollary 4.9.3. *The Weil descent datum Φ on $I^{\phi}(\mathbb{Q})\backslash\mathtt{RZ}_{G,b}\times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p$ is effective.*

[13]Note that we work with contravariant Dieudonné theory, while the formula in [30, §3.41] is deduced via covariant Dieudonné theory.

4.10. **The basic case.** We continue to work in the setting of §3.3.13 and §4.2. Recall the following result of Rapoport and Richartz:

Theorem 4.10.1 ([29, Theorem 3.6]). *Let $\mathscr{S}_{\mathsf{K},[b]} \subset \mathscr{S}_{\mathsf{K}}(\overline{\mathbb{F}}_p)$ be the set of points y such that there exists an isogeny of p-divisible groups $\mathcal{A}_{\mathsf{K},x}[p^\infty] \to \mathcal{A}_{\mathsf{K},y}[p^\infty]$ matching $(t^{\mathrm{univ}}_{\alpha,x})$ and $(t^{\mathrm{univ}}_{\alpha,y})$. Then $\mathscr{S}_{\mathsf{K},[b]}$ is a locally closed subset, which is closed if $[b]$ is basic.*

We will often regard $\mathscr{S}_{\mathsf{K},[b]}$ as a locally closed reduced subscheme of $\mathscr{S}_{\mathsf{K},\overline{\mathbb{F}}_p}$, and call it a *Newton stratum*. If $[b]$ is basic, then we call $\mathscr{S}_{\mathsf{K},[b]}$ a *basic Newton stratum*.

The following is a corollary of Kisin's theorem on I^ϕ (Theorem 4.8):

Corollary 4.10.2. *Let I^ϕ be the reductive group where $I^\phi(\mathbb{Q})$ is the quasi-isogeny group of $(\mathcal{A}_{\mathsf{K},x}, (t^{\mathrm{univ}}_{\alpha,x}))$, which exists by Kisin's theorem (Theorem 4.8). If x is in a basic Newton stratum $\mathscr{S}_{\mathsf{K},[b]}$, then I^ϕ is an inner form of G.*

Proof. By Kisin's theorem, I^ϕ is an inner form of a Levi subgroup of G with $I^\phi_{\mathbb{Q}_p} = J_b$. But by [30, Corollary 1.14] it follows that J_b is an inner form of $G_{\mathbb{Q}_p}$ if b is basic. □

Proposition 4.10.3. *If $[b]$ is basic then all $\overline{\mathbb{F}}_p$-points of $\mathscr{S}_{\mathsf{K},[b]}$ are isogenous; i.e., there exists a unique isogeny class ϕ consisting of all $\overline{\mathbb{F}}_p$-points in $\mathscr{S}_{\mathsf{K},[b]}$.*

In the PEL case, see the proof of Theorem 6.30 in [30, p.293–294] or [11, Proposition 3.1.8].

Proof. As I^ϕ is an inner form of G, it follows that γ_0 is in the centre of $G(\mathbb{Q})$. It follows without difficulty from the definition that any isogeny class ϕ occurring in a basic Newton stratum $\mathscr{S}_{\mathsf{K},[b]}$ gives rise to a unique equivalence class of Kottwitz triples $\mathfrak{k} = (\gamma_0, (\gamma_\ell)_{\ell\neq p}, b)$. See [18, §4.3.1] for the definitions of Kottwitz triple and the equivalence relations, and [18, Corollary 2.3.1] for the construction of the equivalence class of Kottwitz triple from an isogeny class ϕ. (In the PEL case, this assertion can be extracted from [30, Lemma 6.28], as observed in the proof of [11, Proposition 3.1.8].)

It now follows from [18, Proposition 4.4.13] that there exists only one isogeny class ϕ occurring in $\mathscr{S}_{\mathsf{K},[b]}$. Indeed, [18, Proposition 4.4.13] asserts that the set of isogeny classes which give rise to the Kottwitz triple $\mathfrak{k}$ is in bijection with $\text{Ш}_G(\mathbb{Q}, I^\phi)$ (see §4.4.9 and §4.4.7 in [18] for the definition of $\text{Ш}_G(\mathbb{Q}, I^\phi)$), which is clearly a singleton if b is basic (as I^ϕ is an inner form of G). □

Theorem 4.11. *Let ϕ be the unique isogeny class occurring in $\mathscr{S}_{\mathsf{K},[b]}$. Then the following map of formal schemes*

$$\Theta^\phi : I^\phi(\mathbb{Q})\backslash \mathtt{RZ}_{G,b} \times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p \to (\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{S}_{\mathsf{K},[b]}} \tag{4.11.1}$$

is an isomorphism. In particular, $\mathscr{I}^\phi$ consists of finitely many irreducible components of $\mathscr{S}_{\mathsf{K},[b]}$, and we have $(\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{I}^\phi} = (\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{S}_{\mathsf{K},[b]}}$.

Furthermore, the Weil descent datum Φ on the left hand side of (4.11.1) is effective.

In the PEL case, this theorem was proved in [30, Theorem 6.30].

Proof. It suffices to show that Θ^ϕ (4.11.1) is an isomorphism, for which it suffices to show that it is surjective on the underlying topological spaces since the map Θ^ϕ (4.11.1) is a monomorphism by Theorem 4.7. The Θ^ϕ is clearly surjective on the set of $\overline{\mathbb{F}}_p$-points, so we use induction on the height.

Choose two points $y, \eta \in \mathscr{S}_{\mathsf{K},[b]}$ such that $\overline{\{y\}}$ is contained in $\overline{\{\eta\}}$ as a codimention-1 subset, and assume that y is in the image of Θ^ϕ. We want to show that η is also in the image of Θ^ϕ. For this, it suffices to show that the following map

$$f : S := \operatorname{Spec} \kappa[[u]] \to \mathscr{S}_{\mathsf{K},[b]}$$

factors through Θ^ϕ, where κ is an algebraically closed field, $f_{\mathrm{Spec}\,\kappa}$ lands in y, and $f|_{\mathrm{Spec}\,\kappa((u))}$ lands in η.

Note that there exists a unique isomorphism of "F-isocrystals with tensors":

$$(4.11.2)\qquad (\mathbb{D}(\mathcal{A}_{\mathsf{K},y}[p^\infty])[1/p])_S \cong (\mathbb{D}(\mathcal{A}_{\mathsf{K},S}[p^\infty])[1/p].$$

Indeed, the existence was proved by Katz [14, §2.7], and the uniqueness was proved in [29, Lemma 3.9].[14]

By the induction hypothesis on y, f_κ can be lifted to $(\tilde{f}_\kappa, g\mathsf{K}^p) : S \to \mathtt{RZ}_{G,b} \times G(\mathbb{A}_{\mathrm{f}}^p)/\mathsf{K}^p$, where $\tilde{f}_\kappa$ corresponds to $(\mathcal{A}_{\mathsf{K},y}[p^\infty], \iota_y) \in \mathtt{RZ}_{G,b}(\kappa)$ for some quasi-isogeny ι_y. So we obtain a quasi-isogeny of p-divisible groups

$$\iota_S = \jmath \circ (\iota_y)_S : (\mathcal{A}_{\mathsf{K},x})_S[p^\infty] \dashrightarrow \mathcal{A}_{\mathsf{K},S}[p^\infty],$$

where $\jmath : (\mathcal{A}_{\mathsf{K},y})_S \dashrightarrow \mathcal{A}_{\mathsf{K},S}$ is the unique p-power order quasi-isogeny giving rise to the isomorphism (4.11.2), which exists by [2, Théorème 4.1.1] (*cf.* [7, Main Theorem 1]). It now follows that $(\mathcal{A}_{\mathsf{K},S}[p^\infty], \iota_S)$ defines a map $f : S \to \mathtt{RZ}_b$.

We next show that the map f factors through $\mathtt{RZ}_{G,b}$; indeed, the completion $\hat{f} : \widehat{S} \to \mathtt{RZ}_b$ factors through $\mathtt{RZ}_{G,b}$ (where $\widehat{S} := \mathrm{Spf}\,\kappa[[u]]$) since Θ^ϕ induces an isomorphism $\widehat{\mathtt{RZ}}_{G,b,y} \xrightarrow{\sim} \widehat{\mathscr{S}}_{\mathsf{K},\Theta^\phi(y)}$, and this proves the claim as $\mathtt{RZ}_{G,b}$ is a closed formal subscheme of $\mathtt{RZ}_b$.

Let $\tilde{f} : S \to \mathtt{RZ}_{G,b}$ denote the map corresponding to $(\mathcal{A}_{\mathsf{K},S}[p^\infty], \iota_S)$. Then the composed map

$$S \xrightarrow{(\tilde{f}, g\mathsf{K})} \mathtt{RZ}_{G,b} \times G(\mathbb{A}_{\mathrm{f}}^p)/\mathsf{K}^p \xrightarrow{\Theta^\phi} (\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{S}_{\mathsf{K},[b]}}$$

coincides with f, which can be seen by passing to the completion at y. □

5. Rapoport-Zink uniformisation via rigid geometry

We continue to assume that $(G, \mathfrak{H})$ is a Hodge-type Shimura datum such that G is unramified at p. Using our results in §4 for hyperspecial maximal level at p we can obtain a rigid analytic uniformisation result for other levels at p (Theorem 5.4), generalising the unramified case of [30, Theorem 6.36].

We continue to assume that $p > 2$ without mentioning it.

5.1. Level structures at p for Hodge-type Shimura varieties in characteristic 0. In §3.2.3 and §3.2.4, we described level structures at p and $G(\mathbb{Q}_p)$-action for Hodge-type Shimura varieties in characteristic 0, working with abelian varieties up to isogeny. Here, we reformulate them only using prime-to-p isogeny classes (so that we can relate it to the rigid analytic tower over $\mathtt{RZ}_{G,b}^{\mathrm{rig}}$). We assume that $(G, \mathfrak{H})$ is of Hodge-type with G unramified at p, and make auxiliary choices as in §3.3.1.

Let K_p be an open compact subgroup of $G(\mathbb{Z}_p)$. For example, we may consider $\mathsf{K}_p^{(0)} := G(\mathbb{Z}_p)$ and $\mathsf{K}_p^{(i)} := \ker(G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^i))$ for $i > 0$. Let $\mathsf{K} := \mathsf{K}_p\mathsf{K}^p$, and consider $(\mathcal{A}_{\mathsf{K},E}, \eta_\mathsf{K})$ where $\mathcal{A}_{\mathsf{K},E}$ is viewed up to isogeny and η_K is as in §3.2.3. We can decompose η_K into the prime-to-p part η_{K^p} (3.3.5) and the p-part

$$(5.1.1)\qquad \eta_{\mathsf{K}_p} \in \Gamma(\mathrm{Sh}_{\mathsf{K}_p^{(i)}\mathsf{K}^p}, \mathrm{isom}\left[(V_{\mathbb{Q}_p}, (s_\alpha)), (\mathcal{V}_{\mathbb{Q}_p}(\mathcal{A}_{\mathsf{K}_p^{(i)}\mathsf{K}^p,E}), (t_{\alpha,\text{ét},p}^{\mathrm{univ}}))\right]/\mathsf{K}_p).$$

In the isogeny class of $\mathcal{A}_{\mathsf{K},E}$, consider the pull-back of the abelian scheme $\mathcal{A}_{\mathsf{K}_p^{(0)}\mathsf{K}^p,E}$, up to prime-to-$p$ isogeny, that extends to the integral canonical model. We also denote it by $\mathcal{A}_{\mathsf{K},E}$. Then η_{K_p} can be viewed as a right K_p-coset of isomorphisms

[14] *A posteriori* the isomorphism (4.11.2) matches $(t_{\alpha,y}^{\mathrm{univ}})$ and $(t_{\alpha,S}^{\mathrm{univ}})$. This can also be seen directly, as the formation of the isomorphism is functorial and commutes with subquotient, $\otimes$-products, and duals, and also with morphisms between them; *cf.* [29, p.174–175].

$\Lambda \xrightarrow{\sim} T_p(\mathcal{A}_{\mathsf{K},E})$ matching tensors. With such identification, we obtain the following description of $\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$:

$$\mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p} \xrightarrow{\sim} \mathrm{isom}_{\mathrm{Sh}_{\mathsf{K}_p^{(0)}\mathsf{K}^p}}\Big([\Lambda_{\mathbb{Z}_p},(s_\alpha)],[T_p(\mathcal{A}_{\mathsf{K}_p^{(0)}\mathsf{K}^p,E}),(t^{\mathrm{univ}}_{\alpha,\text{ét},p})]\Big)/\mathsf{K}_p, \tag{5.1.2}$$

where the morphism is defined by restricting η_{K_p} to $\Lambda_{\mathbb{Z}_p}$.

When $\mathsf{K}_p = \mathsf{K}_p^{(i)}$ for some i, then (5.1.2) can be interpreted as follows:

$$\mathrm{Sh}_{\mathsf{K}_p^{(i)}\mathsf{K}^p} \xrightarrow{\sim} \mathrm{isom}_{\mathrm{Sh}_{\mathsf{K}_p^{(0)}\mathsf{K}^p}}\Big([\Lambda_{\mathbb{Z}_{(p)}}/p^i,(s_\alpha)],[\mathcal{A}_{\mathsf{K}_p^{(0)}\mathsf{K}^p,E}[p^i],(t^{\mathrm{univ}}_{\alpha,\text{ét},p})]\Big). \tag{5.1.3}$$

For $g \in G(\mathbb{Q}_p)$, assume that ${}^g\mathsf{K}_p \subset G(\mathbb{Z}_p)$. (This can be arranged by replacing K_p by a finite-index open subgroup; namely, $G(\mathbb{Z}_p)\cap {}^g\mathsf{K}_p$.) In §3.2.4, we showed that pulling back by $[g] : \mathrm{Sh}_{{}^g\mathsf{K}_p\mathsf{K}^p} \xrightarrow{\sim} \mathrm{Sh}_{\mathsf{K}_p\mathsf{K}^p}$, we have $[g]^*\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p} \sim \mathcal{A}_{{}^g\mathsf{K}_p\mathsf{K}^p}$ up to isogeny, and changes the level structure at p by "right translation by g". To translate this in terms of the level structure at p described as in (5.1.2), the prime-to-p isogeny class of $[g]^*\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p}$ is the unique one in the isogeny class of $\mathcal{A}_{{}^g\mathsf{K}_p\mathsf{K}^p}$ which matches the $\mathbb{Z}_p$-lattices $\Lambda_{\mathbb{Z}_p}$ and $T_p([g]^*\mathcal{A}_{\mathsf{K}_p\mathsf{K}^p})$ via $[g]^*\eta_{\mathsf{K}_p}$, and then $[g]^*\eta_{\mathsf{K}_p}$ defines a section of the right hand side of (5.1.2).

5.2. **Rigid analytic tower of Hodge-type Rapoport-Zink spaces.** Since $\mathtt{RZ}_{G,b}$ is locally formally of finite type over $\mathrm{Spf}\, W$, it is possible to associate the "rigid analytic generic fibre", denoted by $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$.

We use the notation from §5.1, such as $\mathsf{K}_p^{(i)} \subset G(\mathbb{Z}_p)$, and set $\mathtt{RZ}^{\mathsf{K}_p^{(0)}}_{G,b} := \mathtt{RZ}^{\mathrm{rig}}_{G,b}$. For any $\mathsf{K}_p \subset \mathsf{K}_p^{(0)}$, we now define, in a way analogous to (5.1.2), the following rigid analytic étale cover of $\mathtt{RZ}^{\mathrm{rig}}_{G,b}$:

$$\mathtt{RZ}^{\mathsf{K}_p}_{G,b} := \mathrm{isom}_{\mathtt{RZ}^{\mathrm{rig}}_{G,b}}\Big([\Lambda_{\mathbb{Z}_p},(s_\alpha)],[T_p(X_{G,b}),(t^{\mathrm{univ}}_{\alpha,\text{ét},p})]\Big)/\mathsf{K}_p, \tag{5.2.1}$$

where $X_{G,b}$ is the universal p-divisible group over $\mathtt{RZ}_{G,b}$, and $T_p(X_{G,b}) = \{X_{G,b}[p^n]^{\mathrm{rig}}\}$ is the $\mathbb{Z}_p$-local system over $\mathtt{RZ}^{\mathsf{K}_p}_{G,b}$; i.e., the Tate module of $X_{G,b}$

When $\mathsf{K}_p = \mathsf{K}_p^{(i)}$ for some i, then we have

$$\mathtt{RZ}^{\mathsf{K}_p^{(i)}}_{G,b} := \mathrm{isom}_{\mathtt{RZ}^{\mathrm{rig}}_{G,b}}\Big([\Lambda_{\mathbb{Z}_{(p)}}/p^i,(s_\alpha)],[X_{G,b}[p^i],(t^{\mathrm{univ}}_{\alpha,\text{ét},p})]\Big). \tag{5.2.2}$$

It is possible to extend the Galois action of $G(\mathbb{Z}_p)$ on the tower $\{\mathtt{RZ}^{\mathsf{K}_p}_{G,b}\}_{\mathsf{K}_p}$ naturally to a $G(\mathbb{Q}_p)$-action in a way that is analogous to the case of Shimura varieties as discussed in §5.1; *cf.* [15, §7.4].

5.3. **Rigid analytic Rapoport-Zink uniformisation.** We write $\mathsf{K} := \mathsf{K}_p\mathsf{K}^p$ with $\mathsf{K}_p = G(\mathbb{Z}_p)$. For an isogeny class ϕ of $\overline{\mathbb{F}}_p$-points of $\mathscr{S}_{\mathsf{K}}$, we set

$$\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}}(\phi) := \Big((\widehat{\mathscr{S}_{\mathsf{K},W}})_{/\mathscr{I}^\phi}\Big)^{\mathrm{rig}}. \tag{5.3.1}$$

If $\mathscr{I}^\phi$ is a finite collection of irreducible subvarieties of $\mathscr{S}_{\mathsf{K},\overline{\mathbb{F}}_p}$ (e.g., if ϕ is an isogeny class in a basic Newton stratum), then $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}}(\phi)$ is the tube of $\mathscr{I}^\phi$ in $\widehat{\mathscr{S}_{\mathsf{K},W}}$. In general, $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}}(\phi)$ is a union of tubes of the irreducible subvarieties $Z \in \mathscr{I}^\phi$.

Since the construction of rigid analytic generic fibre is functorial, we obtain the following maps of rigid analytic spaces over K_0 from Theorem 4.7 for $\mathsf{K} := \mathsf{K}_p^{(0)}\mathsf{K}^p$:

$$\Theta^{\phi,\mathrm{rig}} : I^\phi(\mathbb{Q})\backslash \mathtt{RZ}^{\mathrm{rig}}_{G,b} \times G(\mathbb{A}^p_{\mathrm{f}})/\mathsf{K}^p \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}}(\phi) \tag{5.3.2}$$

Furthermore, the rigid analytic spaces and the maps in (5.3.2) descend over $E_{\mathfrak{p}}$ by Corollary 4.9.3.

From now on, assume that $\mathsf{K} := \mathsf{K}_p\mathsf{K}^p \subset G(\mathbb{A}_f)$ such that $\mathsf{K}_p \subset \mathsf{K}_p^{(0)} = G(\mathbb{Z}_p)$, and $\mathsf{K} \subset G(\mathbb{A}_f)$ is a "small enough". We let $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K},K_0}$ and $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K},E_\mathfrak{p}}$ respectively denote the rigid analytifications of $\mathrm{Sh}_{\mathsf{K},K_0}$ and $\mathrm{Sh}_{\mathsf{K},E_\mathfrak{p}}$.[15]

Definition 5.3.3. We let $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}}(\phi)$ denote the preimage of $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p^{(0)}\mathsf{K}^p}(\phi)$ via the natural projection map $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p\mathsf{K}^p,K_0} \to \mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p^{(0)}\mathsf{K}^p,K_0}$. Equivalently, by (5.1.2) we have

$$\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p\mathsf{K}^p}(\phi) \cong \mathrm{isom}_{\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p^{(0)}\mathsf{K}^p}(\phi)}\Big([\Lambda_{\mathbb{Z}_p},(s_\alpha)],[T_p(\mathcal{A}_{\mathsf{K}_p^{(0)}\mathsf{K}^p}),(t^{\mathrm{univ}}_{\alpha,\text{ét},p})]\Big)/\mathsf{K}_p.$$

Since $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p^{(0)}\mathsf{K}^p}(\phi)$ is defined over $E_\mathfrak{p}$ (by Corollary 4.9.3), it follows that $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p\mathsf{K}^p}(\phi)$ is also defined over $E_\mathfrak{p}$.

By matching the definitions of the coverings $\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p\mathsf{K}^p}(\phi) \to \mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}_p^{(0)}\mathsf{K}^p}(\phi)$ (Definition 5.3.3) and $\mathtt{RZ}^{\mathsf{K}_p}_{G,b} \to \mathtt{RZ}^{\mathrm{rig}}_{G,b}$ (5.2.1), we obtain the following theorem:

Theorem 5.4. *Assume that $\mathsf{K} := \mathsf{K}_p\mathsf{K}^p \subset G(\mathbb{A}_f)$ such that $\mathsf{K}_p \subset \mathsf{K}_p^{(0)} = G(\mathbb{Z}_p)$. Then, we can lift $\Theta^{\phi,\mathrm{rig}}$ (5.3.2) to*

$$\Theta^\phi_{\mathsf{K}} : I^\phi(\mathbb{Q})\backslash \mathtt{RZ}^{\mathsf{K}_p}_{G,b} \times G(\mathbb{A}^p_f)/\mathsf{K}^p \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}}(\phi),$$

which also descends over $E_\mathfrak{p}$. Furthermore, by varying K_p and K^p, the isomorphism $\{\Theta^\phi_{\mathbf{K}}\}$ is equivariant for the $G(\mathbb{A}_f)$-action. (On the left hand side, $G(\mathbb{Q}_p)$ acts naturally on $\{\mathtt{RZ}^{\mathsf{K}_p}_{G,b}\}_{\mathsf{K}_p}$, and $G(\mathbb{A}^p_f)$ acts by left translation on $\{G(\mathbb{A}^p_f)/\mathsf{K}^p\}_{\mathsf{K}^p}$. On $\{\mathrm{Sh}^{\mathrm{rig}}_{\mathsf{K}}(\phi)\}_{\mathsf{K}}$, the $G(\mathbb{A}_f)$-action is the restriction on the natural $G(\mathbb{A}_f)$-action on $\{\mathrm{Sh}_{\mathsf{K}}\}_{\mathsf{K}}$.)

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