GALOIS DEFORMATION THEORY FOR NORM FIELDS AND FLAT DEFORMATION RINGS

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Abstract. Let K be a finite extension of $\mathbb{Q}_p$, and choose a uniformizer $\pi \in K$, and put $K_\infty := K(\sqrt[p^\infty]{\pi})$. We introduce a new technique using restriction to $\mathrm{Gal}(\overline{K}/K_\infty)$ to study flat deformation rings. We show the existence of deformation rings for $\mathrm{Gal}(\overline{K}/K_\infty)$-representations "of height $\leqslant h$" for any positive integer h, and we use them to give a variant of Kisin's proof of connected component analysis of a certain flat deformation rings, which was used to prove Kisin's modularity lifting theorem for potentially Barsotti-Tate representations. Our proof does not use the classification of finite flat group schemes.

This $\mathrm{Gal}(\overline{K}/K_\infty)$-deformation theory has a good positive characteristics analogue of crystalline representations in the sense of Genestier-Lafforgue. In particular, we obtain a positive characteristic analogue of crystalline deformation rings, and can analyze their local structure.

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1. Introduction

Since the pioneering work of Wiles on the modularity of semi-stable elliptic curves over $\mathbb{Q}$, there has been huge progress on modularity lifting. Notably, Kisin [Kis09b, Kis09a] (later improved by Gee [Gee06, Gee09]) proved a very general modularity lifting theorem for potentially Barsotti-Tate representations, which had enormous impacts on this subject. (For the precise statement of the theorem, see the aforementioned references.)

One of the numerous noble innovations that appeared in Kisin's result is his improvement of Taylor-Wiles patching argument. The original patching argument required relevant local deformation rings to be formally smooth, which is a very strong requirement. Under Kisin's improved patching, we only need to show that the generic fiber of local deformation rings are formally smooth with correct dimension, and we need to have some control of their connected components. (See [Kis07, Corollary 1.4] for the list of sufficient conditions on local deformation rings to prove modularity lifting.) It turns out that the most difficult part among them (and the hurdle to proving modularity lifting for more general classes of p-adic Galois representations) is to "control" the connected components of certain p-adic

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deformation rings at places over p. (The relevant local deformation rings here are "flat deformation rings".)

The main purpose of this paper is to give another proof of the following theorem of Kisin. Let us fix some notations. Let K be a finite extension of $\mathbb{Q}_p$ and set $\mathcal{G}_K$ to be the absolute Galois group of K. Choose a complete discrete valuation ring $\mathscr{O}$ whose residue field $\mathbb{F}$ is a finite field of characteristic p. Let $\bar{\rho}: \mathcal{G}_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation, and let $R^\square$ be the framed deformation $\mathscr{O}$-algebra of $\bar{\rho}$ (whose existence was shown by Mazur[Maz89]). Let $R^{\square,\mathbf{v}}_{\mathrm{cris}}$ be the unique torsion-free quotient of $R^\square$ whose A-points classifies crystalline lifts of $\bar{\rho}$ with "Hodge type $(0,1)$" for any finite $\mathbb{Q}_p$-algebra A. (See §4.4 for the definition.) By [Kis06, Corollary 2.2.6] and [Ray74, Proposition 2.3.1], $R^{\square,\mathbf{v}}_{\mathrm{cris}}$ differs only by p^∞-torsion from the flat framed deformation ring of $\bar{\rho}$ with the deformation condition that the inertia action on the determinant should be given by the p-adic cyclotomic character.

Theorem 1.1 (Kisin; Gee, Imai)**.** *For finite local $\mathbb{Q}_p$-algebras A and A', consider maps $\xi : R^{\square,\mathbf{v}}_{\mathrm{cris}} \to A$ and $\xi' : R^{\square,\mathbf{v}}_{\mathrm{cris}} \to A'$. Let ρ_ξ and $\rho_{\xi'}$ denote the lifts of $\bar{\rho}$ corresponding to ξ and ξ', respectively. Then ξ and ξ' are supported on the same connected component of $\operatorname{Spec} R^{\square,\mathbf{v}}_{\mathrm{cris}}[\frac{1}{p}]$ if and only if either both ρ_ξ and $\rho_{\xi'}$ do not admit a non-zero unramified quotient (i.e. non-ordinary) or both ρ_ξ and $\rho_{\xi'}$ admit a rank-1 unramified quotient which lift the same (mod p) character.*

Note that proving Theorem 1.1 when $p = 2$ is the main extra difficulty in proving the 2-adic potentially Barsotti-Tate modularity lifting theorem (as opposed to the p-adic modularity lifting theorem with $p > 2$).

Kisin's original argument crucially uses a kind of "resolution" of flat deformation rings constructed via the classification of finite flat group schemes[1], which is proved by Kisin [Kis09b, Corollary 2.3.6] when $p > 2$, and Kisin [Kis09a, Theorem 1.3.9] for connected finite flat group schemes when $p = 2$.[2] (This step is where the main difficulty arises when $p = 2$, as opposed to when $p > 2$.) The remaining argument to prove Theorem 1.1 is rather "linear-algebraic", and is carried out in [Kis09b, §2] with improvements by Gee [Gee06] and Imai [Ima08].[3]

The purpose of this paper is to give another proof of Theorem 1.1 that does not use finite flat group schemes, and instead relies more on "linear-algebraic" tools from p-adic Hodge Theory developed in [Kis06]. The key idea is to introduce deformation rings of height $\leqslant 1$ (Theorem 2.3) and compare it with flat deformation rings (Corollary 4.2.1). Another motivation for removing finite flat group schemes from the proof is that it would be a sensible first step for the modularity lifting theorem with higher weight where no reasonable analogue of finite flat group schemes for torsion representations is available. (We do not claim, however, that our proof gives any indication towards this generalization.)

We point out that our technique is motivated by the author's study of positive characteristic analogue of crystalline deformation rings (using the theory of

[1] In fact, the construction of the resolved deformation space could be carried out using a slightly weaker statement.

[2] This classification of finite flat group schemes when $p = 2$ is now proved in [Kim10] without the connectedness assumption, but in order to prove Theorem 1.1 it is enough to know the classification of *connected* finite flat group schemes which was already proved by Kisin [Kis09a, Theorem 1.3.9].

[3] The most of the proof appears in Kisin [Kis09b, §2] while he only completed the proof when the residue field of K is $\mathbb{F}_p$. Gee [Gee06] then made some technical improvement and proved Theorem 1.1 assuming $\bar{\rho} \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and Imai [Ima08] removed the final assumption. Note that for proving a modularity lifting theorem we may always assume $\bar{\rho} \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so Gee's improvement suffices for this purpose.

The papers [Gee06, Ima08] were written under the assumption that $p > 2$, but the same computations work when $p = 2$.

Genestier-Lafforgue [GL10] and Hartl [Har10, Har09]). We include a section (§5) to sketch this positive characteristic deformation theory.

1.2. **Structure and overview of the paper.** Let K be a finite extension of $\mathbb{Q}_p$, $K_\infty = K(\sqrt[p^\infty]{\pi})$ for a chosen uniformizer $\pi \in K$, $\mathcal{G}_K := \mathrm{Gal}(\overline{K}/K)$ and $\mathcal{G}_{K_\infty} := \mathrm{Gal}(\overline{K}/K_\infty)$. It was observed (first by Breuil) that a certain class $\mathcal{G}_{K_\infty}$-representations (more precisely, $\mathcal{G}_{K_\infty}$-representations of height $\leqslant 1$, defined in Definition 2.1.3) plays an important role in classifying finite flat group schemes and p-divisible groups over $\mathscr{O}_K$. In this paper, instead of using the classification of finite flat group schemes, we directly work with "$\mathcal{G}_{K_\infty}$-deformations of height $\leqslant 1$" (defined in §2.2).

In §2 we prove that the "$\mathcal{G}_{K_\infty}$-deformation functor of height $\leqslant 1$" is representable. (See §2.4 for an explanation why this is not an "obvious" theorem.) In §3 we study the structure of this $\mathcal{G}_{K_\infty}$-deformation ring of height $\leqslant 1$ by "resolving" the deformation space via an analogue of moduli of finite flat group schemes (as in Kisin [Kis09b, §2]). In §4, we relate the "Barsotti-Tate deformation ring" and the "$\mathcal{G}_{K_\infty}$-deformation ring of height $\leqslant 1$", and deduce Theorem 1.1. The main input in §4 is Proposition 4.3.1, which can be read off from the literature when $p > 2$, and is proved when $p = 2$ by the author in [Kim10, Proposition 5.6]. In the last section §5, we explain the positive characteristic analogue of this deformation theory, which inspired the author to study the p-adic $\mathcal{G}_{K_\infty}$-deformations of height $\leqslant 1$.

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2. Deformation rings of height $\leqslant h$

Let k be a finite extension of $\mathbb{F}_p$, $W(k)$ its ring of Witt vectors, and $K_0 := W(k)[\frac{1}{p}]$. Let K be a finite totally ramified extension of K_0 and let us fix its algebraic closure $\overline{K}$. We *fix* a uniformizer $\pi \in K$. and choose $\pi^{(n)} \in \overline{K}$ so that $(\pi^{(n+1)})^p = \pi^{(n)}$ and $\pi^{(0)} = \pi$. Put $K_\infty := \bigcup_n K(\pi^{(n)})$, $\mathcal{G}_K := \mathrm{Gal}(\overline{K}/K)$, and $\mathcal{G}_{K_\infty} := \mathrm{Gal}(\overline{K}/K_\infty)$. We refer to [Kis06] for the motivation of considering K_∞.

2.1. **Etale φ-modules and Kisin modules.** Let us consider a ring R equipped with an endomorphism $\sigma : R \to R$. (We will often assume that σ is finite flat.) By (φ, R)-*module* (often abbreviated as a φ-*module*, if R is understood), we mean a finitely presented R-module M together with an R-linear morphism $\varphi_M : \sigma^* M \to M$, where σ^* denotes the scalar extension by σ. A *morphism* between to (φ, R)-modules is a φ-compatible R-linear map. For any R-algebra R' equipped with an endomorphism σ' over σ, the "scalar extension" $M \otimes_R R'$ has a natural (φ, R')-module structure.

Let $\mathfrak{S} := W(k)[[u]]$ where u is a formal variable. Let $\mathscr{O}_\mathcal{E}$ be the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$, and $\mathcal{E} := \mathscr{O}_\mathcal{E}[\frac{1}{p}]$. Note that $\mathscr{O}_\mathcal{E}$ is a complete discrete valuation ring with uniformiser p and $\mathscr{O}_\mathcal{E}/(p) \cong k(\!(u)\!)$.[4] We extend the Witt vectors Frobenius to $\mathfrak{S}$, $\mathscr{O}_\mathcal{E}$, and $\mathcal{E}$ by sending u to u^p, and denote them by σ. (We write $\sigma_\mathfrak{S}$ instead, if we need to specify that it is an endomoprhism on $\mathfrak{S}$, for example.) Note that σ is finite and flat. We denote by $\sigma^*(\cdot)$ the scalar extension by σ. We fix an Eisenstein

[4] We should view the residue field $k(\!(u)\!)$ as the norm field for the extension K_∞/K. See [Win83] for more details.

polynomial $\mathcal{P}(u) \in W(k)[u]$ with $\mathcal{P}(\pi) = 0$ and $\mathcal{P}(0) = p$, and view it as an element of $\mathfrak{S}$.

Definition 2.1.1. An *étale φ-module* is a $(\varphi, \mathscr{O}_{\mathcal{E}})$-module (M, φ_M) such that $\varphi_M : \sigma^* M \xrightarrow{\sim} M$ is an isomorphism. We say an étale φ-module M is *free* (respectively, *torsion*) if the underlying $\mathscr{O}_{\mathcal{E}}$-module is free (respectively, p^∞-torsion).

For a non-negative integer h, a *φ-module of height* $\leqslant h$ is a $(\varphi, \mathfrak{S})$-module such that the underlying $\mathfrak{S}$-module $\mathfrak{M}$ is free and $\mathrm{coker}(\varphi_{\mathfrak{M}})$ is killed by $\mathcal{P}(u)^h$. A *torsion φ-module of height* $\leqslant h$ is a $(\varphi, \mathfrak{S})$-module such that the underlying $\mathfrak{S}$-module $\mathfrak{M}$ is p^∞-torsion with no non-zero u-torsion and $\mathrm{coker}(\varphi_{\mathfrak{M}})$ is killed by $\mathcal{P}(u)^h$.

Note that a nonzero p^∞-torsion $\mathfrak{S}$-module is of projective dimension $\leqslant 1$ if and only if it has no non-zero u-torsion, and by [Kis06, Lemma 2.3.4] any torsion φ-module of height $\leqslant h$ is obtained as a φ-equivariant quotient of φ-module of height $\leqslant h$.

Let $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$ denote the p-adic completion of strict henselization of $\mathscr{O}_{\mathcal{E}}$. By the universal property of strict henselization, $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$ has a natural $\mathcal{G}_{K_\infty}$-action and a ring endomorphism σ. For a finitely generated $\mathbb{Z}_p$-module T with continuous $\mathcal{G}_{K_\infty}$-action, define

$$\underline{D}_{\mathcal{E}}(T) := (T \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\mathcal{G}_{K_\infty}}, \tag{2.1.2a}$$

equipped with the φ-structure induced from σ on $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$. For an étale φ-module M, define

$$\underline{T}_{\mathcal{E}}(M) := (M \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1}, \tag{2.1.2b}$$

viewed as a $\mathcal{G}_{K_\infty}$-module via its natural action on $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}}$.

By Fontaine [Fon90, §A 1.2], $\underline{T}_{\mathcal{E}}$ and $\underline{D}_{\mathcal{E}}$ define quasi-inverse exact equivalences of categories between the categories of étale φ-modules and the category of finitely generated $\mathbb{Z}_p$-module with continuous $\mathcal{G}_{K_\infty}$-action, which respects all the natural operations and preserves ranks and lengths whenever applicable.

Let $\mathfrak{M}$ denote either a φ-module of height $\leqslant h$ or a torsion φ-module of height $\leqslant h$. Then $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}}$ is an étale φ-module[5], so we may associate $\mathcal{G}_{K_\infty}$-representation to such $\mathfrak{M}$ as follows:

$$\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}) := \underline{T}_{\mathcal{E}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}})(h), \tag{2.1.2c}$$

where $T(h)$ denotes the "Tate twist"; i.e., twisting the $\mathcal{G}_{K_\infty}$-action on T by $\chi_{\mathrm{cyc}}^h|_{\mathcal{G}_{K_\infty}}$. It is a non-trivial theorem of Kisin [Kis06, Proposition 2.1.12] that this functor $\underline{T}_{\mathfrak{S}}^{\leqslant h}$ from the category of φ-module of height $\leqslant h$ to the category of $\mathcal{G}_{K_\infty}$-representations is fully faithful. Note that $\underline{T}_{\mathfrak{S}}^{\leqslant h}$ is not in general fully faithful on the category of *torsion* φ-module of height $\leqslant h$

Definition 2.1.3. A $\mathbb{Z}_p$-lattice $\mathcal{G}_{K_\infty}$-representation[6] T is *of height* $\leqslant h$ if there exists a φ-module $\mathfrak{M}$ of height $\leqslant h$ such that $T \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})$.

A p-adic $\mathcal{G}_{K_\infty}$-representation V is *of height* $\leqslant h$ if there exists a $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice which is of height $\leqslant h$ (or equivalently by [Kis06, Lemma 2.1.15][7], if any $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice which is of height $\leqslant h$).

A torsion $\mathcal{G}_{K_\infty}$-representation[8] T is *of height* $\leqslant h$ if there exists a torsion φ-module $\mathfrak{M}$ of height $\leqslant h$ such that $T \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})$. (We say that such $\mathfrak{M}$ is a *$\mathfrak{S}$-module model of height* $\leqslant h$ for T.)

[5]Note that $\mathcal{P}(u)$ is a unit in $\mathscr{O}_{\mathcal{E}}$.

[6]i.e., a finite free $\mathbb{Z}_p$-module with continuous $\mathcal{G}_{K_\infty}$-action

[7]In fact, we need a slight refinement of [Kis06, Lemma 2.1.15]; namely, replacing "finite height" in the statement by "height $\leqslant h$". The proof can be easily modified to prove this refinement.

[8]i.e., a finite torsion $\mathbb{Z}_p$-module with continuous $\mathcal{G}_{K_\infty}$-action

The motivation of this definition is Kisin's theorem [Kis06, Proposition 2.1.5] which asserts that the restriction to $\mathcal{G}_{K_\infty}$ of a p-adic semi-stable $\mathcal{G}_K$-representation with Hodge-Tate weights in $[0,h]$ is of height $\leqslant h$.

Lemma 2.1.4. *A torsion $\mathcal{G}_{K_\infty}$-representation T is of height $\leqslant h$ if and only if $T \cong \widetilde{T}/\widetilde{T}'$ for some $\widetilde{T}$ and $\widetilde{T}'$ which are $\mathbb{Z}_p$-lattice $\mathcal{G}_{K_\infty}$-representations of height $\leqslant h$.*

Proof. Note that for any exact sequence $(\dagger)\ :\ 0 \to \widetilde{\mathfrak{M}}' \to \widetilde{\mathfrak{M}} \to \mathfrak{M} \to 0$ with $\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{M}}' \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ and $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, the sequence $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\dagger)$ is exact; This is a consequence of the exactness of $\underline{T}_{\mathcal{E}}$ defined in (2.1.2b), which is proved in [Fon90, §A 1.2]. The "if" direction now follows from this.

To show the "only if" direction, one needs to show that any $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ can be put in some exact sequence $(\dagger)$ as above, but this could be done by the essentially same proof of [Kis06, Lemma 2.3.4].[9] □

2.2. **Deformations of height $\leqslant h$.** Let $\mathbb{F}$ be a finite field of characteristic p, and $\bar{\rho}_\infty : \mathcal{G}_{K_\infty} \to \mathrm{GL}_d(\mathbb{F})$ a representation. Let $\mathscr{O}$ be a p-adic discrete valuation ring with residue field $\mathbb{F}$. Let $\mathfrak{AR}_{\mathscr{O}}$ be the category of artin local $\mathscr{O}$-algebras A whose residue field is $\mathbb{F}$, and let $\widehat{\mathfrak{AR}}_{\mathscr{O}}$ be the category of complete local noetherian $\mathscr{O}$-algebras with residue field $\mathbb{F}$.

Let $D_\infty, D_\infty^\square : \widehat{\mathfrak{AR}}_{\mathscr{O}} \to (\mathbf{Sets})$ be the deformation functor and framed deformation functor for $\bar{\rho}_\infty$. For the definition, see the standard references such as [Maz97, Maz89, Gou01]. Contrary to local and global deformation functors we usually consider, these functors *cannot* be represented by complete local noetherian rings since the tangent spaces $D_\infty(\mathbb{F}[\epsilon])$ and $D_\infty^\square(\mathbb{F}[\epsilon])$ are infinite dimensional $\mathbb{F}$-vector spaces. See §2.4 for more details.

We say that a deformation $\rho_{\infty,A}$ over $A \in \mathfrak{AR}_{\mathscr{O}}$ is *of height* $\leqslant h$ if it is a torsion $\mathcal{G}_{K_\infty}$-representation of height $\leqslant h$ as a torsion $\mathbb{Z}_p[\mathcal{G}_{K_\infty}]$-module; or equivalently, if there exists $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ and an isomorphism $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}) \cong \rho_{\infty,A}$ as $\mathbb{Z}_p[\mathcal{G}_{K_\infty}]$-modules. For $A \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$, we say that $\rho_{\infty,A}$ is *of height* $\leqslant h$ if $\rho_{\infty,A} \otimes A/\mathfrak{m}_A^n$ is a deformation of height $\leqslant h$ for $n \gg 1$.[10] When $A \in \mathfrak{AR}_{\mathscr{O}}$, both definitions clearly coincide. When A is finite flat over $\mathbb{Z}_p$, a deformation $\rho_{\infty,A}$ over A is of height $\leqslant h$ if and only if $\rho_{\infty,A}$ is of height $\leqslant h$ as a $\mathbb{Z}_p$-lattice $\mathcal{G}_{K_\infty}$-representation (in the sense of Definition 2.1.3), by [Liu07, Theorem 2.4.1].

Let $D_\infty^{\leqslant h} \subset D_\infty$ and $D_\infty^{\square,\leqslant h} \subset D_\infty^\square$ respectively denote subfunctors of deformations and framed deformations of height $\leqslant h$. The following theorem is one of the main result of this paper:

Theorem 2.3. *The functor $D_\infty^{\leqslant h}$ always has a hull. If $\mathrm{End}_{\mathcal{G}_{K_\infty}}(\bar{\rho}_\infty) \cong \mathbb{F}$ then $D_\infty^{\leqslant h}$ is representable (by $R_\infty^{\leqslant h} \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$). The functor $D_\infty^{\square,\leqslant h}$ is representable (by $R_\infty^{\square,\leqslant h} \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$) with no assumption on $\bar{\rho}_\infty$. Furthermore, the natural inclusions $D_\infty^{\leqslant h} \hookrightarrow D_\infty$ and $D_\infty^{\square,\leqslant h} \hookrightarrow D_\infty^\square$ of functors are relatively representable by surjective maps in $\widehat{\mathfrak{AR}}_{\mathscr{O}}$.*

We call $R_\infty^{\square,\leqslant h}$ the *universal framed deformation ring of height* $\leqslant h$ and $R_\infty^{\leqslant h}$ the *universal deformation ring of height* $\leqslant h$ if it exists. We prove this theorem for the rest of this section beginning §2.4.

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[9] We will only need this result when $h=1$ which is proved in [Kis06, Lemma 2.3.4]. In general, one just need to modify the proof as follows: using the same notation as in *loc.cit.*, take $\widetilde{L}$ to be a finite free $\mathfrak{S}/\mathcal{P}(u)^h$-module which admits a $\mathfrak{S}/\mathcal{P}(u)^h$-surjection $\widetilde{L} \twoheadrightarrow L := \mathrm{coker}(1\otimes\varphi_{\mathfrak{M}})$.

[10] By Lemma 2.4.1, it is equivalent to require that $\rho_{\infty,A} \otimes A/\mathfrak{m}_A^n$ is a deformation of height $\leqslant h$ for each n.

Remark 2.3.1. Let A be any finite algebra over $\mathrm{Frac}\,\mathscr{O}$, and $\rho_{\infty,A} : \mathcal{G}_{K_\infty} \to \mathrm{GL}_d(A)$ be any lift of $\bar\rho_\infty$ (i.e., there exists some finite $\mathscr{O}$-subalgebra $A^\circ \subset A$ and $\mathcal{G}_{K_\infty}$-stable A°-lattice in $\rho_{\infty,A}$ which lifts $\bar\rho_\infty$. Then, by [Liu07, Theorem 2.4.1] $\rho_{A,\infty}$ arises as a pull back of the universal (framed) deformation of height $\leqslant h$ if and only if $\rho_{A,\infty}$ is of height $\leqslant h$ as a $\mathbb{Q}_p$-representation.

2.4. **Resumé of Mazur's and Ramakrishna's theory.** Schlessinger [Sch68, Thm 2.11] gave a set of criteria (H1) – (H4) for a functor $D : \mathfrak{AR}_{\mathscr{O}} \to (\mathbf{Sets})$ to be representable. For a profinite group Γ and a continuous $\mathbb{F}$-linear Γ-representation $\bar\rho$, Mazur [Maz89, §1.2] showed that the framed deformation functor $D^\square_{\bar\rho}$ of $\bar\rho$ satisfies all the Schlessinger criteria *except* the finiteness of the "tangent space" $D^\square_{\bar\rho}(\mathbb{F}[\epsilon])$, and the same for the deformation functor $D_{\bar\rho}$ if $\mathrm{End}_\Gamma(\bar\rho) \cong \mathbb{F}$. When Γ is either an absolute Galois group for a finite extension of $\mathbb{Q}_p$, or a certain quotient of the absolute Galois group of any finite extension of $\mathbb{Q}$, Mazur obtained the finiteness of the tangent space from so-called p-finiteness [Maz89, §1.1], but it is very unlikely to hold for more general class of Γ.

Unfortunately, $\mathcal{G}_{K_\infty}$ does not satisfy the p-finiteness, and in fact the tangent space $D_\infty(\mathbb{F}[\epsilon])$ is infinite even when $\bar\rho_\infty$ is 1-dimensional. To see this, note that $D_\infty(\mathbb{F}[\epsilon]) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathcal{G}_{K_\infty}, \mathbb{F})$ when $\bar\rho_\infty$ is 1-dimensional, This is infinite from the norm field isomorphism $\mathcal{G}_{K_\infty} \cong \mathrm{Gal}(k(\!(u)\!)^{\mathrm{sep}}/k(\!(u)\!))$ and the existence of infinitely many Artin-Schreier cyclic p-extensions of $k(\!(u)\!)$. For a finite dimensional $\bar\rho_\infty$, one sees that the deformation and framed deformation functors D_∞ and $D^\square_\infty$ never satisfies (H3) from deforming the determinant[11], and in particular these 'unrestricted' deformation functors are never represented by a complete local noetherian ring.

Now, let us look at the subfunctors $D^{\leqslant h}_\infty \subset D_\infty$ and $D^{\square,\leqslant h}_\infty \subset D^\square_\infty$ which consist of deformations of height $\leqslant h$ (as defined in §2.2). We first state the following lemma:

Lemma 2.4.1. *Any subquotients and direct sums of torsion $\mathcal{G}_{K_\infty}$-representations of height $\leqslant h$ is of height $\leqslant h$.*

Proof. The assertion about direct sums is obvious. Now consider a short exact sequence $0 \to M' \to M \to M'' \to 0$ of p^∞-torsion étale φ-modules and assume that there is a φ-sbable $\mathfrak{S}$-submodule $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ in M such that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = M$. Let $\mathfrak{M}''$ be the image of $\mathfrak{M}$ by $M \twoheadrightarrow M''$ and $\mathfrak{M}'$ the kernel of the natural map $\mathfrak{M} \to \mathfrak{M}''$. One can check that $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects in $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ such that $\mathfrak{M}' \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = M'$ and $\mathfrak{M}'' \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}} = M''$. Now the proposition follows from the exactness of $\underline{D}_{\mathcal{E}}$ and $\underline{T}_{\mathcal{E}}$. □

Lemma 2.4.1 implies that the condition of being of height $\leqslant h$ is closed under fiber products. It immediately follows (*cf.* the proof of [Ram93, Theorem 1.1]) that the functor $D^{\square,\leqslant h}_\infty$ satisfies all the Schlessinger's criteria except the finiteness of $D^{\square,\leqslant h}_\infty(\mathbb{F}[\epsilon])$; and the same for $D^{\leqslant h}_\infty$ if we have $\mathrm{End}_{\mathcal{G}_{K_\infty}}(\bar\rho_\infty) \cong \mathbb{F}$. So to prove the representability assertion of Theorem 2.3 it remains to check the finiteness[12] of $D^{\leqslant h}_\infty(\mathbb{F}[\epsilon])$ and $D^{\square,\leqslant h}_\infty(\mathbb{F}[\epsilon])$. Before doing this, let us digress to show the relative representability of the subfunctor $D^{\leqslant h}_\infty \subset D_\infty$, which "essentially" follows from Lemma 2.4.1.

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[11]For any $\mathbb{F}[\epsilon]$-deformation $\det(\bar\rho_\infty) + \epsilon\cdot c$ of $\det(\bar\rho_\infty)$ (where $c : \mathcal{G}_K \to F$ is a cocycle), the deformation $\bar\rho_\infty + \epsilon\cdot\tilde c$ with $\tilde c := \begin{pmatrix} c & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}$ has determinant $\det(\bar\rho_\infty) + \epsilon\cdot c$.

[12]Even though $D^\square_\infty(\mathbb{F}[\epsilon])$ is infinite, one can hope that the subspace $D^{\square,\leqslant h}_\infty(\mathbb{F}[\epsilon])$ is finite.

Proposition 2.5. *The subfunctor $D_\infty^{\leqslant h} \subset D_\infty$ is relatively representable by surjective maps in $\widehat{\mathfrak{AR}}_{\mathscr{O}}$. In other words, for any given deformation ρ_A over $A \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$, there exists a universal quotient $A^{\leqslant h}$ of A over which the deformation is of height $\leqslant h$.*

Proof. Consider a functor $\mathfrak{h}_A : \mathfrak{AR}_{\mathscr{O}} \to (\mathbf{Sets})$ defined by $\mathfrak{h}_A(B) := \mathrm{Hom}_{\mathscr{O}}(B, A)$ for $B \in \mathfrak{AR}_{\mathscr{O}}$, and a subfunctor $\mathfrak{h}_A^{\leqslant h} \subset \mathfrak{h}_A$ defined as below:

$$\mathfrak{h}_A^{\leqslant h}(B) := \{f : B \to A \text{ such that } \rho_A \otimes_{A,f} B \text{ is of height } \leqslant h\},$$

where $B \in \mathfrak{AR}_{\mathscr{O}}$. Since $\mathfrak{h}_A$ is prorepresentable and the subfunctor $\mathfrak{h}_A^{\leqslant h}$ is closed under subquotients and direct sums, it follows that $\mathfrak{h}_A^{\leqslant h}$ is prorepresentable, say by a quotient $A^{\leqslant h}$ of A. It is clear that $A^{\leqslant h}$ satisfies the desired properties. (*cf.* the proof of [Ram93, Theorem 1.1].) ☐

Now let us verify (H3) for $D_\infty^{\leqslant h}$ and $D_\infty^{\square,\leqslant h}$, thus prove the representability assertion of Theorem 2.3.

Proposition 2.6. *The tangent spaces $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ and $D_\infty^{\square,\leqslant h}(\mathbb{F}[\epsilon])$ are finite-dimensional $\mathbb{F}$-vector spaces.*

Proof. Let us first fix the notation.

2.6.1. *Notations and Definitions.* Let A be a p-adically separated and complete topological ring[13], (for example, finite $\mathbb{Z}_p$-algebras or any ring A with $p^N \cdot A = 0$ for some N). Set $\mathfrak{S}_A := \mathfrak{S}\widehat{\otimes}_{\mathbb{Z}_p} A := \varprojlim_\alpha \mathfrak{S}\widehat{\otimes}_{\mathbb{Z}_p} A/I_\alpha$ where $\{I_\alpha\}$ is a basis of open ideals in A. We define a ring endomorphism $\sigma : \mathfrak{S}_A \to \mathfrak{S}_A$ (and call it the *Frobenius endomorphism*) by A-linearly extending the Frobenius endomorphism $\sigma_{\mathfrak{S}}$. We also put $\mathscr{O}_{\mathcal{E},A} := \mathscr{O}_{\mathcal{E}}\widehat{\otimes}_{\mathbb{Z}_p} A := \varprojlim_\alpha \mathscr{O}_{\mathcal{E}}\widehat{\otimes}_{\mathbb{Z}_p} A/I_\alpha$ and similarly define an endomorphism $\sigma : \mathscr{O}_{\mathcal{E},A} \to \mathscr{O}_{\mathcal{E},A}$.

Let $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ be the category of finite free $\mathfrak{S}_A$-modules $\mathfrak{M}_A$ equipped with a $\mathfrak{S}_A$-linear map $\varphi_{\mathfrak{M}_A} : \sigma^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ such that $\mathcal{P}(u)^h$ annihilates $\mathrm{coker}(\varphi_{\mathfrak{M}_A})$. If A is finite artinean $\mathbb{Z}_p$-algebra, then $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ is precisely a torsion $(\varphi, \mathfrak{S})$-module of height $\leqslant h$ equipped with a φ-compatible A-action such that $\mathfrak{M}_A$ is finite free over $\mathfrak{S}_A$.

Let $(\mathrm{ModFI}/\mathscr{O}_{\mathcal{E}})_A^{\text{ét}}$ be the category of finite free $\mathscr{O}_{\mathcal{E},A}$-modules M_A equipped with a $\mathscr{O}_{\mathcal{E},A}$-linear isomorphism $\varphi_{M_A} : \sigma^*(M_A) \xrightarrow{\sim} M_A$. If A is finite artinean $\mathbb{Z}_p$-algebra, then one can check that $\underline{T}_{\mathcal{E}}$ and $\underline{D}_{\mathcal{E}}$, defined in (2.1.2), induce rank-preserving quasi-inverse exact equivalences of categories between the category of A-representations of $\boldsymbol{\mathcal{G}}_{K_\infty}$ and $(\mathrm{ModFI}/\mathscr{O}_{\mathcal{E}})_A^{\text{ét}}$.[14]

Lemma 2.6.2. *Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$, and $\bar\rho$ a $\boldsymbol{\mathcal{G}}_{K_\infty}$-representation over $\mathbb{F}$ which is of height $\leqslant h$ as a torsion $\boldsymbol{\mathcal{G}}_{K_\infty}$-representation (in the sense of Definition 2.1.3). Then there exists $\mathfrak{M}_{\mathbb{F}} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ such that $\bar\rho \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_{\mathbb{F}})$.*

Proof. Put $M := \underline{D}_{\mathcal{E}}^{\leqslant h}(\bar\rho(-h))$ and let $\mathfrak{M}_{\mathbb{F}} := \mathfrak{M}^+ \subset M$ be the maximal $\mathfrak{S}$-submodule of height $\leqslant h$, which exists by [CL09, Proposition 3.2.3]. Then the φ-compatible $\mathbb{F}$-action on M (induced by the scalar multiplication on $\bar\rho$) induces a φ-compatible $\mathbb{F}$-action on $\mathfrak{M}_{\mathbb{F}}$, which makes $\mathfrak{M}_{\mathbb{F}}$ a projective $\mathfrak{S}_{\mathbb{F}}$-module. (Note that $\mathfrak{S}_{\mathbb{F}}$ is a product of copies of a discrete valuation ring.) To show $\mathfrak{M}_{\mathbb{F}} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ it is left to show that $\mathfrak{M}_{\mathbb{F}}$ is free over $\mathfrak{S}_{\mathbb{F}}$, but this follows because the endomorphism $\sigma : \mathfrak{S}_{\mathbb{F}} \to \mathfrak{S}_{\mathbb{F}}$ transitively permutes the orthogonal idempotents of $\mathfrak{S}_{\mathbb{F}}$. ☐

[13] For us topological rings are always linearly topologized. Later we need to consider coefficient rings that are not finite $\mathbb{Z}_p$-algebras such as $A = \mathbb{F}[t]$, especially for analyzing the connected components of the generic fiber of a deformation ring.

[14] The relevant freeness follows from length consideration and Nakayama lemma.

Remark 2.6.3. Lemma 2.6.2 does not fully generalize to a $\mathcal{G}_{K_\infty}$-representation ρ_A of height $\leqslant h$ over a finite artinean $\mathbb{Z}_p$-algebra A. Assume that $he \geqslant p$ where e is the absolute ramification index of K, and consider $\mathbb{F}[\epsilon]$ where $\epsilon^2 = 0$. Let M be a rank-1 free $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-module equipped with $\varphi_M(\sigma^*\mathbf{e}) = (\mathcal{P}(u)^h + \frac{1}{u}\epsilon)\mathbf{e}$ for an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\mathbf{e} \in M$. Let $\mathfrak{M}$ be a $\mathfrak{S}_\mathbb{F}$-span of $\{\mathbf{e}, \frac{1}{u}\epsilon\mathbf{e}\}$ in M. Then $\mathfrak{M} \subset M$ is a $\mathfrak{S}$-submodule of height $\leqslant h$ (using that $he \geqslant p$), but one can check that there cannot exist a $\mathfrak{S}$-submodule of height $\leqslant h$ which is rank-1 free over $\mathfrak{S}_{\mathbb{F}[\epsilon]}$.[15] Note also that $\mathfrak{M}$ above is the maximal $\mathfrak{S}$-submodule of height $\leqslant h$ and has a φ-compatible $\mathbb{F}[\epsilon]$-action induced from M, but $\mathfrak{M}$ is not projective over $\mathfrak{S}_{\mathbb{F}[\epsilon]}$. This is where the proof of Lemma 2.6.2 fails.

Now we can begin the proof of Proposition 2.6. Since $D_\infty^{\square,\leqslant h}(\mathbb{F}[\epsilon])$ is a torsor of $\widehat{\mathrm{GL}_d}(\mathbb{F}[\epsilon])/(1+\epsilon\,\mathrm{Ad}(\bar\rho_\infty)^{\mathcal{G}_{K_\infty}})$ over $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ (where $\widehat{\mathrm{GL}_d}$ is the formal completion of GL_d at the identity section), it is enough to show that the set $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ is finite.

2.6.4. *Setup.* Let $\overline{M} := \underline{D}_\mathcal{E}(\bar\rho_\infty(-h))$ and consider (M,ι), where $M \in (\mathrm{ModFI}\,/\mathscr{O}_\mathcal{E})^{\text{ét}}_{\mathbb{F}[\epsilon]}$ and $\iota : \overline{M} \cong M \otimes_{\mathbb{F}[\epsilon]} \mathbb{F}$ is a φ-compatible $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-linear isomorphism. Two such lifts (M,ι) and (M',ι') are *equivalent* if there exists an isomorphism $f : M \xrightarrow{\sim} M'$ with $(f \bmod \epsilon)\circ\iota = \iota'$. Fontaine's theory of étale φ-modules [Fon90, §A 1.2] implies that $\underline{T}_\mathcal{E}(\cdot)(h)$ and $\underline{D}_\mathcal{E}(\cdot(-h))$ induce inverse bijections between $D_\infty(\mathbb{F}[\epsilon])$ and the set of equivalent classes of (M,ι).

Now assume that there is a φ-stable $\mathfrak{S}_\mathbb{F}$-lattice $\mathfrak{M} \subset M$ of height $\leqslant h$. (Note that we do not require $\mathfrak{M}$ to be a $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-submodule.) By Lemma 2.6.2, the set of equivalence classes of (M,ι) admitting such $\mathfrak{M} \subset M$ exactly corresponds to $D_\infty^{\leqslant h}(\mathbb{F}[\epsilon])$ via the bijections in the previous paragraph. So Proposition 2.6 is equivalent to the following claim:

Claim 2.6.5. *There exists only finitely many equivalence classes of* (M,ι) *where* M *admits a* φ*-stable* $\mathfrak{S}_\mathbb{F}$*-lattice which is of height* $\leqslant h$.

2.6.6. *Strategy and Outline.* One possible approach to prove Claim 2.6.5 is to fix a $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis for $\overline{M}$ and a lift to an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis for each deformation M once and for all, and identify M with the "φ-matrix" with respect to the fixed basis and interpret the equivalence relations in terms of the "φ-matrix." Then the problem turns into showing the finiteness of equivalence classes of matrices with some constraints – namely, having some "integral structure"; more precisely, having a φ-stable $\mathfrak{S}_\mathbb{F}$-lattice with height $\leqslant h$ (but not necessarily a $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice; *cf.* Lemma 2.6.2 and Remark 2.6.3). So the fixed basis has to "reflect" the integral structure.

This approach faces the following obstacles. Firstly, the deformations M we consider do not necessarily allow any φ-stable $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice with height $\leqslant h$ as we have seen at Remark 2.6.3. In other words, we cannot expect, in general, to find a $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\{e_i\}$ for M in such a way that $\{e_i, \epsilon e_i\}$ generates a $\mathfrak{S}_\mathbb{F}$-lattice of height $\leqslant h$. In §2.6.7–§2.6.9 we show that a weaker statement is true. Roughly speaking, we show that there is an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\{\mathbf{e}_i\}$ for M so that there exists a φ-stable $\mathfrak{S}_\mathbb{F}$-lattice with height $\leqslant h$ with a $\mathfrak{S}_\mathbb{F}$-basis only involving "uniformly" u-adically bounded denominators as coefficients relative to the $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis $\{\mathbf{e}_i, \epsilon\cdot\mathbf{e}_i\}$ of M.

Secondly, we may have more than one φ-stable $\mathfrak{S}_\mathbb{F}$-lattice with height $\leqslant h$ for $\overline{M}$ or for M, especially when he is large. In particular, a fixed $\mathfrak{S}_\mathbb{F}$-lattice for $\overline{M}$ may not be nicely related to any φ-stable $\mathfrak{S}_\mathbb{F}$-lattice with height $\leqslant h$ for some lift $M \in (\mathrm{ModFI}\,/\mathscr{O}_\mathcal{E})^{\text{ét}}_{\mathbb{F}[\epsilon]}$. We get around this issue by varying the basis for $\overline{M}$ among finitely many choices. This step is carried out in §2.6.11. In fact, we only need

[15] One way to see this is by directly computing the "φ-matrix" for any $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\mathbf{e}' \in M$, and show that it cannot divide $\mathcal{P}(u)^h$.

finitely many choices of bases because there are only finitely many $\mathfrak{S}_{\mathbb{F}}$-lattices of height $\leqslant h$ for a fixed $\overline{M}$, thanks to [CL09, Proposition 3.2.3].

Once we get around these technical problems, we show the finiteness by a σ-conjugacy computation of matrices. This is the key technical step and crucially uses the assumption that the $\mathbb{F}[\epsilon]$-deformations we consider (or rather, the corresponding étale φ-module M) admits a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice in M with height $\leqslant h$. See Claim 2.6.12 for more details.

2.6.7. Let M correspond to some $\mathbb{F}[\epsilon]$-deformation of height $\leqslant h$. Even though there may not exist any φ-stable $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice with height $\leqslant h$ for M, we can find a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}$ with height $\leqslant h$ such that $\mathfrak{M}$ is stable under multiplication by ϵ.[16] In fact, the maximal $\mathfrak{S}$-submodule $\mathfrak{M}^+ \subset M$ among the ones with height $\leqslant h$ does the job. (The existence of $\mathfrak{M}^+$ is by [CL09, Proposition 3.2.3].)

2.6.8. For a $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M} \subset M$ of height $\leqslant h$ which is stable under the ϵ-multiplication, we can find a $\mathfrak{S}_{\mathbb{F}}$-basis which can be "nicely" written in terms of some $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis of M, as follows. Let $\overline{\mathfrak{M}}$ be the image of $\mathfrak{M} \to \overline{M}$ induced by the natural projection $M \to \overline{M}$, which is a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice in $\overline{M}$ with height $\leqslant h$. Now, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{M} & \longrightarrow & \overline{\mathfrak{M}} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \epsilon\cdot M & \longrightarrow & M & \longrightarrow & M_{\mathbb{F}} & \longrightarrow & 0, \end{array}$$

where $\mathfrak{N} := \ker[\mathfrak{M} \twoheadrightarrow \overline{\mathfrak{M}}]$ is a φ-stable $\mathfrak{S}_{\mathbb{F}}$-lattice with height $\leqslant h$ in M. We choose a $\mathfrak{S}_{\mathbb{F}}$-basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ of $\overline{\mathfrak{M}}$. Viewing them as a $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis of $\overline{M}$, we lift $\{\mathbf{e}_i\}$ to an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis of M (again denoted by $\{\mathbf{e}_i\}$). By the assumption from the previous step, we have $\bigoplus_{i=1}^n \mathfrak{S}_{\mathbb{F}}\cdot(\epsilon\mathbf{e}_i) \subset \mathfrak{N}$, where both are $\mathfrak{S}_{\mathbb{F}}$-lattices of height $\leqslant h$ for $\epsilon\cdot M$. It follows that $(\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i$ form a $\mathfrak{S}_{\mathbb{F}}$-basis of $\mathfrak{N}$ for some non-negative integers r_i. Therefore, $\{\mathbf{e}_i, (\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i\}$ is a $\mathfrak{S}_{\mathbb{F}}$-basis of $\mathfrak{M}$.

2.6.9. In this step, we find an upper bound for the non-negative integers r_i only depending on $\overline{\mathfrak{M}}$ and the choice of $\mathfrak{S}_{\mathbb{F}}$-basis of $\overline{\mathfrak{M}}$. Since $\mathfrak{N}$ is a φ-stable submodule, it contains

$$\varphi_M\left(\sigma^*\left(\frac{1}{u^{r_i}}\epsilon\mathbf{e}_i\right)\right) = \left(\frac{1}{u^{pr_i}}\epsilon\right)\cdot\varphi_{\overline{\mathfrak{M}}}(\sigma^*\mathbf{e}_i) = \frac{1}{u^{pr_i}}\epsilon\cdot\sum_{j=1}^n \alpha_{ij}\mathbf{e}_j, \tag{2.6.10}$$

where $\alpha_{ij} \in \mathfrak{S}_{\mathbb{F}}$ satisfy $\varphi_{\overline{\mathfrak{M}}}(\sigma^*\mathbf{e}_i) = \sum_{j=1}^n \alpha_{ij}\mathbf{e}_j$. Note that we obtain the first identity because $\varphi_M(\sigma^*\mathbf{e}_i)$ lifts $\varphi_{\overline{\mathfrak{M}}}(\sigma^*\mathbf{e}_i)$ and the ϵ-multiple ambiguity in the lift disappears when we multiply against ϵ. Since any element of $\mathfrak{N}$ is a $\mathfrak{S}_{\mathbb{F}}$-linear combination of $(\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i$, we obtain inequalities $\mathrm{ord}_u(\alpha_{ij}) - pr_i \geq -r_j$ for all i, j from the above equation (2.6.10). Let $r := \max_j\{r_j\}$ and we obtain $pr_i \leq r + \min_j\{\mathrm{ord}_u(\alpha_{ij})\}$ for all i. (Note that the right side of the inequality is always finite.) Now, by taking the maximum among all i, we obtain

$$r \leq \frac{1}{p-1}\max_i\big\{\min_j\{\mathrm{ord}_u(\alpha_{ij})\}\big\} < \infty$$

This shows that the non-negative integers r_i has an upper bound which only depends on the matrices entries for $\varphi_{\overline{\mathfrak{M}}}$ with respect to the $\mathfrak{S}_{\mathbb{F}}$-basis of $\overline{\mathfrak{M}}$.

[16] This means that $\mathfrak{M}$ is a φ-module over $\mathfrak{S}_{\mathbb{F}[\epsilon]}$ and is projective over $\mathfrak{S}_{\mathbb{F}}$, but $\mathfrak{M}$ does not have to be a projective $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-module. Hence, such $\mathfrak{M}$ may not be an object in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\mathbb{F}[\epsilon]}$. This actually occurs: $\mathfrak{M} \cong \mathfrak{S}_{\mathbb{F}}\cdot\mathbf{e} \oplus \mathfrak{S}_{\mathbb{F}}\cdot(\frac{1}{u}\epsilon\mathbf{e})$ discussed in Remark 2.6.3 is such an example.

2.6.11. *Recapitulation.* Let $\{\overline{\mathfrak{M}}^{(a)}\}$ denote the set of all the $\mathfrak{S}_{\mathbb{F}}$-lattices of height $\leqslant h$ in $\overline{M}$. This is a finite set by [CL09, Proposition 3.2.3]. For each $\overline{\mathfrak{M}}^{(a)}$, we fix a $\mathfrak{S}_{\mathbb{F}}$-basis $\{\mathbf{e}_i^{(a)}\}$ and let $\alpha^{(a)} = (\alpha_{ij}^{(a)}) \in \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ be the "φ-matrix" with respect to $\{\mathbf{e}_i^{(a)}\}$; i.e., $\varphi_{\overline{\mathfrak{M}}^{(a)}}(\sigma^*\mathbf{e}_i^{(a)}) = \sum_{i=1}^n \alpha_{ij}^{(a)}\mathbf{e}_j^{(a)}$. We also view $\{\mathbf{e}_i^{(a)}\}$ as a $\mathscr{O}_{\mathcal{E},\mathbb{F}}$-basis for $\overline{M}$ and $(\alpha_{ij}^{(a)})$ is the matrix for $\varphi_{\overline{M}}$ with respect to $\{\mathbf{e}_i^{(a)}\}$. Note that $(\alpha_{ij}^{(a)})$ is invertible over $\mathscr{O}_{\mathcal{E},\mathbb{F}}$ since $\overline{M} = \overline{\mathfrak{M}}^{(a)}[\frac{1}{u}]$ is an étale φ-module. We pick an integer $r^{(a)} \geq \frac{1}{p-1}\max_i\big\{\min_j\{\mathrm{ord}_u(\alpha_{ij})\}\big\}$, for each index a.

For any M which corresponds to a deformation of height $\leqslant h$, we may find a $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M} \subset M$ of height $\leqslant h$ which is stable under ϵ-multiplication. (See §2.6.7.) Then the image of $\mathfrak{M}$ inside $\overline{M}$ is equal to one of $\overline{\mathfrak{M}}^{(a)}$. Pick such $\overline{\mathfrak{M}}^{(a)}$, and lift the chosen basis $\{\mathbf{e}_i^{(a)}\}$ to an $\mathscr{O}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis for M. Then $\mathfrak{M}$ admits a $\mathfrak{S}_{\mathbb{F}}$-basis of form $\{\mathbf{e}_i^{(a)}, (\frac{1}{u^{r_i}}\epsilon)\mathbf{e}_i^{(a)}\}$ for some integers $r_i \leq r^{(a)}$ (§2.6.8–§2.6.9).

Let us consider the matrix representation of φ_M with respect to the basis $\{\mathbf{e}_i^{(a)}\}$. We have $\varphi_M(\mathbf{e}_i^{(a)}) = \sum_i(\alpha_{ij}^{(a)} + \epsilon\beta_{ij}^{(a)})\mathbf{e}_j^{(a)}$ for some $\beta^{(a)} = (\beta_{ij}^{(a)}) \in \mathrm{Mat}_n(\mathscr{O}_{\mathcal{E},\mathbb{F}})$ because φ_M lifts $\varphi_{\overline{M}}$. Furthermore we have that $\beta \in \frac{1}{u^{r^{(a)}}}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ since $\mathfrak{M} \subset M$ is φ-stable. We say two such matrices β and β' are *equivalent* if there exists a matrix $X \in \mathrm{Mat}_n(\mathscr{O}_{\mathcal{E},\mathbb{F}})$ such that $\beta' = \beta + (\alpha^{(a)}\cdot\sigma(X) - X\cdot\alpha^{(a)})$. This equation is obtained from the following:

$$(\alpha^{(a)} + \epsilon\beta') = (\mathrm{Id}_n + \epsilon X)^{-1}\cdot(\alpha^{(a)} + \epsilon\beta)\cdot\sigma(\mathrm{Id}_n + \epsilon X),$$

which defines the equivalence of two étale φ-modules whose φ-structures are given by $(\alpha^{(a)} + \epsilon\beta)$ and $(\alpha^{(a)} + \epsilon\beta')$, respectively.

Now, the theorem is reduced to the verification of the following claim: *for each a, there exist only finitely many equivalence classes of matrices $\beta \in \frac{1}{u^{r^{(a)}}}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$.* Indeed, by varying both a and the equivalence classes of β, we cover all the possible lifts M of "height $\leqslant h$" up to equivalence, hence the theorem is proved.

From now on, we fix a and suppress the superscript $(\cdot)^{(a)}$ everywhere. For example, $\overline{\mathfrak{M}} := \overline{\mathfrak{M}}^{(a)}$, $r := r^{(a)}$, and $\alpha := \alpha^{(a)}$. Proving the following claim is the last step of the proof.

Claim 2.6.12. *For any $X \in u^c\,\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ with $c > 2he$, the matrices β and $\beta + X$ are equivalent.*[17]

This claim provides a surjective map from $\left(\frac{1}{u^r}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})\right) / (u^c\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}}))$ onto the set of equivalence classes of β's, and the former is a finite set[18], thus we conclude the proof of Proposition 2.6.

We prove the claim by "successive approximation." Let $\gamma = u^{he}\cdot\alpha^{-1}$. Note that $\gamma \in \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ since $\overline{\mathfrak{M}}$ is of height $\leqslant h$ and $\mathcal{P}(u)$ has image in $\mathfrak{S}_{\mathbb{F}} \cong (k \otimes_{\mathbb{F}_p} \mathbb{F})[[u]]$ with u-order e. We set $Y^{(1)} := \frac{1}{u^{he}}\cdot(X\gamma)$, which is in $u^{c-he}\,\mathrm{Mat}_n(\mathfrak{S}_F)$ by the assumption on X. Then $\beta + X$ is equivalent to

$$(\beta + X) + (\alpha\cdot\sigma(Y^{(1)}) - Y^{(1)}\alpha) = \beta + \alpha\cdot\sigma(Y^{(1)}) =: \beta + X^{(1)}$$

with $X^{(1)} \in u^{c^{(1)}}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$, where $c^{(1)} := p(c - he) > c$. Now for any positive integer i, we recursively define the following

$$Y^{(i)} := \frac{1}{u^{he}}\cdot(X^{(i-1)}\gamma), \qquad X^{(i)} := \alpha\cdot\sigma(Y^{(i)}), \qquad c^{(i)} := p(c^{(i-1)} - he).$$

[17] The inequality $c > 2he$ is used to ensure $p(c - he) > c$. So $c = 2he$ also works unless $p = 2$.

[18] We crucially used the fact that we can bound the denominator.

One can check that $c^{(i)} > c(i-1)(> 2he)$, $X^{(i)} \in u^{c^{(i)}}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$, and $Y^{(i)} \in u^{c^{(i-1)}-he}\,\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$. Since $c^{(i)} \to \infty$ as $i\to\infty$, it follows that the infinite sum $Y := \sum_{i=1}^\infty Y^{(i)}$ converges and $X^{(i)} \to 0$ as $i\to\infty$. Therefore we see that $\beta + X$ is equivalent to

$$\begin{aligned}(\beta+X)+(\alpha\cdot\sigma(Y)-Y\cdot\alpha) &= (\beta+X)+\left(\alpha\cdot\sigma\big(\sum_{i=1}^\infty Y^{(i)}\big)-\big(\sum_{i=1}^\infty Y^{(i)}\big)\cdot\alpha\right)\\ &= \lim_{i\to\infty}(\beta+X^{(i)})=\beta,\end{aligned}$$

so we are done. □

3. Generic fibers of deformation rings of height $\leqslant 1$

Let $\bar\rho_\infty$ be a $\mathcal{G}_{K_\infty}$-representation over $\mathbb{F}$ and let $R_\infty^{\square,\leqslant 1}$ denote the framed deformation ring of height $\leqslant 1$. In this section we study the generic fiber $R_\infty^{\square,\leqslant 1}[\frac{1}{p}]$, such as formal smoothness, dimension, and connected components. The main idea is to "resolve" $R_\infty^{\square,\leqslant 1}$ via some suitable closed subscheme of affine grassmannian which parametrize certain "nice" $\mathfrak{S}$-module models of height $\leqslant 1$ for framed deformations of $\bar\rho_\infty$. This technique is inspired by Kisin's construction of moduli of finite flat group schemes [Kis09b, §2] (and also [Kis08, §1]). In fact, it is even possible to summarize this section by "repeating [Kis09b, §2] for our setting", due to the similarities of linear algebraic structures involved. (For this reason, we try to state minimal number of definitions and results enough to indicate the flavor of the argument, and give references for proofs.[19]) Note, however, that we "resolve" a framed $\mathcal{G}_{K_\infty}$-deformation ring $R_\infty^{\square,\leqslant 1}$ instead of a framed flat deformation ring.

With some extra work, many results in this section generalize to (framed) $\mathcal{G}_{K_\infty}$-deformation rings of height $\leqslant h$ for any h, possibly except the connectedness of non-ordinary loci. See [Kim09, §11] for the statements and proofs. All the results apply to (unframed) $\mathcal{G}_{K_\infty}$-deformation rings if they exist.

3.1. **Definition: moduli of $\mathfrak{S}$-modules of height $\leqslant h$.** Let h be a positive integer, and we will later specialize to the case when $h=1$. Consider a deformation ρ_R of $\bar\rho_\infty$ over $R\in\widehat{\mathfrak{AR}}_{\mathscr{O}}$ which is of height $\leqslant h$ (i.e. $\rho_R\otimes_R R/\mathfrak{m}_R^n$ is of height $\leqslant h$ for each n). The main examples to keep in mind are universal framed or unframed deformation of height $\leqslant h$.

We use the notation introduced in §2.6.1. Put $M_R := \varprojlim M_n$ where $M_n \in (\mathrm{ModFI}\,/\mathscr{O}_{\mathcal{E}})^{\text{ét}}_{R/\mathfrak{m}_R^n}$ is such that $\underline{T}_{\mathcal{E}}(M_n)(h) \cong \rho_R\otimes_R R/\mathfrak{m}_R^n$ for each n. For any R-algebra A, we view $M_R\otimes_R A$ as an étale φ-module by A-linearly extending φ_{M_R}.

For a complete local noetherian ring R, let $\mathfrak{Aug}_R$ be the category of pairs (A,I) where A is an R-algebra and $I\subset A$ is an ideal with $I^N=0$ for some N which contains $\mathfrak{m}_R\cdot A$. Note that an artin local R-algebra A can be viewed as an element in $\mathfrak{Aug}_{\mathscr{O}}$ by setting $I:=\mathfrak{m}_A$. A morphism $(A,I)\to(B,J)$ in $\mathfrak{Aug}_R$ is an R-morphism $A\to B$ which takes I into J. We define a functor $D^{\leqslant h}_{\mathfrak{S},\rho_R}:\mathfrak{Aug}_R\to(\mathbf{Sets})$ by putting $D^{\leqslant h}_{\mathfrak{S},\rho_R}(A,I)$ the set of φ-stable $\mathfrak{S}_A$-lattices in $M_R\otimes_R A$ which are of height $\leqslant h$. In [Kis08] this functor is denoted by $L^{\leqslant h}_{\rho_R}$.

Proposition 3.2. *There exists a projective R-scheme $\mathscr{GR}^{\leqslant h}_{\rho_R}$ and a $\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathscr{GR}^{\leqslant h}_{\rho_R}}$-lattice $\underline{\mathfrak{M}}^{\leqslant h}_{\rho_R}\subset M_R\otimes_R\mathcal{O}_{\mathscr{GR}^{\leqslant h}_{\rho_R}}$ of height $\leqslant h$ which represents $D^{\leqslant h}_{\mathfrak{S},\rho_R}$ in the following sense: there exists a natural isomorphism $\mathscr{GR}^{\leqslant h}_{\rho_R}\xrightarrow{\sim}D^{\leqslant h}_{\mathfrak{S},\rho_R}$ such that for any*

[19]Some references are written under the assumptions that $p>2$, but this will not be used in the proof we cite later.

$(A,I)\in \mathfrak{Aug}_R$ *it sends an A-point* $\eta\in \mathscr{GR}^{\leqslant h}_{\rho_R}(A)$ *to* $\eta^*(\underline{\mathfrak{M}}^{\leqslant h}_{\rho_R})\in D^{\leqslant h}_{\mathfrak{S},\rho_R}(A,I)$. *(We call* $\underline{\mathfrak{M}}^{\leqslant h}_{\rho_R}$ *a* universal $\mathfrak{S}$-lattice of height $\leqslant h$ *for* ρ_R.*)*

Moreover, for any map $R\to R'$ *of complete local noetherian rings, there exists a unique isomorphism* $\mathscr{GR}^{\leqslant h}_{\rho_R}\otimes_R R'\xrightarrow{\sim}\mathscr{GR}^{\leqslant h}_{\rho_R\otimes_R R'}$, *which pulls back* $\underline{\mathfrak{M}}^{\leqslant h}_{\rho_R\otimes_R R'}$ *to* $\underline{\mathfrak{M}}^{\leqslant h}_{\rho_R}\otimes_R R'$ *inside of* $(M_R\otimes_R \mathcal{O}_{\mathscr{GR}^{\leqslant h}_{\rho_R}})\otimes_R R'$.

This proposition is proved in [Kis08, Proposition 1.3; Corollary 1.5.1; Corollary 1.7] possibly except the base change assertion which is straightforward, so we omit the proof.

3.3. **Generic Fiber of** $\mathscr{GR}^{\leqslant h}_{\rho_R}$**.** For a finite $\mathbb{Q}_p$-algebra A, we define $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ to be the category of φ-modules $\mathfrak{M}_A$ such that for some finite $\mathbb{Z}_p$-subalgebra $A^\circ\subset A$ there exists $\mathfrak{M}_{A^\circ}\in(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A^\circ}$ with $\mathfrak{M}_{A^\circ}[\frac{1}{p}]\cong\mathfrak{M}_A$. For such $\mathfrak{M}_A$, we define $\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_A):=\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_{A^\circ})\otimes_{A^\circ}A$, which is naturally an A-representation of height $\leqslant h$. Note that $\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_A)$ is independent of the choice of $\mathfrak{M}_{A^\circ}$.

Let A be a local R-algebra with residue field E which is finite over $\mathbb{Q}_p$, and let A^+ denote the preimage of $\mathscr{O}_E$ under the natural projection $A\twoheadrightarrow E$. By the valuative criterion, any R-map $\operatorname{Spec}A\to\mathscr{GR}^{\leqslant h}_{\rho_R}$ factors through $\operatorname{Spec}A^+$, so in turn, it factors through $\operatorname{Spec}A^\circ$ where A° is an R-subalgebra in A^+ which is finite over $\mathscr{O}$ and such that $A^\circ[\frac{1}{p}]=A$. This implies that $\mathscr{GR}^{\leqslant h}_{\rho_R}(A)$ is naturally isomorphic to the set of φ-stable $\mathfrak{S}_A$-lattices $\mathfrak{M}_A$ of $M_R\otimes_R A$ such that $\mathfrak{M}_A\in(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$.

Proposition 3.4. *The structure morphism* $\mathscr{GR}^{\leqslant h}_{\rho_R}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\to\operatorname{Spec}(R\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$ *is an isomorphism. If the map* $\operatorname{Spf}R\to D^{\leqslant h}_\infty$ *of functors on* $\mathfrak{AR}_{\mathscr{O}}$ *defined by a deformation* ρ_R *is formally smooth, then* $R[\frac{1}{p}]$ *is formally smooth over* $\operatorname{Frac}(\mathscr{O})$. *In particular,* $R^{\square,\leqslant h}_\infty[\frac{1}{p}]$ *is formally smooth over* $\operatorname{Frac}(\mathscr{O})$.

Proof. The first assertion is a direct consequence of [Kis08, Proposition 1.6.4]. For the second assertion, it is enough to check the formal smoothness at each maximal ideal because the formal smoothness locus is open. (*cf.* [Gro67, 0_{IV}, (22.6.6)].) Using the assumption on R, we are reduced to showing that for any square-zero thickening $A\twoheadrightarrow\overline{A}$ of finite $\mathbb{Q}_p$-algebra and any $\mathfrak{M}_{\overline{A}}\in(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\overline{A}}$, there exists $\mathfrak{M}_A\in(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ which lifts $\mathfrak{M}_{\overline{A}}$. When $h=1$ this readily follows from [Kis06, Proposition 2.2.2] and [Kis09b, Lemma 2.3.9]. In general, we can proceed as follows.

Choose finite $\mathbb{Z}_p$-subalgebras $\overline{A^\circ}\subset\overline{A}$ and $A^\circ\subset A$, so that A° is a square-zero thickening of $\overline{A^\circ}$, and there exists a φ-stable $\mathfrak{S}_{\overline{A^\circ}}$-lattice $\mathfrak{M}_{\overline{A^\circ}}\in(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\overline{A^\circ}}$ of $\mathfrak{M}_{\overline{A}}$. We will lift $\mathfrak{M}_{\overline{A^\circ}}$ to some $\mathfrak{M}_{A^\circ}\in(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A^\circ}$. Put $\overline{\omega}:=\operatorname{coker}(\varphi_{\mathfrak{M}_{\overline{A^\circ}}})$, and the proof of [Kis09b, Lemma 1.2.2(2)] can be adapted to show that $\overline{\omega}$ is projective over $\overline{A}^\circ$. Now consider a projective A-module ω which lifts $\overline{\omega}$, a free $\mathfrak{S}_{A^\circ}$-module $\mathfrak{M}_{A^\circ}$ which lifts $\mathfrak{M}_{\overline{A^\circ}}$, and a A°-linear surjection $\mathfrak{M}_{A^\circ}/(\mathcal{P}(u)^h)\twoheadrightarrow\omega$ which lifts $\mathfrak{M}_{\overline{A^\circ}}/(\mathcal{P}(u)^h)\twoheadrightarrow\overline{\omega}$. Let $\mathfrak{N}$ denote the kernel of the natural projection $\mathfrak{M}_{A^\circ}\twoheadrightarrow\omega$. Then since ω is A-flat, $\mathfrak{N}$ surjects onto the image of $\varphi_{\mathfrak{M}_{\overline{A^\circ}}}$ under the natural projection $\mathfrak{M}_{A^\circ}\twoheadrightarrow\mathfrak{M}_{\overline{A^\circ}}$; so in turn, the surjective map $\sigma^*\mathfrak{M}_{A^\circ}\twoheadrightarrow\sigma^*\mathfrak{M}_{\overline{A^\circ}}\xrightarrow{\sim}\varphi_{\mathfrak{M}_{\overline{A^\circ}}}(\sigma^*\mathfrak{M}_{\overline{A^\circ}})$ can be lifted to a map $\sigma^*\mathfrak{M}_{A^\circ}\twoheadrightarrow\mathfrak{N}$. (Note that $\varphi_{\mathfrak{M}_{\overline{A^\circ}}}$ is injective. *cf.* [Kis09b, Lemma 1.2.2(1)].) By post-composing it with the natural inclusion, we obtain a $\mathfrak{S}_{A^\circ}$-map $\varphi_{\mathfrak{M}_{A^\circ}}:\sigma^*\mathfrak{M}_{A^\circ}\to\mathfrak{M}_{A^\circ}$ which lifts $\varphi_{\mathfrak{M}_{\overline{A^\circ}}}$, and its image contains $\mathcal{P}(u)^h\mathfrak{M}_{A^\circ}$ by construction. □

3.5. **Hodge type and local structure.** From now on, we assume that $h=1$ and $\bar\rho_\infty$ is 2-dimensional. Consider a lift ρ_R of $\bar\rho_\infty$ over $R\in\widehat{\mathfrak{AR}}_{\mathscr{O}}$ as a $\boldsymbol{\mathcal{G}}_{K_\infty}$-representation.

Define a subfunctor $D^{\mathbf{v}}_{\mathfrak{S},\rho_R} \subset D^{\leqslant 1}_{\mathfrak{S},\rho_R}$ as follows: for any $(A,I) \in \mathfrak{Aug}_R$ the subset $D^{\mathbf{v}}_{\mathfrak{S},\rho_R}(A,I)$ consists of all $\mathfrak{M}_A$'s with the property that $\mathrm{im}(\varphi_{\mathfrak{M}_A})/\mathcal{P}(u)\mathfrak{M}_A \subset \mathfrak{M}_A/\mathcal{P}(u)\mathfrak{M}_A$ is its own annihilator under some $\mathscr{O}_K \otimes_{\mathbb{Z}_p} A$-linear symplectic pairing on $\mathfrak{M}_A/\mathcal{P}(u)\mathfrak{M}_A$. This notion does not depend on the choice of the pairing. The following lemma follows from the same argument as the proof of [Kis09a, Theorem 2.3.9(1)].

Lemma 3.5.1. *The subfunctor $D^{\mathbf{v}}_{\mathfrak{S},\rho_R} \subset D^{\leqslant 1}_{\mathfrak{S},\rho_R}$ can be represented by a closed subscheme $\mathscr{GR}^{\mathbf{v}}_{\rho_R} \subset \mathscr{GR}^{\leqslant 1}_{\rho_R}$.*

Remark 3.5.2. With a little more work, one can show that $\operatorname{Spec} R^{\mathbf{v}}[\frac{1}{p}] \subset \operatorname{Spec} R[\frac{1}{p}]$ is an equi-dimensional union of connected components, and when $R = R^{\square,\leqslant 1}_{\infty}$, then the dimension of $R^{\mathbf{v}}[\frac{1}{p}] = R^{\square,\mathbf{v}}_{\infty}[\frac{1}{p}]$ is $4 + [K:\mathbb{Q}_p]$. We do not use this result later.

The following is a consequence of [Kis09a, Lemma 2.3.4].

Lemma 3.5.3. *Let $\operatorname{Spec} R^{\mathbf{v}}$ denote the scheme-theoretic image of $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$ in $\operatorname{Spec} R$. Then for any finite $\mathbb{Q}_p$-algebra A, a $\mathbb{Z}_p$-morphism $\xi : R \to A$ factors through $R^{\mathbf{v}}$ if and only if $\det(\rho_R \otimes_{R,\xi} A)|_{I_{K_\infty}} = \chi_{\mathrm{cyc}}|_{I_{K_\infty}}$.*

Let $\mathscr{GR}^{\mathbf{v}}_{\rho_R,0}$ denote the fiber over the closed point of $\operatorname{Spec} R$ under the structure morphism $\mathscr{GR}^{\mathbf{v}}_{\rho_R} \to \operatorname{Spec} R$. For a scheme X, we let $H_0(X)$ denote the set of connected components of X.

Proposition 3.5.4. *Assume that the morphism $\operatorname{Spf} R \to D^{\leqslant 1}_{\infty}$ induced by ρ_R is formally smooth. Then the natural maps below*

$$H_0(\mathscr{GR}^{\mathbf{v}}_{\rho_R} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \to H_0(\mathscr{GR}^{\mathbf{v}}_{\rho_R}) \leftarrow H_0(\mathscr{GR}^{\mathbf{v}}_{\rho_R,0})$$

are bijective.

We only indicate the main idea and references as the proof is not very different from the situation considered in Kisin [Kis09b]. We first show that $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$ is $\mathscr{O}$-flat and $\mathscr{GR}^{\mathbf{v}}_{\rho_R} \otimes_{\mathscr{O}} \mathscr{O}/\mathfrak{m}_{\mathscr{O}}$ is reduced under the assumption of the proposition. This can be proved by constructing a local model diagram (for smooth topology) with "Deligne-Pappas local model" as in the proof of [Kis07, Proposition 2.7] or [Kis09a, Theorem 2.3.9]. Now one can apply the same proof of [Kis09b, Corollary 2.4.10] (which uses the theorem on formal functions) to deduce the proposition.

3.6. Let $\mathfrak{M}$ be an object either in$(\mathrm{Mod}/\mathfrak{S})^{\leqslant 1}$ or in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant 1}_A$ where A is either p-adically separated and complete or finite over $\mathbb{Q}_p$. Note that there exists a unique map $\psi_{\mathfrak{M}} : \mathfrak{M} \to \sigma^*\mathfrak{M}$ such that $\psi_{\mathfrak{M}} \circ \varphi_{\mathfrak{M}}$ and $\varphi_{\mathfrak{M}} \circ \psi_{\mathfrak{M}}$ are given by multiplication by $\mathcal{P}(u)$. We say $\mathfrak{M}$ is *étale* if $\varphi_{\mathfrak{M}}$ is an isomorphism, and *of Lubin-Tate type (of height* 1*)* if $\psi_{\mathfrak{M}}$ is an isomorphism. When $\mathfrak{M} \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant 1}_A$ is of $\mathfrak{S}_A$-rank 2, we say that $\mathfrak{M}$ is *ordinary* if $\mathfrak{M}$ is an extension of a rank-1 Lubin-Tate type $(\varphi, \mathfrak{S}_A)$-module $\mathfrak{M}^{\mathcal{LT}}$ by a rank-1 étale $(\varphi, \mathfrak{S}_A)$-module $\mathfrak{M}^{\text{ét}}$.

Assume that ρ_R is a $\mathcal{G}_{K_\infty}$-representation of R-rank 2 with height $\leqslant 1$, as before. We define the following subfunctor $D^{\mathrm{ord}}_{\mathfrak{S},\rho_R} \subset D^{\mathbf{v}}_{\mathfrak{S},\rho_R}$ by requiring that $\mathfrak{M}_A$ should be ordinary, and let $D^{\mathrm{ss}}_{\mathfrak{S},\rho_R} \subset D^{\mathbf{v}}_{\mathfrak{S},\rho_R}$ be the complement of $D^{\mathrm{ord}}_{\mathfrak{S},\rho_R}$ (where the superscript 'ss' stands for 'supersingular').

The following proposition is a direct consequence of Proposition 3.5.4 and the semilinear algebraic statement [Kis09b, Proposiiton 2.4.14] applied to the universal $\mathfrak{S}$-lattice of height $\leqslant 1$.

Proposition 3.7. *The subfunctors $D^{\mathrm{ord}}_{\mathfrak{S},\rho_R}, D^{\mathrm{ss}}_{\mathfrak{S},\rho_R} \subset D^{\mathbf{v}}_{\mathfrak{S},\rho_R}$ are represented by $\mathscr{GR}^{\mathrm{ord}}_{\rho_R}$ and $\mathscr{GR}^{\mathrm{ss}}_{\rho_R}$, respectively, which are unions of connected components of $\mathscr{GR}^{\mathbf{v}}_{\rho_R}$.*

Finally, we state the main result of this section.

Proposition 3.8. *Assume that the morphism* $\operatorname{Spf} R \to D_{\infty}^{\leqslant 1}$ *induced by* ρ_R *is formally smooth. Then* $\operatorname{Spec} R^{\mathrm{ss}}[\frac{1}{p}]$ *is geometrically connected. Furthermore* $\operatorname{Spec} R^{\mathrm{ord}}[\frac{1}{p}]$ *is geometrically connected unless* $\bar\rho_\infty \sim \bar\psi_1 \oplus \bar\psi_2$ *where* $\bar\psi_1$ *and* $\bar\psi_2$ *are distinct unramified characters. In the exceptional case there are exactly two components (which are geometrically connected), and two lifts are in the same component if and only if the* $\boldsymbol{\mathcal{G}}_{K_\infty}$*-stable lines on which* I_{K_∞} *act via* $\chi_{\mathrm{cyc}}|_{I_{K_\infty}}$ *reduce to the same character (either* $\bar\psi_1$ *or* $\bar\psi_2$*).*

Idea of the proof. Since the statement is insensitive of $\mathscr{O}$ and the setting is compatible with scalar extension of $\mathscr{O}$ by its finite integral extension, it is enough to show the connectedness instead of the geometric connectedness. By Proposition 3.4 and Proposition 3.5.4, the theorem follows from the corresponding connectedness assertions on $\mathscr{GR}^{\mathrm{ord}}_{\rho_R,0}$ and $\mathscr{GR}^{\mathrm{ss}}_{\rho_R,0}$ (using the notation of Proposition 3.5.4). To show the connectedness of $\mathscr{GR}^{\mathrm{ss}}_{\rho_R,0}$, one explicitly constructs a chain of rational curves that connects any two closed points via the same affine grassmannian computation done in Kisin [Kis09b, Proposition 2.5.6] (improved by Gee [Gee06] and Imai [Ima08]); note that the linear algebra involved in $\mathscr{GR}^{\mathrm{ss}}_{\rho_R,0}$ is very similar to that of moduli of finite flat group schemes.

The scheme $\mathscr{GR}^{\mathrm{ord}}_{\rho_R,0}$ can be computed by exactly the same method as [Kis09b, Proposition 2.5.15], possibly except that we need to prove the following assertion: *For any finite* $\mathbb{F}$*-algebra* A *and a* $\boldsymbol{\mathcal{G}}_{K_\infty}$*-stable* A*-line* L_A *in* $\bar\rho_\infty \otimes_{\mathbb{F}} A$ *such that* $L_A(-1)$ *is unramified, there exists at most one* φ*-stable* $\mathfrak{S}$*-submodule* $\mathfrak{M}_A \subset \underline{D}_{\mathcal{E}}(\bar\rho_\infty(-1)) \otimes_{\mathbb{F}} A$ *which is ordinary torsion* φ*-module height* $\leqslant 1$ *and whose étale submodule corresponds to* L_A. To prove this, we first observe that the set of such $\mathfrak{M}$'s is partially ordered by inclusion, so there exist maximal and minimal objects $\mathfrak{M}^+$ and $\mathfrak{M}^-$ by [CL09, Proposition 3.2.3]. Since $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are extensions of common torsion φ-modules[20] and the inclusion $\mathfrak{M}^- \hookrightarrow \mathfrak{M}^+$ respects the extension, it is an isomorphism by 5-lemma (applied to the abelian category of $\mathfrak{S}$-modules). □

4. Application to "Barsotti-Tate deformation rings"

In this section, we finally relate $\boldsymbol{\mathcal{G}}_{K_\infty}$-deformation rings and crystalline deformation rings. The main result of this section is Corollary 4.2.1 which shows that a "Barsotti-Tate deformation ring" are isomorphic to a suitable $\boldsymbol{\mathcal{G}}_{K_\infty}$-deformation ring of height $\leqslant 1$ via the map defined by "viewing $\boldsymbol{\mathcal{G}}_K$-deformations as $\boldsymbol{\mathcal{G}}_{K_\infty}$-deformations".

4.1. **crystalline and semi-stable deformation rings.** Let $\mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_K)$ (respectively, $\mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_K)$) denote the category of p-adic crystalline (respectievly, semi-stable) $\boldsymbol{\mathcal{G}}_K$-representations V such that $\mathrm{gr}^w\, \underline{D}^*_{\mathrm{dR}}(V) = 0$ for $w \notin [0,h]$. Let $\mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_K)$ (respectively, $\mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_K)$) denote the category of $\mathbb{Z}_p$-lattice $\boldsymbol{\mathcal{G}}_K$-representations T such that $T[\frac{1}{p}] \in \mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_K)$ (respectively, $T[\frac{1}{p}] \in \mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_K)$).

Let $\mathrm{Rep}^{[0,h]}_{\mathrm{cris,tor}}(\boldsymbol{\mathcal{G}}_K)$ (respectively, $\mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\boldsymbol{\mathcal{G}}_K)$) denote the category of finite p^∞-torsion $\boldsymbol{\mathcal{G}}_K$-modules T which admit a $\boldsymbol{\mathcal{G}}_K$-equivariant surjection $\widetilde{T} \twoheadrightarrow T$ where $\widetilde{T} \in \mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_K)$ (respectively, $\widetilde{T} \in \mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_K)$). Note that the subcategories $\mathrm{Rep}^{[0,h]}_{\mathrm{cris,tor}}(\boldsymbol{\mathcal{G}}_K)$ and $\mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\boldsymbol{\mathcal{G}}_K)$ are obviously closed under subquotients and direct sums inside the category of all finite p^∞-torsion $\boldsymbol{\mathcal{G}}_K$-modules.

[20] Namely, both $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are extensions of a unique Lubin-Tate type torsion $\mathfrak{S}$-module model for $(\bar\rho_\infty \otimes_{\mathbb{F}} A)/L_A$ by a unique étale torsion $\mathfrak{S}$-module model for L_A.

Let $\bar{\rho} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ be a representation. Define a subfunctor $D^{\square,[0,h]}_{\mathrm{cris}}$ of the framed deformation functor $D^{\square}$ of $\bar{\rho}$ by requiring that a framed deformation ρ_A over $A \in \mathfrak{AR}_{\mathscr{O}}$ is in $D^{\square,[0,h]}_{\mathrm{cris}}(A)$ if and only if $\rho_A \in \mathrm{Rep}^{[0,h]}_{\mathrm{cris,tor}}(\mathcal{G}_K)$ as a torsion $\mathbb{Z}_p[\mathcal{G}_K]$-module (i.e., ignoring the A-action). By repeating the argument in §2.4, we obtain the universal quotient $R^{\square,[0,h]}_{\mathrm{cris}}$ of the universal framed deformation ring $R^{\square}$ which represents $D^{\square,[0,h]}_{\mathrm{cris}}$. We can similarly define a subfunctor $D^{\square,[0,h]}_{\mathrm{st}} \subset D^{\square}$ using $\mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\mathcal{G}_K)$ and obtain the universal quotient $R^{\square,[0,h]}_{\mathrm{st}}$ or $R^{\square}$. Note also that one can define subfunctors $D^{[0,h]}_{\mathrm{cris}}$, $D^{[0,h]}_{\mathrm{st}} \subset D$, which turn out to be relatively representable.

The following is a difficult theorem of Tong Liu [Liu07]:

Theorem 4.1.1. *Let $\xi : R^{\square} \to A$ be an $\mathscr{O}$-algebra map where A is an finite algebra over* $\mathrm{Frac}\,\mathscr{O}$*, and let $\rho_\xi := \rho^{\square} \otimes_{R^{\square},\xi} A$ where $\rho^{\square}$ is the universal framed deformation. Then ξ factors through the quotient $R^{\square,[0,h]}_{\mathrm{cris}}$ (respectively, $R^{\square,[0,h]}_{\mathrm{st}}$) if and only if $\rho_\xi \in \mathrm{Rep}^{[0,h]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$, where ρ_ξ is viewed as a p-adic $\mathcal{G}_K$-representation by forgetting the A-action.*

Remark 4.1.2. If $\rho \in \mathrm{Rep}^{[0,1]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$, then there exists a p-divisible group G over $\mathscr{O}_K$ such that $V_p(G) \cong \rho$. (This is proved by Breuil [Bre00, Théorème 1.4] when $p > 2$ and by Kisin [Kis06, Corollary 2.2.6] when $p = 2$.) Combining T. Liu's theorem (Theorem 4.1.1), we see that $D^{\square,[0,1]}_{\mathrm{cris}}(A)$ is same as the flat (framed) deformation functor.

Set $\bar{\rho}_\infty := \bar{\rho}|_{\mathcal{G}_{K_\infty}}$. Let $R^{\square,\leqslant h}_{\infty}$ be the universal framed deformation ring of $\bar{\rho}_\infty$ with height $\leqslant h$.

Lemma 4.1.3. *Restricting to $\mathcal{G}_{K_\infty}$ induces the following natural morphisms:*

$$\mathrm{res}^{\leqslant h}_{\mathrm{cris}} : R^{\square,\leqslant h}_{\infty} \to R^{\square,[0,h]}_{\mathrm{cris}}, \text{ and } \mathrm{res}^{\leqslant h}_{\mathrm{st}} : R^{\square,\leqslant h}_{\infty} \to R^{\square,[0,h]}_{\mathrm{st}}.$$

Furthermore, $\mathrm{res}^{\leqslant h}_{\mathrm{cris}} \otimes \mathbb{Q}_p$ induces surjections on the completions at each maximal ideal of $R^{\square,\leqslant h}_{\infty} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The same holds for deformation rings (without framing) if $\mathrm{End}_{\mathcal{G}_{K_\infty}}(\bar{\rho}_\infty) \cong \mathbb{F}$.

Proof. Let us first show that the $\mathcal{G}_{K_\infty}$-restriction of any $T \in \mathrm{Rep}^{[0,h]}_{\mathrm{st,tor}}(\mathcal{G}_K)$ is a torsion $\mathcal{G}_{K_\infty}$-representation of height $\leqslant h$. Choose a presentation $T \cong \widetilde{T}/\widetilde{T}'$ such that $\widetilde{T}, \widetilde{T}' \in \mathrm{Rep}^{[0,h]}_{\mathrm{st},\mathbb{Z}_p}(\mathcal{G}_K)$. Kisin [Kis06, Proposition 2.1.5, Lemma 2.1.15] showed that $\widetilde{T}$ and $\widetilde{T}'$ are of height $\leqslant h$, so it follows from Lemma 2.1.4 that $T|_{\mathcal{G}_{K_\infty}}$ is a torsion representation of height $\leqslant h$. Finally, the assertion on $\mathrm{res}^{\leqslant h}_{\mathrm{cris}} \otimes \mathbb{Q}_p$ directly follows from the full faithfulness of the $\mathcal{G}_{K_\infty}$-restriction on the category of p-adic crystalline $\mathcal{G}_K$-representations [Kis06, Corollary 2.1.14]. □

Let $\mathrm{Rep}^{\leqslant 1}_{\mathrm{tor}}(\mathcal{G}_{K_\infty})$ denote the the category of torsion $\mathcal{G}_{K_\infty}$-representations of height $\leqslant 1$ (Definition 2.1.3). The following theorem and its immediate corollary (Corollary 4.2.1) is the key step for connecting Proposition 3.8 and Theorem 1.1, which is done in §4.4:

Theorem 4.2. *Restricting the $\mathcal{G}_K$-action to $\mathcal{G}_{K_\infty}$ induces an equivalence of categories $\mathrm{Rep}^{[0,1]}_{\mathrm{cris,tor}}(\mathcal{G}_K) \to \mathrm{Rep}^{\leqslant 1}_{\mathrm{tor}}(\mathcal{G}_{K_\infty})$.*

Corollary 4.2.1. *The natural map $\underline{\mathrm{res}}^{\leqslant 1}_{\mathrm{cris}} : \mathrm{Spec}\, R^{\square,[0,1]}_{\mathrm{cris}} \to \mathrm{Spec}\, R^{\square,\leqslant 1}_{\infty}$ (defined in Lemma 4.1.3) is an isomorphism. The same holds for unframed deformation rings if they exist.*

Remark 4.2.2. Note that $\mathrm{res}_{\mathrm{cris}}^{\leqslant h} : \mathrm{Spec}\, R_{\mathrm{cris}}^{\square,[0,h]} \to \mathrm{Spec}\, R_{\infty}^{\square,\leqslant h}$ is *not* in general an isomorphism (even after inverting p); the dimensions of the source and the target are not same at a maximal ideal of $R_{\mathrm{cris}}^{\square,[0,h]}[\frac{1}{p}]$ which corresponds to a lift which has two Hodge-Tate weights that differ at least by 2. See [Kis08, Theorem 3.3.8] and [Kim09, Corollary 11.3.11] for the dimension formulas.

4.3. **Proof of Theorem 4.2.** Theorem 4.2 can be easily deduced from the following proposition:

Proposition 4.3.1. *If $V \in \mathrm{Rep}_{\mathrm{cris},\mathbb{Q}_p}^{[0,1]}(\mathcal{G}_K)$ then any $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice in V is $\mathcal{G}_K$-stable.*

Sketch of the proof. This proposition when $p > 2$ can be read off from the literature (*cf.* [Kis06, Theorem 2.2.7]), and when $p = 2$ this proposition is the main arithmetic ingredient of [Kim10] (as opposed to geometric ingredients). So we only sketch the idea, and for the full details we refer to [Kim10, §5] or the references cited below.

For $V \in \mathrm{Rep}_{\mathrm{cris},\mathbb{Q}_p}^{[0,1]}(\mathcal{G}_K)$ as in the statement, consider a $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice $T \subset V$. By [Kis06, Lemma 2.1.15] there exists $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$ such that $T \cong \underline{T}_{\mathfrak{S}}^{\leqslant 1}(\mathfrak{M})^*(1)$.[21] On the other hand, we can associate, to $\mathfrak{M}$, a $\mathcal{G}_K$-stable lattice T' in V via Breuil's theory of strongly divisible modules. To explain, let S be the p-adically completed divided powers envelop of $W(k)[u]$ with respect to $(\mathcal{P}(u))$. By the recipe given in [Kis06, §2.2.3] and [Bre00, Proposition 5.1.3][22], we can view $S \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}$ as a strongly divisible module. (*cf.* Definition 2.2.1 and Theorem 2.2.3 in [Bre02].) Now, we obtain a $\mathcal{G}_K$-stable $\mathbb{Z}_p$-lattice $T' := T_{\mathrm{st}}^*(S \otimes_{\sigma,\mathfrak{S}} \mathfrak{M})$ in V, where T_{st}^* is defined below Definition 2.2.4 in [Bre02].

When $p > 2$, we can show that $T = T'$ (so T is $\mathcal{G}_K$-stable). This is proved in the first paragraph of the proof of [Kis06, Theorem 2.2.7]. When $p = 2$, we only have $T \subseteq T'$ which can be proper inclusion, but we can show that T is a $\mathcal{G}_K$-stable sublattice in T'. This is proved in [Kim10, Proposition 5.6]. □

Using Proposition 4.3.1, let us prove Theorem 4.2; i.e., the $\mathcal{G}_{K_\infty}$-restriction functor $\mathrm{Rep}_{\mathrm{cris,tor}}^{[0,1]}(\mathcal{G}_K) \to \mathrm{Rep}_{\mathrm{tor}}^{\leqslant 1}(\mathcal{G}_{K_\infty})$ is an equivalence of categories. By Lemma 2.1.4, torsion $\mathcal{G}_{K_\infty}$-representations of height $\leqslant h$ are precisely those that can be written as $T \cong \widetilde{T}/\widetilde{T}'$ where $\widetilde{T}$ and $\widetilde{T}'$ are $\mathbb{Z}_p$-lattice $\mathcal{G}_{K_\infty}$-representations of height $\leqslant 1$. Set $V := \widetilde{T}[\frac{1}{p}] = \widetilde{T}'[\frac{1}{p}]$. By [Kis06, Proposition 2.2.2], V can be uniquely extended to a crystalline representation (denoted by the same letter V) which satisfies the condition in Proposition 4.3.1. This shows the essential surjectivity.

It is left to show that for $T, T' \in \mathrm{Rep}_{\mathrm{cris,tor}}^{[0,1]}(\mathcal{G}_K)$, any $\mathcal{G}_{K_\infty}$-equivariant map $f : T \to T'$ is $\mathcal{G}_K$-equivariant. It is enough to handle the case when f is surjective; once we show this, then any injective $\mathcal{G}_{K_\infty}$-map $T'' \to T$ is $\mathcal{G}_K$-equivariant since $T \twoheadrightarrow T/T''$ is $\mathcal{G}_K$-equivariant, and any map f can be factored as a composition of a surjective map and an injective map Now, assume that f is surjective, and choose a $\mathcal{G}_K$-stable lattice $\widetilde{T}$ in V as in Proposition 4.3.1 such that there is a $\mathcal{G}_K$-surjection $\widetilde{T} \twoheadrightarrow T$. Set $\widetilde{T}' := \ker[\widetilde{T} \twoheadrightarrow T \xrightarrow{f} T']$ which is a $\mathcal{G}_{K_\infty}$-stable $\mathbb{Z}_p$-lattice in V. Then by Proposition 4.3.1, the $\widetilde{T}'$ is also $\mathcal{G}_K$-stable in V, hence f is $\mathcal{G}_K$-equivariant.

4.4. **Deducing Theorem 1.1 from Theorem 4.2.** Let $R_{\mathrm{cris}}^{\square,\mathbf{v}}$ denote the $\mathscr{O}$-flat quotient of $R_{\mathrm{cris}}^{\square,[0,1]}$ such that for any finite $\mathrm{Frac}\,\mathscr{O}$-algebra A, a crystalline A-deformation ρ_A

[21]Note that $\underline{T}_{\mathfrak{S}}^{\leqslant 1}(\mathfrak{M})^*(1) = \underline{T}_{\mathcal{E}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}})^*$ is the usual contravariant version of the functor, which is more convenient here.

[22]Although $p > 2$ is assumed through the paper [Bre00], the statement and proof of Proposition 5.1.3 is valid even when $p = 2$.

defines an A-point of $R^{\square,\mathbf{v}}_{\mathrm{cris}}$ if and only if ρ_A defines an A-point of $R^{\square,[0,1]}_{\mathrm{cris}}$ and the following additional condition is satisfied:

$$\det \rho_A|_{I_K} \sim \chi_{\mathrm{cyc}}|_{I_K}. \tag{4.4.1a}$$

Indeed, the condition (4.4.1a) for $A \in \mathfrak{AR}_{\mathscr{O}}$ defines a universal quotient of $R^{\square,\mathbf{v}}_{\mathrm{cris}}$, and we further quotient out all p^∞-torsion to obtain $R^{\square,\mathbf{v}}_{\mathrm{cris}}$.

Note that the condition (4.4.1a) for a finite $\mathbb{Q}_p$-algebra A is equivalent to the following:

$$\det \rho_A|_{I_{K_\infty}} \sim \chi_{\mathrm{cyc}}|_{I_{K_\infty}}, \tag{4.4.1b}$$

since a p-adic crystalline I_K-representation is uniquely determined by its restriction to I_{K_∞} by Kisin [Kis06, Corollary 2.1.14]. Therefore, it follows from Theorem 4.2 and Lemma 3.5.3 that the map $\underline{\mathrm{res}}^{\mathrm{cris}}$ induces an isomorphism $R^{\square,\mathbf{v}}_{\infty} \xrightarrow{\sim} R^{\square,\mathbf{v}}_{\mathrm{cris}}$ (where $R^{\square,\mathbf{v}}_{\infty}$ is defined in Lemma 3.5.3).

Now, in order to obtain Theorem 1.1 (granting Theorem 4.2), we are left to show that the "ordinarity" and "supersingularity" (which will be defined below) are preserved by $\mathcal{G}_{K_\infty}$-restriction. Let us first recall the definitions. Let A be finite flat over either $\mathbb{Z}_p$ or $\mathbb{Q}_p$, and let T_A be a free A-module of rank 2 equipped with continuous $\mathcal{G}_{K_\infty}$-action which is of height $\leqslant 1$ and such that I_{K_∞} acts on $\det T_A$ via $\chi_{\mathrm{cyc}}|_{I_{K_\infty}}$. We say that such T_A is *ordinary* if it admits a non-zero unramified quotient (necessarily unique and of rank 1), and is *supersingular* otherwise. When A is finite over $\mathbb{Q}_p$, a $\mathcal{G}_{K_\infty}$-representation T_A over A is ordinary if and only if the unique $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant 1}_A$ that gives rise to T_A is ordinary in the sense of §3.6.

For A as above, let T_A be a free A-module of rank 2 equipped with continuous $\mathcal{G}_K$-action such that $T_A[\frac{1}{p}] \in \mathrm{Rep}^{[0,1]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$ as a p-adic representation and I_K acts on $\det T_A$ via $\chi_{\mathrm{cyc}}|_{I_K}$. We similarly define ordinary and supersingular representations, and the definition can be interpreted in terms of the associated filtered φ-module.

Set $R^{\square,\mathrm{ord}}_{\mathrm{cris}} := R^{\square,[0,1]}_{\mathrm{cris}} \otimes_{R^{\square,\leqslant 1}_{\infty}} R^{\square,\mathrm{ord}}_{\infty}$ and $R^{\square,\mathrm{ss}}_{\mathrm{cris}} := R^{\square,[0,1]}_{\mathrm{cris}} \otimes_{R^{\square,\leqslant 1}_{\infty}} R^{\square,\mathrm{ss}}_{\infty}$.

Proposition 4.4.2. *Let A be a finite $\mathbb{Q}_p$-algebra. Then a map $\xi : R^{\square,[0,1]}_{\mathrm{cris}} \to A$ factors through $R^{\square,\mathrm{ord}}_{\mathrm{cris}}$ (respectively, $R^{\square,\mathrm{ss}}_{\mathrm{cris}}$) if and only if the corresponding framed A-deformation ρ_ξ is an ordinary $\mathcal{G}_K$-representation (respectively, ρ_ξ is a supersingular $\mathcal{G}_K$-representation).*

By this proposition, the statement of Theorem 1.1 is clearly reduced to Theorem 4.2 and Proposition 3.8.

Proof. By definition, $\xi : R^{\square,[0,1]}_{\mathrm{cris}} \to A$ a map factors through the quotient $R^{\square,\mathrm{ord}}_{\mathrm{cris}}$ (respectively, $R^{\square,\mathrm{ss}}_{\mathrm{cris}}$) if and only if $\rho_\xi|_{\mathcal{G}_{K_\infty}}$ is ordinary (respectively, $\rho_\xi|_{\mathcal{G}_{K_\infty}}$ is supersingular). By [Kis06, Corollary 2.1.14], we have $\det \rho_\xi|_{I_K} \cong \chi_{\mathrm{cyc}}|_{I_K}$ if and only if $\det \rho_\xi|_{I_{K_\infty}} \cong \chi_{\mathrm{cyc}}|_{I_{K_\infty}}$. Now the proposition follows from Lemma 4.4.3. □

Lemma 4.4.3. *Let A be a finite local $\mathbb{Q}_p$-algebra, and V_A an A-representation of $\mathcal{G}_K$ such that $V_A \in \mathrm{Rep}^{[0,1]}_{\mathrm{cris},\mathbb{Q}_p}(\mathcal{G}_K)$ as a p-adic representation. Then the maximal unramified $A[\mathcal{G}_{K_\infty}]$-quotient $V_A^{\text{ét}}$ of V_A is free over A and is a $\mathcal{G}_K$-equivariant quotient. (So $V_A^{\text{ét}}$ is also the maximal unramified $\mathcal{G}_K$-quotient.)*

Proof. By [Kis06, Proposition 2.2.2] the natural projection $V_A \twoheadrightarrow V_A^{\text{ét}}$ is $\mathcal{G}_K$-equivariant, so it remains to show that $V_A^{\text{ét}}$ is free over A.

It follows from [Kis08, Corollary 1.6.3] (*cf.* Proposition 3.4) that there exist a $\mathbb{Z}_p$-subalgebra $A^\circ \subset A$ with $A^\circ[\frac{1}{p}] = A$ and $\mathfrak{M}_{A^\circ} \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant 1}_{A^\circ}$ such that $T_{A^\circ}[\frac{1}{p}] \cong V_A|_{\mathcal{G}_{K_\infty}}$ where $T_{A^\circ} := \underline{T}^{\leqslant 1}_{\mathfrak{S}}(\mathfrak{M}_{A^\circ})$. Then the image of T_{A° in $V_A^{\text{ét}}$ is free over A° by [Kis09b, Proposition 1.2.11], and its A-span is $V_A^{\text{ét}}$. This proves that $V_A^{\text{ét}}$ is free over A. □

Remark 4.4.4. Indeed, Lemma 4.4.3 still holds when $V_A \in \mathrm{Rep}_{\mathrm{st},\mathbb{Q}_p}^{[0,h]}(\mathcal{G}_K)$ as a p-adic representation, so Proposition 4.4.2 can be generalized to this setting (with suitable the definitions of ordinary and supersingular representations of $\mathcal{G}_K$ and $\mathcal{G}_{K_\infty}$). For the statements and the proofs see Lemma 11.4.19 and Proposition 11.4.18 in [Kim09].

5. Positive characteristic analogue of crystalline deformation rings

In this section, we introduce a class of $\mathrm{Gal}(k(\!(u)\!)^{\mathrm{sep}}/k(\!(u)\!))$-representation with coefficients in some equi-characteristic local field which could be thought of as an analogue of crystalline representations, and develop a deformation theory for them. Such representations are (implicitly) introduced by Genestier-Lafforgue [GL10], and its torsion version also appeared in Abrashkin [Abr09]. A useful observation is that the linear algebra objects that give rise to such Galois representations have very similar structure to various $(\varphi,\mathfrak{S})$-modules we saw in Kisin theory. Considering the norm field isomorphism $\mathcal{G}_{K_\infty} \cong \mathrm{Gal}(k(\!(u)\!)^{\mathrm{sep}}/k(\!(u)\!))$ [Win83], it is not too surprising that the $\mathcal{G}_{K_\infty}$-deformation theory has an analogue in positive characteristic.

5.1. **Notations/Definitions.** Let $\mathscr{O}_0 := \mathbb{F}_q[[\pi_0]]$ be a complete discrete valuation ring of characteristic p. For this section, let $K := k(\!(u)\!)$ and $\mathscr{O}_K := k[[u]]$ where k is a finite extension of $\mathbb{F}_q$. (So K is no more a finite extension of $\mathbb{Q}_p$.) We fix a finite map $\iota : \mathscr{O}_0 \to \mathscr{O}_K$ over $\mathbb{F}_q$. Roughly speaking, $\mathscr{O}_0$ will play the role of $\mathbb{Z}_p$, and $\pi_0 \in \mathscr{O}_0$ will play the role of p.

Put $\mathcal{G}_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We will study a certain class of $\mathcal{G}_K$-representations over $\mathscr{O}_0$, $\mathrm{Frac}(\mathscr{O}_0)$, or finite algebras thereof. It is defined in terms of linear-algebraic objects called *(effective) local shtukas* over $\mathscr{O}_K$, which we introduce below. Local shtukas have many analogous features to $(\varphi,\mathfrak{S})$-modules of finite height in Kisin theory, so we use similar notations to Kisin theory to emphasize the analogy.

Let $\mathfrak{S} := \mathscr{O}_K[[\pi_0]]$ and $\mathscr{O}_{\mathcal{E}} := K[[\pi_0]]$. We define a partial q-Frobenius endomorphism σ for each of these rings so that it acts as the qth power map on K and $\sigma(\pi_0) = \pi_0$. This σ lifts the qth power map modulo π_0, and fixes $\mathscr{O}_0$. We also set $\mathcal{E} := K(\!(\pi_0)\!)$ and extend σ on $\mathcal{E}$. Then σ fixes $\mathrm{Frac}(\mathscr{O}_0)$

Let $u_0 := \iota(\pi_0) \neq 0$ where $\iota : \mathscr{O}_0 \to \mathscr{O}_K$ is the map we fixed earlier. Put $\mathcal{P}(u) := \pi_0 - u_0 \in \mathfrak{S}$ and let $e := \mathrm{ord}_u(u_0)$. Clearly we have $\mathfrak{S}/(\mathcal{P}(u)) \cong \mathscr{O}_K$, which is a totally ramified ring extension of $k[[\pi_0]]$. This shows that $\mathcal{P}(u)$ is a $\mathfrak{S}^\times$-multiple of some Eisenstein polynomial in $k[[\pi_0]][u]$ with degree e.

5.1.1. An *étale φ-module* is a $(\varphi,\mathscr{O}_{\mathcal{E}})$-module[23] $(\mathcal{M},\varphi_M)$ such that φ_M is an isomorphism.The main motivation for considering étale φ-modules is that we have an equivalence of categories between the category of étale φ-modules and the category of $\mathcal{G}_K$-representations . Let $\widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}} := K^{\mathrm{sep}}[[\pi_0]]$, and we let $\mathcal{G}_K$ act on it through the coefficients, and define the partial q-Frobenius endomorphism σ so that it acts as the qth power map on K^{sep} and $\sigma(\pi_0) = \pi_0$. For an étale φ-module M we define

$$\underline{T}_{\mathcal{E}}(M) := (M \otimes_{\mathscr{O}_{\mathcal{E}}} \widehat{\mathscr{O}}_{\mathcal{E}^{\mathrm{ur}}})^{\varphi=1}. \tag{5.1.2}$$

This induces an exact equivalence of categories between the category of étale φ-modules and the category of finitely generated $\mathscr{O}_0$-module with continuous $\mathcal{G}_K$-action. One can define the quasi-inverse $\underline{D}_{\mathcal{E}}$ in a similar fashion to (2.1.2a). Furthermore, they respect all the natural operations, and they preserve rank and length

[23]The notion of φ-module is defined in §2.1. Note that we use $\mathscr{O}_{\mathcal{E}}$ defined in §5.1, not the one in §2.1.

whenever applicable. The proof is identical to the proof of the p-adic case [Fon90, §A 1.2].[24]

Definition 5.1.3. Consider the following étale φ-module $M_{\mathcal{LT}} := \mathscr{O}_{\mathcal{E}} \cdot \mathbf{e}$ equipped with $\varphi_{M_{\mathcal{LT}}}(\sigma^*\mathbf{e}) = \mathcal{P}(u)^{-1}\mathbf{e}$. Let $\chi_{\mathcal{LT}} : \mathcal{G}_K \to \mathscr{O}_0^\times$ denote the character that defines the $\mathcal{G}_K$-action on $\underline{T}_{\mathcal{E}}(M_{\mathcal{LT}})$. For any $\mathscr{O}_0[\mathcal{G}_K]$-module V, we let $V(n)$ the $\mathscr{O}_0[\mathcal{G}_K]$-module whose $\mathcal{G}_K$-action is twisted by $\chi_{\mathcal{LT}}^n$.

This character $\chi_{\mathcal{LT}}$ is equivalent to the character obtained from the π_0-adic Tate module of the Lubin-Tate formal $\mathscr{O}_0$-module over $\mathscr{O}_K$. See [And93] for the proof. Note that when K is a finite extension of $\mathbb{Q}_p$, we can obtain $\chi_{\mathrm{cyc}}|_{\mathcal{G}_{K_\infty}}$ from the étale φ-module defined analogously as above. Compare with [Kis09a, Lemma 2.3.4].

5.1.4. For a non-negative integer h, an *effective local shtuka (over $\mathscr{O}_K$) of height $\leqslant h$* is a finite free $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a $\mathfrak{S}$-linear morphism $\varphi_{\mathfrak{M}} : \sigma^*\mathfrak{M} \to \mathfrak{M}$ such that $\mathrm{coker}(\varphi_{\mathfrak{M}})$ is killed by $\mathcal{P}(u)^h$. The original definition of effective local shtuka (over $\mathscr{O}_K$) requires $\mathrm{coker}(\varphi_{\mathfrak{M}})$ to be flat over $\mathscr{O}_K$, but this is automatic because it is a $\mathcal{P}(u)$-power torsion $\mathfrak{S}$-module of projective dimension $\leqslant 1$. Note that effective local shtukas can be defined over any $\mathscr{O}_0$-scheme (not just over $\mathrm{Spf}\, \mathscr{O}_K$), and there are more general objects called *local shtukas* which are defined by allowing $\varphi_{\mathfrak{M}}$ to have a pole at $\mathcal{P}(u)$. See [GL10, Definition 0.1] or [Har10, Definition 2.1.1] for more general definition.

Since $\mathcal{P}(u)$ is a unit in $\mathscr{O}_{\mathcal{E}}$, the scalar extension $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}}$ is naturally an étale φ-module. So we can associate a $\mathcal{G}_K$-representation to an effective local shtuka $\mathfrak{M}$ of height $\leqslant h$ as follows:

$$\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}) := \underline{T}_{\mathcal{E}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}})(h) \tag{5.1.5}$$

We state the following fundamental and non-trivial result on this functor $\underline{T}_{\mathfrak{S}}^{\leqslant h}$. Compare with [Kis06, Proposition 2.1.12, Lemma 2.1.15].

Proposition 5.1.6.

(1) *The functor $\underline{T}_{\mathfrak{S}}^{\leqslant h}$ from the category of effective local shtukas of height $\leqslant h$ to the category of $\mathscr{O}_0$-representations of $\mathcal{G}_K$ is fully faithful.*
(2) *Let $V := \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})[\frac{1}{\pi_0}]$, then for any $\mathcal{G}_K$-stable $\mathscr{O}_0$-lattice $T' \subset V$ there exists an effective local shtuka $\mathfrak{M}'$ of height $\leqslant h$ such that $T' \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}')$.*

Proof. The proof of (1) is very similar to the proof of its p-adic analogue [Kis06, Proposition 2.1.12], except that one needs to work with "weakly admissible isocrystals with Hodge-Pink structure" instead of filtered φ-modules, and apply [GL10, Théorème 7.3] instead of [Kis06, Lemma 1.3.13]. The detail is worked out in [Kim09, Theorem 5.2.3].

The claim (2) easily follows from [GL10, Lemme 2.3] by the same way as its p-adic analogue [Kis06, Lemma 2.1.15] is proved. □

A finite free $\mathscr{O}_0$-module equipped with continuous $\mathcal{G}_K$-action is called *$\mathscr{O}_0$-lattice $\mathcal{G}_K$-representation.* A finitely generated π_0^∞-torsion $\mathscr{O}_0$-module equipped with continuous $\mathcal{G}_K$-action is called *π_0^∞-torsion $\mathcal{G}_K$-representation.*

Definition 5.1.7. Let h be a non-negative integer. An $\mathscr{O}_0$-lattice $\mathcal{G}_K$-action T is called *of height $\leqslant h$* if there exists an effective local shtuka $\mathfrak{M}$ of height $\leqslant h$ such that $T \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})$. A continuous $\mathcal{G}_K$-representation V over $\mathrm{Frac}(\mathscr{O}_0)$ is called *of height $\leqslant h$* if it admits a $\mathcal{G}_K$-stable $\mathscr{O}_0$-lattice $T \subset V$ which is of height $\leqslant h$; or equivalently by Proposition 5.1.6(2), any $\mathcal{G}_K$-stable $\mathscr{O}_0$-lattice $T \subset V$ is of height

[24]See [Kim09, §5.1] for the full proof, but the positive characteristic version of the theory of étale φ-modules must have been known for a while

$\leqslant h$. A π_0^∞-torsion $\mathcal{G}_K$-representation T is called *of height* $\leqslant h$ if there exist $\mathscr{O}_0$-lattice $\mathcal{G}_K$-representations $\widetilde{T}' \subset \widetilde{T}$ of height $\leqslant h$ such that $T \cong \widetilde{T}/\widetilde{T}'$.

It easily follows from Definition 5.1.3 that $\chi^r_{\mathcal{LT}}$ for $0 \leqslant r \leqslant h$ is of height $\leqslant h$. It is not difficult to show that any unramified $\mathcal{G}_K$-representation is of height $\leqslant 0$ (hence, of height $\leqslant h$ for any non-negative h). See, for example, [Kim09, Proposition 5.2.10] for the proof.

Proposition 5.1.6 suggests that $\mathcal{G}_K$-representations of height $\leqslant h$ should enjoy similar properties to those enjoyed by $\mathcal{G}_{K_\infty}$-representation of height $\leqslant h$ in the setting of Kisin theory. On the other hand, $\mathcal{G}_K$-representations of height $\leqslant h$ can also be regarded as a positive characteristic analogue of crystalline representations with Hodge-Tate weights in $[0, h]$, for the following reasons.[25] Effective local shtukas arise naturally by completing global objects at "places of good reduction" such as t-motives, elliptic sheaves, and Drinfeld shtukas. (See [Har10, Example 2.1.2] for more details.) It has been known for experts that there exists a natural anti-equivalence of categories between the category of effective local shtukas of height $\leqslant 1$ and the category of strict π_0-divisible groups (using the terminology of [Fal02]), and if $\mathfrak{M}$ is the effective local shtuka of height $\leqslant 1$ which corresponds to a strict π_0-divisible group G then $(\underline{T}^{\leqslant 1}_{\mathfrak{S}}(\mathfrak{M}))^*(1)$ is naturally isomorphic to the π_0-adic Tate module of G. This is generalized by Hartl [Har09, §3] to any effective local shtukas.[26] See [Kim09, §7.3] for the proof.

5.1.8. For a non-negative integer h, a *torsion shtuka of height* $\leqslant h$ is a finitely generated π_0^∞-torsion u-torsion-free $\mathfrak{S}$-module $\mathfrak{M}$ equipped with an $\mathfrak{S}$-linear morphism $\varphi_{\mathfrak{M}} : \sigma^*\mathfrak{M} \to \mathfrak{M}$ such that $\mathrm{coker}(\varphi_{\mathfrak{M}})$ is killed by $\mathcal{P}(u)^h$. We let $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ denote the category of torsion shtukas of height $\leqslant h$ with the obvious notion of morphisms. There is a natural analogue of Cartier duality. (See [Kim09, §8.3] for more details.)

Let $\mathfrak{M}$ be a torsion shtuka of height $\leqslant h$. Since $\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}}$ is a π_0^∞-torsion étale φ-module, one can associate a π_0^∞-torsion $\mathcal{G}_K$-representation $\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}) := \underline{T}_{\mathcal{E}}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathscr{O}_{\mathcal{E}})(h)$. The same proof as in [Kis06, Lemma 2.3.4] shows that any torsion shtuka of height $\leqslant h$ can be obtained as the cokernel of an isogeny $\widetilde{\mathfrak{M}}' \to \widetilde{\mathfrak{M}}$ of effective local shtukas of height $\leqslant h$. Now, exactness of $\underline{T}^{\leqslant h}_{\mathfrak{S}}$ implies that any π_0^∞-torsion $\mathcal{G}_K$-representation T is of height $\leqslant h$ (in the sense of Definition 5.1.7) comes from a torsion shtuka of height $\leqslant h$.

5.1.9. We finally remark that the analogue of the "limit theorem" holds; i.e., an $\mathscr{O}_0$-lattice $\mathcal{G}_K$-representation obtained as a limit of π_0^∞-torsion $\mathcal{G}_K$-representation of height $\leqslant h$ is again of height $\leqslant h$ (as an $\mathscr{O}_0$-lattice $\mathcal{G}_K$-representation). The proof is "identical" to the proof of its p-adic analogue [Liu07, Theorem 2.4.1].

5.2. **Deformation theory.** Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_q$ (which is the residue field of $\mathscr{O}_0$), and $\bar\rho : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ a representation. Let $\mathscr{O}$ be a finite extension of $\mathscr{O}_0$ with residue field $\mathbb{F}$. Let $\mathfrak{AR}_{\mathscr{O}}$ be the category of artin local $\mathscr{O}$-algebras A whose residue field is $\mathbb{F}$, and similarly let $\widehat{\mathfrak{AR}}_{\mathscr{O}}$ be the category of complete local noetherian $\mathscr{O}$-algebras with residue field $\mathbb{F}$.

Let $D, D^\square : \widehat{\mathfrak{AR}}_{\mathscr{O}} \to (\mathbf{Sets})$ be the deformation functor and framed deformation functor for $\bar\rho$. Since the tangent spaces of these functors are infinite-dimensional (as explained in §2.2), they cannot be represented by complete local noetherian $\mathscr{O}$-algebras.

[25] We remark that in positive characteristic $K_\infty := K(\sqrt[q^\infty]{u})$ is a purely inseparable field extension of K, so the gap between $\mathcal{G}_K$ and $\mathcal{G}_{K_\infty}$ collapses.

[26] Note that not all the π_0-divisible groups come from effective local shtukas – the π_0-divisible groups that come from effective local shtukas are called *divisible Anderson modules* in [Har09, §3].

We say that a deformation ρ_A over $A \in \mathfrak{AR}_{\mathscr{O}}$ is *of height* $\leqslant h$ if it is a π_0^∞-torsion $\boldsymbol{\mathcal{G}}_K$-representation of height $\leqslant h$ as a π_0^∞-torsion $\boldsymbol{\mathcal{G}}_K$-representation; or equivalently, if there exist $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ and an $\mathscr{O}_0[\boldsymbol{\mathcal{G}}_K]$-isomorphism $\rho_A \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M})$. For $A \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$, we say that ρ_A is *of height* $\leqslant h$ if $\rho_A \otimes A/\mathfrak{m}_A^n$ is a deformation of height $\leqslant h$ for each n. When $A \in \mathfrak{AR}_{\mathscr{O}}$, both definitions are compatible because the condition of being height $\leqslant h$ is closed under subquotient. (The proof is the same as Lemma 2.4.1.) When A is finite flat over $\mathscr{O}_0$, a deformation ρ_A over A is of height $\leqslant h$ if and only if ρ_A is of height $\leqslant h$ as a $\mathscr{O}_0$-lattice $\boldsymbol{\mathcal{G}}_K$-representation, as remarked in §5.1.9.

Let $D^{\leqslant h} \subset D$ and $D^{\square,\leqslant h} \subset D^{\square}$ respectively denote subfunctors of deformations and framed deformations of height $\leqslant h$. In this setting, we have the analogue of Theorem 2.3:

Theorem 5.2.1. *The functor $D^{\leqslant h}$ has a hull, and if $\mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\bar\rho) \cong \mathbb{F}$ then $D^{\leqslant h}$ is representable (by $R^{\leqslant h} \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$). The functor $D^{\square,\leqslant h}$ is representable (by $R^{\square,\leqslant h} \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$) with no assumption on $\bar\rho$. Furthermore, the natural inclusions $D^{\leqslant h} \hookrightarrow D$ and $D^{\square,\leqslant h} \hookrightarrow D^{\square}$ of functors are relatively representable.*

We call $R^{\square,\leqslant h}$ the *universal framed deformation ring of height* $\leqslant h$ and $R^{\leqslant h}$ the *universal deformation ring of height* $\leqslant h$ if it exists.

The proof of Theorem 2.3 can easily be adapted. The main step is to show the finiteness of the tangent space, but the same proof of Proposition 2.6 works if we replace $\mathfrak{S}$, $\mathscr{O}_{\mathcal{E}}$, $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ by their positive characteristic analogues as introduced in §5.1 and the pth power map is replaced by the qth power map in suitable places. See [Kim09, §11.7] for the full details.

5.3. **Moduli of torsion shtukas of height** $\leqslant h$**.** Let h be a positive integer, and let A be a π_0-adically separated and complete topological $\mathscr{O}_0$-algebra, (for example, finite $\mathscr{O}_0$-algebras or any $\mathscr{O}_0$-algebra A with $\pi_0^N \cdot A = 0$ for some N). We can define $\mathfrak{S}_A$, $fo_{\mathcal{E},A}$, $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$, and $(\mathrm{ModFI}/\mathscr{O}_{\mathcal{E}})_A^{\text{ét}}$ in a manner similar to §2.6.1 but using $\mathfrak{S}$ and $\mathscr{O}_{\mathcal{E}}$ defined in §5.1.

Consider a deformation ρ_R of $\bar\rho$ over $R \in \widehat{\mathfrak{AR}}_{\mathscr{O}}$ which is of height $\leqslant h$ (i.e. $\rho_R \otimes_R R/\mathfrak{m}_R^n$ is of height $\leqslant h$ for each n). The main examples to keep in mind are universal framed deformation of height $\leqslant h$. Put $M_R := \varprojlim M_n$ where $M_n \in (\mathrm{ModFI}/\mathscr{O}_{\mathcal{E}})_{R/\mathfrak{m}_R^n}^{\text{ét}}$ is such that $\underline{T}_{\mathcal{E}}(M_n)(h) \cong \rho_R \otimes_R R/\mathfrak{m}_R^n$ for each n. For any R-algebra A, we view $M_R \otimes_R A$ as an étale φ-module by A-linearly extending φ_{M_R}.

For a complete local noetherian ring R, let $\mathfrak{Aug}_R$ be the category of pairs (A, I) where A is an R-algebra and $I \subset A$ is an ideal with $I^N = 0$ for some N such that $\mathfrak{m}_R \cdot A \subseteq I$. Note that an artin local R-algebra A can be viewed as an element in $\mathfrak{Aug}_{\mathscr{O}}$ by setting $I := \mathfrak{m}_A$. A morphism $(A, I) \to (B, J)$ in $\mathfrak{Aug}_R$ is an R-morphism $A \to B$ which takes I into J. We define a functor $D_{\mathfrak{S},\rho_R}^{\leqslant h} : \mathfrak{Aug}_R \to (\mathbf{Sets})$ by putting $D_{\mathfrak{S},\rho_R}^{\leqslant h}(A, I)$ the set of φ-stable $\mathfrak{S}_A$-lattices in $M_R \otimes_R A$ which are objects in $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$.

With this setting, we have an analogue of Proposition 3.2.

Proposition 5.3.1. *The functor $D_{\mathfrak{S},\rho_R}^{\leqslant h}$ can be represented by a projective R-scheme $\mathscr{GR}_{\rho_R}^{\leqslant h}$ and a $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathscr{GR}_{\rho_R}^{\leqslant h}}$-lattice $\underline{\mathfrak{M}}_{\rho_R}^{\leqslant h} \subset M_R \otimes_R \mathcal{O}_{\mathscr{GR}_{\rho_R}^{\leqslant h}}$. (We call $\underline{\mathfrak{M}}_{\rho_R}^{\leqslant h}$ a* universal $\mathfrak{S}$-lattice of height $\leqslant h$ *for ρ_R.) Moreover, the formation of $\mathscr{GR}_{\rho_R}^{\leqslant h}$ and $\underline{\mathfrak{M}}_{\rho_R}^{\leqslant h}$ commute with scalar extension $R \to R'$.*

Indeed, the proof of its p-adic analogue (Proposition 3.2) works verbatim in the positive characteristic setting. The proof is also worked out in Proposition 11.1.9, Corollary 11.1.11 of [Kim09] for the positive characteristic setting.

The discussions in §3 also applies to this positive characteristic setting. For example, the structure morphism $\mathscr{G}\mathscr{R}^{\leqslant h}_{\rho_R} \to \operatorname{Spec} R$ becomes an isomorphism after inverting π_0 (Proposition 3.4, also using Proposition 5.1.6(1)); $R^{\square,\leqslant h}[\frac{1}{\pi_0}]$ is formally smooth (Proposition 3.4); and in the rank-2 case the condition of having "ordinary" local shtuka model defines a union of connected components in $R^{\square,\leqslant h}[\frac{1}{\pi_0}]$ (Proposition 3.7).

When $\bar{\rho}$ is 2-dimensional and $h = 1$, one can define the $\mathscr{O}$-flat quotient $R^{\square,\mathbf{v}}$ of $R^{\square,\leqslant 1}$ in the similar fashion to §3.5[27], whose generic fiber classifies lifts such that I_K acts via $\chi_{\mathcal{LT}}$ on the determinant. (*cf.* [Kis09a, Lemma 2.3.4].) Then the direct analogue of the connected component result (Proposition 3.8) holds for the positive characteristic deformation ring $R^{\square,\mathbf{v}}[\frac{1}{\pi_0}]$. Furthermore, the argument in [Kis08, §3] can be adapted to show that $R^{\square,\mathbf{v}}[\frac{1}{\pi_0}]$ is equi-dimensional of dimension $4 + [K : \mathbb{F}_q((u_0))]$, which is strongly analogous to the p-adic case. (Compare with [Kis08, Theorem 3.3.8] and [Kim09, §11.3.17].) All these results can be generalized to the case with $h > 1$ except the connectedness of the "supersingular locus" in $\operatorname{Spec} R^{\square,\mathbf{v}}[\frac{1}{\pi_0}]$ (with the suitable definition of $R^{\square,\mathbf{v}}$).

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