



# On pricing of interest rate derivatives

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## Abstract

At present, there is an explosion of practical interest in the pricing of interest rate (IR) derivatives. Textbook pricing methods do not take into account the leptokurticity of the underlying IR process. In this paper, such a leptokurtic behavior is illustrated using London interbank offered rate data, and a possible martingale pricing scheme is discussed.

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## 1. Introduction

In financial theory and practice, interest rates are a very important subject which can be approached from several different perspectives. The classical theoretical approach models the term structure of interest rates using stochastic processes. Various models have been proposed and can be found in Refs. [1–3]. Although they provide analytical formulas for the pricing of interest rate derivatives, the implied deformations of the term structure have a Brownian motion component and are often rejected by empirical data (see Ref. [4]). The inadequacies of the Gaussian model for the description of financial time series have been reported since a long time ago by Mandelbrot [5], but thanks to the availability of large sets of financial data, the interest on this point has risen recently [6,7]. In particular, the fat-tail property of the empirical distribution of price changes has been widely documented and is a crucial feature for monitoring the

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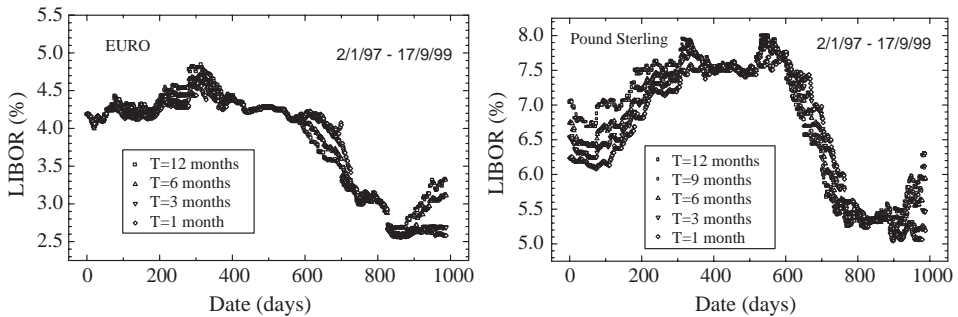


Fig. 1. LIBOR interest rates  $r(T, t)$  as a function of the current date  $t$ , for the maturities  $T = 1, 3, 6$  and 12 months and EURO currency (left) and Pound Sterling currency (right).

extreme risks and for accurately pricing interest rate derivatives. An important recent development in the pricing of interest rate derivatives is the emergence of models that incorporate lognormal volatilities for forward London interbank offered rate (LIBOR) or forward swap rates while keeping interest rates stable [8]. To our knowledge, up to now, no universally accepted theory has been obtained for the description of interest rates data [9].

In this framework, we have empirically studied the probability density distribution of LIBOR, in order to characterize the stochastic behavior of the daily fluctuations. In Section 2, we present the data set and the data analysis. Section 3 contains a short discussion of a possible IR derivative pricing scheme using martingale methods.

## 2. Empirical findings

LIBOR stands for the London interbank offered rate and is the rate of interest at which banks are willing to offer deposits to other prime banks, in marketable size, in the London interbank market. British Bankers' Association (BBA) LIBOR [10] is the most widely used benchmark or reference rate. It is used as the basis for settlement of interest rate contracts on many of the world's major future and option exchanges as well as most over the counter and lending transactions. BBA LIBOR is compiled each working day and broadcast through 10 international distribution networks. BBA LIBOR fixings are provided in seven international currencies: Pound Sterling, US Dollar, Japanese Yen, Swiss Franc, Canadian Dollar, Australian Dollar, EURO. LIBOR rates are fixed for each currency at monthly maturities from 1 to 12 months. Rates shall be contributed in decimal to at least two decimal places but no more than five. In the following, we have analyzed a data set of LIBOR interest rates  $r(T, t)$ , where  $T$  is the maturity date and  $t$  the current date, for EURO and Pound Sterling. These data are shown in Fig. 1 where  $t$  goes from January 2, 1997 to September 17, 1999, and  $T$  assumes the following values: 1, 3, 6, 9 and 12 months for the Pound Sterling and 1, 3, 6, 12 months for the EURO. In Fig. 2, the 1-month LIBOR is compared to the interest rates fixed by Central Banks at that time, namely the repurchase agreement (REPO) data. It is quite evident

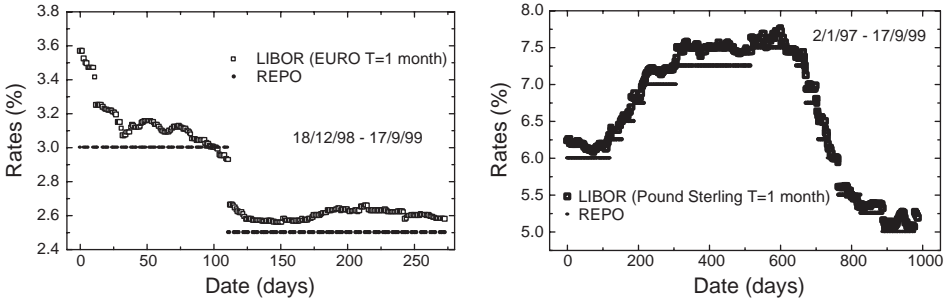


Fig. 2. LIBOR compared to REPO for EURO (left) and for Sterling Pound (right).

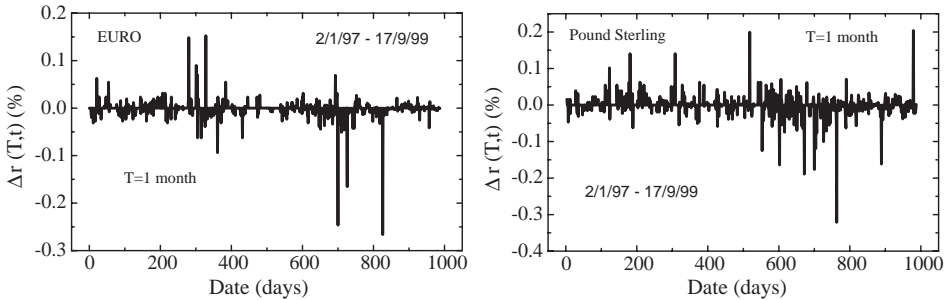


Fig. 3. 1-month LIBOR increments as a function of the current date for EURO (left) and for Sterling Pound (right).

that BBA LIBOR follows the trend determined by the decisions of Central Banks. In order to roughly eliminate these trends, in Fig. 3 the interest rates differences  $\Delta r(T, t) = r(T, t + \Delta t) - r(T, t)$ , with  $\Delta t$  being 1 day and  $T = 1$  month, are plotted as a function of the current date, for the EURO and the Pound Sterling, respectively. A similar behavior is also found for the other maturities. Some large oscillations of  $\Delta r$  are induced by Central Banks. In any case,  $\Delta r$  heavily fluctuates around zero. We focus the attention on the probability distribution behavior of the interest rates increments  $\Delta r(T, t)$ . To this purpose, we estimate  $\Psi(\Delta r)$ , the complementary cumulative distribution function of the daily interest rates increments, defined as

$$\Psi(\Delta r) = 1 - \int_{-\infty}^{\Delta r} p(\eta) d\eta, \tag{1}$$

where  $p$  is the probability density of  $\Delta r(T, t)$ . Because LIBOR data are supplied with only few decimal digits, it is interesting to examine the effects of different data cut-offs in the behavior of  $\Psi(\Delta r)$ . In Fig. 4, we plot the complementary cumulative distribution function for a simulated Gaussian stochastic process using data characterized by three different decimal digit precisions. It turns out that the numerical rounding does not influence the results. Fig. 5 shows the tail distribution behaviors in the case of EURO

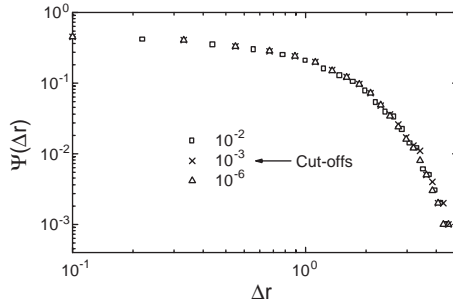


Fig. 4. The complementary cumulative distribution function of a simulated Gaussian stochastic process using different decimal digit precisions (cut-offs).

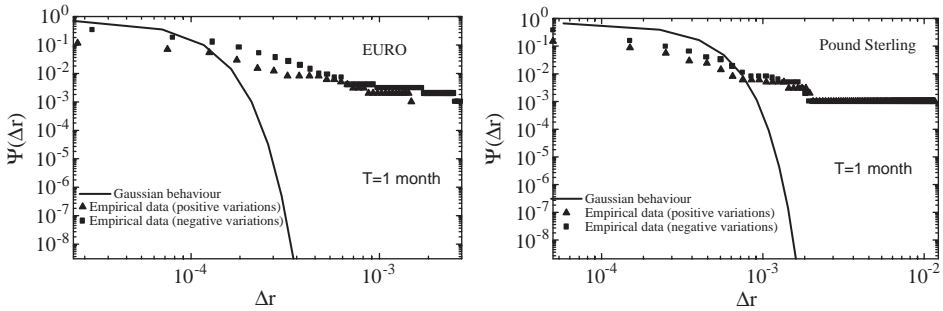


Fig. 5. The complementary cumulative distribution function of LIBOR increments for EURO (left) and Sterling Pound (right).  $T = 1$  month and  $\Delta t = 1$  day.

and Sterling Pound, respectively. In particular, in Fig. 5 (left side), we report the empirical results obtained estimating the probability density function of both positive and negative LIBOR increments with  $\Delta t = 1$  day and  $T = 1$  month. These empirical curves are slightly asymmetric and the negative variations are more probable than the positive one. In the same figure, these two curves are compared with the equivalent (i.e. with the same average and standard deviation) Gaussian complementary cumulative distribution. The non-Gaussian behavior is also evident from Fig. 5 (right side). In both Figs. 5, the empirical LIBOR data exhibit a *fat tail* or *leptokurtic* character, which is present for the other maturities as well. These observations indicate that the random behavior of  $\Delta r(T, t)$  is non-Gaussian and that using a Gaussian probability density function leads to underestimating the probability of large fluctuations. For a better understanding of the deviations from a pure Brownian motion and what kind of stochastic process we are dealing with, we analyze the power spectral density behavior. The power spectra [11],  $S(f)$ , for both  $r(1, t)$  and  $\Delta r(1, t)$  are reported in Fig. 6 for the EURO and in Fig. 7 for the Sterling Pound. For  $r$  the spectral density shows a power law behavior. A linear fit gives a slope value  $\alpha = -1.80 \pm 0.02$  and  $\alpha = -1.79 \pm 0.01$  for

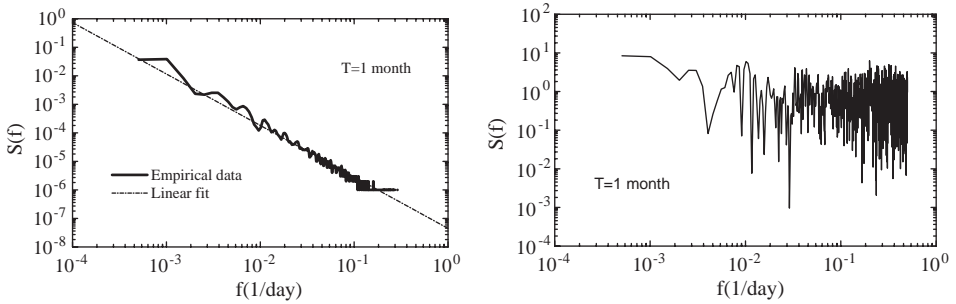


Fig. 6. Power spectrum of  $r(T, t)$  (left) and of  $\Delta r(T, t)$  (right) for the EURO.  $T = 1$  month and  $\Delta t = 1$  day.

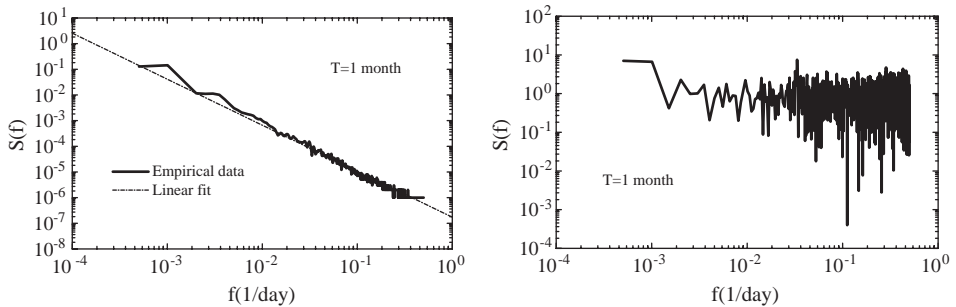


Fig. 7. Power spectrum of  $r(T, t)$  (left) and of  $\Delta r(T, t)$  (right) for the Sterling Pound.  $T = 1$  month and  $\Delta t = 1$  day.

EURO and Sterling Pound, respectively. A similar result holds for the other maturities. Therefore, we argue that the power spectrum analysis for  $r(T, t)$  indicates a stochastic process with spectral components decreasing as  $S(f) \sim f^\alpha$  [12]. The power spectrum for the increments (Figs. 6 and 7 (right)) is flat, typical of a white noise process. These results are also corroborated by a similar analysis performed on Eurodollars interest rates for a longer time period [13].

### 3. Discussion

In the previous section, we have shown that the daily increment of the interest rate series is non-Gaussian, non-Brownian and follows a leptokurtic distribution. The problem arises of how derivatives written on interest rates can be evaluated, given that the usual Gaussian white-noise assumption of many models is not satisfied. Indeed, a first partial answer is that the central limit theorem ensures that, after a sufficiently long time, the increment distribution will tend to a Gaussian distribution. However, if the time horizon of derivative evaluation is not appropriate, deviations from the Gaussian behavior may lead to a dramatic underestimate of large increments with a consequent

improper risk coverage as well as option price estimate. Although well studied in mainstream finance, this problem has received much attention in recent times, within the community of physicists working on financial problems. In particular, Bouchaud and Sornette [14] suggested the direct use of the historical probability measure, rather than the equivalent martingale measure for evaluating options. In this way, one gets an option price depending on the expected rate of returns, a consequence which is not fully desirable due to the subjective character of that rate. Assessing trends is a difficult task, as they depend on decision taken by Central Banks (as shown in Fig. 2) and are based on macroeconomic effects. Thus, martingale methods could prove more reliable. As early as 1977, some years after the seminal paper of Black and Scholes, Parkinson generalized their approach to option pricing and explicitly took into account leptokurtic distributions [15]. More recently, Boyarchenko and Levendorskii have studied the problem of option pricing in the presence of a specific distribution which seems to fit well the empirical data in many instances: the truncated Lévy distribution [16]. Wim Schoutens has recently published a book on Lévy processes in finance devoted to the extension of martingale methods to a large class of leptokurtic distributions [17].

The method can be described from a heuristic point of view. Let  $S(t)$  denote the stochastic process underlying a contingent claim  $C(S, t)$ ; thus,  $S(t)$  can be a price process, an interest rate process, etc. Let  $X(t) = \log S(t)$  be the corresponding logarithmic process. Let further  $p(x, t)$  be the probability density of finding the value  $x$  of the random variable  $X$  at time  $t$  (this is a conditional probability density in  $x$  with respect to suitable initial conditions; here,  $X(0) = 0$ ). This density defines the probability measure  $\mathbf{P}$ . It is possible to show that, if the following relation holds true:

$$E_{\mathbf{P}}\{\exp[aX(t)]\} = \int_{-\infty}^{+\infty} \exp(ax)p(x, t) dx = \exp[g(a)t], \quad (2)$$

where  $E_{\mathbf{P}}$  denotes the expectation operator with respect to  $\mathbf{P}$ ,  $a$  is a complex number and  $g(a)$  a complex function of  $a$ , then the process  $\zeta(t, a) = \exp[aX(t) - g(a)t]$  is a martingale with respect to the measure  $\mathbf{P}$ . Therefore, as a consequence of Girsanov's theorem, we can build an equivalent martingale measure  $\mathbf{Q}_{T,a}$  such that the Radon–Nikodym derivative  $d\mathbf{Q}_{T,a}/d\mathbf{P}$  is given by

$$\frac{d\mathbf{Q}_{T,a}}{d\mathbf{P}} = \zeta(T, a). \quad (3)$$

In order to price a contingent claim written on  $S(t) = S(0)\exp[X(t)]$ , we require that the discounted process  $S_d(t)$  is a martingale with respect to the measure  $\mathbf{Q}_{T,a}$ . This is equivalent to the requirement that the process  $\zeta(t, a)S_d(t)$  is a martingale with respect to  $\mathbf{P}$ . In this paper, we have shown that interest rates are fluctuating variables. However, just for the sake of simplicity, let us consider a fixed interest rate  $r$ . In this particular case, the martingale condition is equivalent to the following equation:

$$g(a + 1) - g(a) - r = 0. \quad (4)$$

In other words, if it is possible to determine a single value of  $a$  such that Eq. (4) is satisfied, the measure  $\mathbf{Q}_{T,a}$  exists and is unique. For instance, if  $S(t)$  is described by geometric Brownian motion with drift  $\mu$  and volatility  $\sigma$ , we have  $g(a) = \mu a - (i a \sigma)^2/2$

and there is a unique solution of Eq. (4):  $a = -(\mu + \sigma^2/2 - r)/\sigma^2$ . The reader is referred to Ref. [16] for the case of truncated Lévy processes. Then, under the requirement that  $C(S, t)$  be a martingale with respect to the measure  $\mathbf{Q}_{T,a}$ , one can find the price of the contingent claim. If  $C_d(S, t)$  is the discounted process, we have

$$C_d(S, t) = \xi^{-1}(t, a) E_{\mathbf{P}}[\xi(T, a) C_d(S, T) | F_t], \quad (5)$$

where  $F_t$  is the appropriate filtration.

The technique has been already applied to derivatives written on interest rates, here, we outline the generalization of a popular IR model: the Heath, Jarrow and Morton (HJM) model, following Eberlein and Reible [18]. Within this model, the zero-coupon bond price  $P(T, t)$  is given by

$$P(T, t) = P(T, 0) \exp \left[ \int_0^t r(s, s) ds \right] \frac{\exp \left[ \int_0^t \sigma(T, s) dW_s \right]}{E \left\{ \exp \left[ \int_0^t \sigma(T, s) dW_s \right] \right\}}, \quad (6)$$

where  $\sigma(T, t)$  is the bond volatility structure,  $W_t$  is the Wiener process,  $E$  is the expectation operator and  $r(s) := r(s, s)$ . Note that  $r(T, t)$  has been interpreted as the instantaneous forward rate  $f(T, t)$ . The Wiener process can be replaced by a leptokurtic Lévy process  $L_t$ , such that the expectation in the denominator is finite. In this case, it is possible to show that the discounted bond-price process is a martingale and that the martingale measure is unique [19]. Also, the European vanilla call option price on a bond maturing at time  $T$  can be obtained [18]. Along these lines, we believe, it is possible to develop a consistent option pricing theory taking into account the leptokurtic character of the empirical short- to mid-term interest rate distributions.

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