Local behavior of embedded constant mean curvature disks

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Abstract. Let Σ be a simply connected constant mean curvature surface embedded in $\mathbb{R}^3$. If the curvature is large at some point $x \in \Sigma$ then Σ contains a multi-valued graph near x. This is a generalization of a Colding and Minicozzi's result for minimal surfaces.

0. **Introduction.**

This paper is an announcement of the result described in the abstract and a complete proof will be given elsewhere.

Section 1 is a short overview of constant mean curvature surfaces. In sections 2 and 3 we state and motivate the result, describe what a multi-valued graph is and go over the hypotheses of the theorem/result. In section 4 we briefly outline the proof,

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†Lecture given on December 22, 2003, in the Department of Mathematics, at University of Bologna.

This paper is in definitive form and no version of it will be published elsewhere

showing how it is essentially a compactness argument applied on constant mean curvature surfaces containing a multi-valued graph. However, we see there are some problems which can not be solved with a standard approach. In section 5 we introduce the concept of δ-stability for constant mean curvature surfaces, generalization of (strong) stability. That will be a key ingredient in the proof. In sections 6 and 7 we take a closer look at the "not so standard" aspect of the proof.

1. **Constant mean curvature surfaces.**

Let $\Sigma \subset \mathbb{R}^3$ be a 2-dimensional smooth orientable surface (possibly with boundary) with unit normal N_Σ. Given a function ϕ in the space $C_0^\infty(\Sigma)$ of infinitely differentiable (i.e., smooth), compactly supported functions on Σ, consider the one-parameter variation

$$\Sigma_{t,\phi} = \{x + t\phi(x)N_\Sigma(x) | x \in \Sigma\}$$

and let $A(t)$ be the area functional,

$$A(t) = Area(\Sigma_{t,\phi}).$$

The so called first variation formula of area is the equation (integration is with respect to $d$area)

$$A'(0) = \int_\Sigma \phi H, \tag{1.1}$$

where H is the mean curvature of Σ. When H is constant the surface is said to be a *constant mean curvature* (CMC) surface (see [1]) and it is a critical point for the area functional restricted to those variations which preserve the *enclosed volume*, in other words ϕ must satisfy the condition,

$$\int_\Sigma \phi = 0.$$

In general, if Σ is given as graph of a function u it is,

$$H = div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{1.2}$$

Therefore, when H is constant u satisfies a quasi-linear differential equation. In the particular case where the mean curvature H is identically zero the surface Σ is said to be a *minimal* surface (see [2] and [3]). Concrete examples of constant mean curvature surfaces are spheres, cylinders and Delauney surfaces.

In general, let k_1, k_2 be the principal curvatures on Σ, the constant mean curvature H is the sum $k_1 + k_2$; $|A|^2 = k_1^2 + k_2^2$ is the norm of the second fundamental form squared. Since the Gaussian curvature K_Σ is equal to the product of the principal curvatures $k_1 k_2$, we have the Gauss equation, that is

$$H^2 = |A|^2 + 2K_\Sigma. \tag{1.3}$$

2. **Multi-valued graphs in CMC surfaces.**

The result is the following,

Theorem 2.1. *For each $N \in \mathbb{Z}_+$, $\omega > 1$ and $\varepsilon > 0$ there exist $H > 0$, $C(N, \omega, \varepsilon) > 0$ and $\bar{l} > 1$ so:*
Let $0 \in \Sigma \subset B_{\bar{l}}(0) \subset \mathbb{R}^3$ be an embedded and simply connected constant mean curvature equal to h surface (embedded CMC disk) such that $|h| \leq H$ and $\partial\Sigma \subset \partial B_{\bar{l}}(0)$. If

$$\sup_{\Sigma \cap B_{\bar{l}}(0)} |A|^2 \leq 4C^2 = 4|A|^2(0)$$

then there exists $\bar{R} < \frac{1}{\omega}$ and (after a rotation) an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega\bar{R}} \setminus D_{\bar{R}}$ (with gradient $\leq \varepsilon$ and $\text{dist}_\Sigma(0, \Sigma_g) \leq 4\bar{R}$).

This is a generalization of a Colding and Minicozzi's result for minimal surfaces, Theorem 0.2 of [4]. We are proving the same result for constant mean curvature surfaces. As a matter of fact the constant, $C(N, \omega)$, in Theorem **??** is essentially the same constant. For a minimal surface, Colding and Minicozzi were able to extend the multi-valued graph that forms locally, all the way up to the boundary. It is not known if the same can be done for CMC surfaces.

What is an N-valued graph?

Definition 2.2 (Multi-valued graph)**.** *Let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbb{C}\setminus 0$ with global coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbb{R}$. An N-valued graph of a function u on the annulus $D_s \setminus D_r$ is a single valued graph over $\{(\rho, \theta) | r \leq \rho \leq s, |\theta| \leq N\pi\}$.*

The surface to keep in mind is the helicoid. A parametrization of the helicoid that illustrates the existence of such an N-valued graph is the following

$$(s \sin t, s \cos t, t) \qquad \text{where } (s, t) \in \mathbb{R}^2$$

It is easy to see that it contains the N-valued graph ϕ defined by

$$\phi(\rho, \theta) = \theta \qquad \text{where } (\rho, \theta) \in \mathbb{R}^+ \setminus 0 \times [-N\pi, N\pi].$$

In fact the helicoid is a minimal surface.

3. **About the hypotheses and the rescaling argument.**

Thanks to the upper bound on the norm of the second fundamental form the surface is "uniformly locally flat" (we will see later in the paper what that means). Moreover, $\sup_\Sigma |A|^2 \leq 4C^2$ together with the Gauss equation (**??**) gives a lower bound for the Gaussian curvature, $K_\Sigma \geq \frac{-4C^2 + H^2}{2}$. This lower bound is important because by Bishop comparison theorem implies an upper bound on the area of the intrinsic balls (see [5]).

From Theorem **??** it follows by rescaling that the result is true even when the mean curvature is large but on a smaller ball. That is, surfaces with large constant mean curvature have tiny multy-valued graphs around the origin. The theorem is the following,

Theorem 3.1. *Given $N \in \mathbb{Z}_+$, $\omega > 1$ and $\varepsilon > 0$, there exist $C = C(N, \omega, \varepsilon) > 0$, $H > 0$ and $\bar{l} > 1$ so:*
Let $\Sigma \subset \mathbb{R}^3$ be an embedded simply connected constant mean curvature equal to h surface. If $|h| < \frac{H}{r_0}$ and

$$\sup_{\Sigma \cap B_{r_0 \bar{l}}(0)} |A|^2 \leq 4C^2 r_0^{-2} = 4|A|^2(0)$$

for some $r_0 > 0$, then Σ (after a rotation) contains an N-valued graph over $D_{\omega \bar{R}} \backslash D_{\bar{R}}$ where $\bar{R} < \frac{r_0}{\omega}$ (with gradient $\leq \varepsilon$ and $dist_\Sigma(0, \Sigma_g) \leq 4\bar{R}$).

Note that a rescaling argument applied to a minimal surface gives another minimal surface. To our task, it is necessary that H differs from 0.

4. **Proof by contradiction.**

The proof is a proof by contradiction and fundamentally a compactness argument. We show that the statement

> For each $h > 0$ there exists an embedded and simply connected constant mean curvature surface Σ that does not contain an N-valued graph $\Sigma_g \subset \Sigma$ over $D_{\omega \bar{R}} \backslash D_{\bar{R}}$ for any $\bar{R} < \frac{1}{\omega}$ but such that satisfies the hypotheses of **??**

is false. The idea is the following, let us assume that such a Σ exists for any $h > 0$ then we will be able to construct a sequence of Σ_n, not containing an N-valued graph, that converges in the C^2 convergence (see [6]) to a minimal surface Σ_∞ which satisfy the hypotheses of Colding-Minicozzi and therefore contains an N-valued graph. Clearly, it will be $H(\Sigma_n) \to 0$ as n approaches infinity.
The C^2 convergence together with the fact that the limit of the sequence contains an N-valued graph will imply that one of the element of the sequence contains an N-valued graph, giving the contradiction. However, there are some nontrivial aspects in the proof.

Roughly speaking the C^2 convergence means that Σ_∞ can be covered by a finite number of balls $B_r(x_i)$, $x_i \in \Sigma_\infty$, such that in each ball $\Sigma_n \cap B_r(x_i)$ looks like a bunch of graphs u_n^j over $\mathrm{T}_{x_i}\Sigma_\infty$ and u_n^j converges C^2 to a certain graph u_∞^j. Given that the sectional curvatures are bounded it is not too hard to find a lower bound for the radius of the small ball where everything must look graphical, this is what we called "uniformly locally flat". Once everything is graphical we will be able to extract, using Arzela-Ascoli, a subsequence u_n^j that converges uniformly to a graph u_∞^j and it satisfies

$$\frac{1}{n} = div \left(\frac{\nabla u_n^j}{\sqrt{1 + |\nabla u_n^j|^2}} \right).$$

Using Schauder theory (see [7]) we can prove that u_n^j converges C^2 to u_∞^j and therefore the latter is a minimal graph. This part is the more or less standard argument.

Unfortunately, as it has been said, there is a problem. What has just been described happens locally, on small balls, and it is not enough to prove that the limit is an embedded and simply connected minimal disk (for instance it could be a lamination or not simply connected). In short, the standard approach by itself does not produce the desired simply connected and embedded minimal surface. For the global result it is necessary to prove that the number of graphs in each ball is uniformly bounded and that will require more machinery.

From now on, our goal is to find that upper bound on the number of graphs. In order to do that, we must first introduce the concept of δ-stability.

5. δ-stability.

Let A be the area functional described in section **??**, we showed that $A'(0) = \int_\Sigma \phi H$. A computation shows that if Σ is a CMC surface then

$$A''(0) = -\int_\Sigma \phi L_\Sigma \phi, \qquad \text{where } L_\Sigma\phi = \Delta_\Sigma\phi + |A|^2\phi \tag{5.1}$$

is the second variational operator. Here Δ_Σ is the intrinsic Laplacian on Σ. A CMC surface Σ is said to be (strongly) stable if

$$A''(0) \geq 0, \qquad \text{for all } \phi \in C_0^\infty(\Sigma). \tag{5.2}$$

Applying Stokes' theorem to **??** shows that Σ is stable if and only if

$$\int_\Sigma |A|^2\phi^2 \leq \int_\Sigma |\nabla\phi|^2, \qquad \text{for all } \phi \in C_0^\infty(\Sigma)$$

and that allows us to define δ-stability, namely Σ is said to be δ-stable if

$$(1-\delta)\int_\Sigma |A|^2\phi^2 \leq \int_\Sigma |\nabla\phi|^2, \qquad \text{for all } \phi \in C_0^\infty(\Sigma). \tag{5.3}$$

The next two lemmas will provide a criteria to find δ-stable domains in CMC surfaces.

Lemma 5.1. *There exists $\delta > 0$ so: Let Σ be a CMC surface and u a positive solution of the CMC graph equation over Σ that is $\Sigma^u := \{x + u(x)N_\Sigma(x) | x \in \Sigma\}$ is a CMC surface. If $H_{\Sigma^u} = H_\Sigma$, $< N_{\Sigma^u}, N_\Sigma >\geq 0$ and $|u||A| + |\nabla u| \leq \delta$, then Σ is $\frac{1}{2}$-stable.*

Proof. Roughly, if Σ^u is a graph over Σ it satisfies

$$H_{\Sigma^u} = H_\Sigma + \frac{1}{2}(\triangle u + u|A|^2) + o(|u|, |\nabla u|).$$

Therefore, being $H_{\Sigma^u} = H_\Sigma$, Lu is closer and closer to zero as $|u|, |\nabla u|$ become smaller and smaller. The existence of a positive solution of $Lu = 0$ would imply $A''(0) \geq 0$ for all $\phi \in C_0^\infty(\Sigma)$. In this case there exists a positive function u which is "almost" a solution, therefore $A''(0)$ will be "almost" non-negative for all $\phi \in C_0^\infty(\Sigma)$. □

When does it happen that a piece of a CMC surface is graphical over another piece? It happens when they are close enough (with respect to the Euclidean distance).

From now on Σ will be a CMC surface as in Theorem **??** and, without loss of generality, we may assume $|H(\Sigma)| < 1$, as a matter of fact as small as we want. Let $x, y \in \Sigma$ and $r \geq 0$, $d_\Sigma(x, y)$ is the intrinsic distance (geodesic distance) between x and y while $\mathcal{B}_r(x)$ is the intrinsic ball centered at x of radius r.

Lemma 5.2. *There exists ε_1 so: Let $x, y \in \Sigma$ such that $|x-y| \leq \varepsilon \leq \varepsilon_1$, $d_\Sigma(x,y) \geq 2\varepsilon$ and $< n(x), n(y) >> 0$ then there exists $t > \varepsilon_1$ such that $\mathcal{B}_t(y)$ contains a graph $\{z + u(z)n(z)\}$ over a domain containing $\mathcal{B}_{\frac{t}{4}}(x)$ and $|\nabla u| + |u| \leq \varepsilon C_0$.*

Lemma **??** and Lemma **??** together produce δ-stable domains. Note that the closer x and y are the smaller δ is. Throughout the next two sections we will see why we have introduced δ-stability.

6. The non-existence of large $\frac{1}{2}$-stable domains.

In order to continue with this proof by contradiction, we need the following,

Proposition 6.1. *There exists $l_1 > 0$ so: For any $l \geq l_1$ and $x \in \Sigma$ if $\mathcal{B}_l(x)$ is $\frac{1}{2}$-stable then it is not contained in $\Sigma \cap B_1(0)$.*

Namely, if a stable intrinsic ball is contained in $\Sigma \cap B_1(0)$ then it cannot be too large. Why? The reason lies in the following result of Zhang Sirong, that is Theorem 0.1 in [8],

Theorem 6.2 (Zhang Sirong)**.** *There exists a C such that given any $l > 0$ there exists an $h > 0$ so: if $\mathcal{B}_l(0)$ is a "constant mean curvature equal to h", $\frac{1}{2}$-stable intrinsic ball with trivial normal bundle then $\sup_{\mathcal{B}_{\frac{l}{2}}(0)} |A|^2 \leq \frac{C}{l^2}$.*

This result can be thought as a generalization of [9] and [10]. From this result it follows that if l is big enough and $\mathcal{B}_l(x) \subset \Sigma$ is $\frac{1}{2}$-stable then $\mathcal{B}_{\frac{l}{2}}(x)$ is almost flat. That forces the intrinsic disk to leave the unit ball giving the contradiction. In Theorem **??** we take $\bar{l} > l_1 + 1$. We prove that the number of graphs is uniformly bounded if we consider the connected component of $\Sigma \cap B_1(0)$ containing zero.

7. Uniform bound on the number of pieces.

And now, why is it that the non-existence of large $\frac{1}{2}$-stable intrinsic balls implies a bound on the number of graphs? The main idea is that when two points in Σ are close to each other (Euclidian distance) a little neighborhood of each point is $\frac{1}{2}$-stable. Decreasing the Euclidian distance and increasing the intrinsic distance between the two points gives a large $\frac{1}{2}$-stable domain. This happens because in a small ball the difference of the two graphs satisfies an ordinary elliptic P.D.E. and that means, via the Harnack inequality, that if two points are close the two graphs are close all the way up to the boundary and we can repeat this argument (see [6]). and Figure **??**). This "being close" argument might stop when the two graphs close up, but this does not happen if the intrinsic distance between the two starting point is big. Hence we will have large pieces of the surface which are graphical over each other and therefore $\frac{1}{2}$-stable by Lemma **??** and Lemma **??**. In sum, the existence of two points with big intrinsic distance but small euclidian distance produces a large $\frac{1}{2}$-stable domain.

In fact, Bishop inequality and a lower bound on the area of each piece imply that the more graphs there are in a small ball the larger the intrinsic distance becomes between two points and that finishes the proof by contradiction. In other words, there can not be too many graphs close to each other (that is in a small Euclidean ball) otherwise in the same ball (that is close to each other with respect to the Euclidean distance) there will be two points which are far away with respect to the intrinsic distance and that produces a large $\frac{1}{2}$-stable intrinsic ball. This completes the sketch of the proof. Full details will appear elsewhere.

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Received January 21, 2004.