DUALITY IN CONVEX PROBLEMS OF BOLZA OVER FUNCTIONS OF BOUNDED VARIATION

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Abstract. This paper studies fully convex problems of Bolza in the conjugate duality framework of Rockafellar. We parameterize the problem by a general Borel measure which has a direct interpretation in certain problems of financial economics. We derive a dual representation for the optimal value function in terms of continuous dual trajectories and we give conditions for the existence of solutions. Combined with well-known results on problems of Bolza over absolutely continuous trajectories, we obtain optimality conditions in terms of extended Hamiltonian equations.

Key words. Calculus of variations, convex duality, Hamiltonian conditions, impulse control

AMS subject classifications. 49N15, 49N25, 46N10, 49J24, 49K24, 49J53

1. Introduction. Problems of Bolza were introduced a century ago as a general class of problems in the calculus of variations [2]. In a sequence of papers in the 70's, Rockafellar extended the theory to possibly nonsmooth and extended real-valued convex Lagrangian's and end-point penalties. This extension allows for treating convex problems of optimal control under the same framework. Rockafellar's original formulation was over absolutely continuous arcs [12], but soon after, he generalized it to arcs of bounded variation [17, 20]. In [19] the theory was extended to problems over absolutely continuous arcs but with Lagrangians that are only required to be convex in the velocity variable. We refer the reader to [9, Section 6.5] for a general account of the history of optimal control and the calculus of variations.

The present paper extends the theory of fully[1] convex problems of Bolza in two directions. First, we relax the continuity assumptions on the domain of the Hamiltonian using recent results of Perkkiö [10] on conjugates of convex integral functionals. This extension allows for discontinuous state constraints both in the primal and the dual. Second, we parameterize the primal problem with a general Borel measure that perturbs the derivative rather than the state. We derive a dual representation for the optimal value function for all values of the parameter, not just at the origin. This is of interest in financial economics where the parameter may represent e.g. endowments and/or liabilities of an economic agent. Parameterization in terms of Borel measures allows for discontinuities in cumulative endowments/liabilities.

Given $T > 0$, let X be the space of left-continuous functions $x : \mathbb{R}_+ \to \mathbb{R}^d$ of bounded variation such that x is constant after T. The space X may be identified with $\mathbb{R}^d \times M$ where M is the space of $\mathbb{R}^d$-valued Radon measures on $[0, T]$. Indeed, given $x \in X$ there is a unique $\mathbb{R}^d$-valued Radon measure Dx on $[0, T]$ such that $x_t = x_0 + Dx([0, t))$ for all $t \in [0, T]$ and $x_t = x_0 + Dx([0, T])$ for $t > T$; see e.g. [5, Theorem 3.29]. The value of $x \in X$ on (T, ∞) will be denoted by x_{T+}.

Given an atomless strictly positive[2] Radon measure μ on $[0, T]$, a proper convex normal integrand $K : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \overline{\mathbb{R}}$ and a proper convex lower semicontinuous

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[1]A problem of Bolza is said to be *fully convex* if the Lagrangian is jointly convex in the state and the velocity variables.

[2]A measure μ is strictly positive if $\mu(O) > 0$ for every nonempty open O.

function $k : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}}$, we will study the parametric optimization problem

$$\text{minimize} \quad J_K(x, Dx + u) + k(x_0, x_{T+}) \quad \text{over } x \in X, \qquad (P_u)$$

where $u \in M$ and $J_K : X \times M \to \overline{\mathbb{R}}$ is given by

$$J_K(x, \theta) = \int K_t(x_t, (d\theta^a/d\mu)_t) d\mu_t + \int K_t^\infty(0, (d\theta^s/d|\theta^s|)_t) d|\theta^s|_t.$$

Here θ^a and θ^s denote the absolutely continuous and the singular parts of θ with respect to μ, $|\theta^s|$ denotes the total variation of θ^s and K_t^∞ the recession function of K_t; see the appendix. Throughout this paper, we define the integral of a measurable function as $+\infty$ unless its positive part is integrable. Similarly, the sum of finite collection of extended real numbers is defined as $+\infty$ if any of the terms equals $+\infty$. It follows that J_K as well as the objective in (P_u) are well-defined extended real-valued functions on $X \times M$.

Note that (P_u) can be written as

$$\text{minimize} \quad J_K(\tilde{x} + \tilde{u}, D\tilde{x}) + k(\tilde{x}_0, \tilde{x}_{T+} + \tilde{u}_{T+}) \quad \text{over } \tilde{x} \in X, \qquad (P_{\tilde{u}})$$

where $\tilde{u} \in X$ with $\tilde{u}_0 = 0$, $D\tilde{u} = u$ and $\tilde{x} = x - \tilde{u}$. This is analogous to [15] and [17] where the perturbations were essentially bounded and continuous functions, respectively. However, the equivalence of problems (P_u) and $(P_{\tilde{u}})$ requires that the perturbation $\tilde{u}$ is in the space X of functions of bounded variation. When $u = 0$ and the minimization is restricted to the space AC of absolutely continuous functions with respect to μ, problem (P_u) can be written in the more familiar form

$$\text{minimize} \quad \int K_t(x_t, \dot{x}_t) d\mu_t + k(x_0, x_T) \quad \text{over } x \in AC, \qquad (P_{AC})$$

where $\dot{x}$ denotes the Radon–Nikodym derivative of Dx with respect to μ. Such problems have been extensively studied since [12] (often in the case where μ is the Lebesque measure). Allowing K and k to be extended real-valued, various more traditional problems in calculus of variations and optimal control can be written in the above form; see [12, 22] for details. Problems of the form (P_u) with $u = 0$ extend (P_{AC}) by allowing for discontinuous trajectories. In the context of optimal control, discontinuous trajectories correspond to impulse control. Rockafellar [20] developed a duality theory for problems of the form (P_u) with $u = 0$ in the case where[3] $k = \delta_{\{(a,b)\}}$.

Much as in [12, 15, 17, 20, 22], we will study (P_u) by embedding it in the general conjugate duality framework of [18]. We give sufficient conditions under which the infimum in (P_u) is attained for every u and the *value function*

$$\varphi(u) = \inf_{x \in X} \{J_K(x, Dx + u) + k(x_0, x_{T+})\}$$

of (P_u) has the dual representation

$$\varphi(u) = \sup_{y \in C \cap X} \{\langle u, y \rangle - J_{\tilde{K}}(y, Dy) - \tilde{k}(y_0, y_T)\},$$

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[3]Throughout this paper, δ_C denotes the *indicator function* of a set C, i.e. $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise.

where C denotes the space of continuous functions on $[0, T]$ and $\tilde{K}$ and $\tilde{k}$ are given in terms of the conjugates of K_t and k as

$$\tilde{K}_t(y, v) = K_t^*(v, y) = \sup_{x, u \in \mathbb{R}^d} \{x \cdot v + u \cdot y - K_t(x, u)\},$$

$$\tilde{k}(\tilde{a}, \tilde{b}) = k^*(\tilde{a}, -\tilde{b}) = \sup_{a, b \in \mathbb{R}^d} \{a \cdot \tilde{a} - b \cdot \tilde{b} - k(a, b)\}.$$

This paper relaxes the continuity assumptions made in [17, 20] on the domain of the associated Hamiltonian

$$H_t(x, y) = \inf_{u \in \mathbb{R}^d} \{K_t(x, u) - u \cdot y\}.$$

This turns out to have significant consequences in certain problems of financial economics where the continuity relates to the behavior of financial markets; see e.g. [7, Section 3.6.6]. We also show that our relaxed continuity assumptions allow for optimality conditions in terms of an extended Hamiltonian equation. Combined with the results of [15] on problems of Bolza over absolutely continuous arcs, we obtain necessary and sufficient conditions of optimality in (P_u) with $u = 0$.

2. Conjugate duality. A set-valued mapping $S : [0, T] \to \mathbb{R}^d$ is *measurable* if the preimage $S^{-1}(A) := \{t \in [0, T] \mid S_t \cap A \neq \emptyset\}$ of every open $A \subset \mathbb{R}^d$ is measurable. An extended real-valued function h on $\mathbb{R}^n \times [0, T]$ is a *proper convex normal integrand* if the set-valued mapping $t \mapsto \operatorname{epi} h_t(\cdot)$ is closed convex-valued and measurable, and $h_t(\cdot)$ is proper for all t. By [23, Corollary 14.34], this implies that h is an $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}([0, T])$-measurable function on $\mathbb{R}^d \times [0, T]$, so $t \mapsto h_t(x_t)$ is an $\mathcal{F}$-measurable extended real-valued function and

$$I_h(x) = \int h_t(x_t) d\mu_t$$

is well-defined for every $\mathcal{B}([0, T])$-measurable $x : [0, T] \to \mathbb{R}^d$. For every t, the recession function $x \mapsto h_t^\infty(x)$ is a closed and sublinear convex function; see the appendix. By [23, Exercise 14.54], h^∞ is a convex normal integrand.

We will study (P_u) in the conjugate duality framework of Rockafellar [18]. To this end, write it as

$$\text{minimize} \quad f(x, u) \quad \text{over} \quad x \in X,$$

where

$$f(x, u) = J_K(x, Dx + u) + k(x_0, x_{T+}).$$

Since K is a convex normal integrand, we see that f is well-defined on $X \times M$. The convexity of K and k implies the convexity of f on $X \times M$, which in turn implies that the optimal value function

$$\varphi(u) = \inf_{x \in X} f(x, u)$$

is convex on M; see e.g. [18, Theorem 1].

The bilinear form

$$\langle u, y \rangle := \int y_t \, du_t$$

puts M in separating duality with the space C of $\mathbb{R}^d$-valued continuous functions on $[0, T]$. Indeed, if we equip C with the supremum norm, the Riesz representation theorem says that M may be identified with the Banach dual of C through the representation $y \mapsto \langle u, y \rangle$; see e.g. [5, Theorem 7.17]. Similarly the bilinear form

$$\langle x, v \rangle := x_0 \cdot v_{-1} + \int v_t dx_t$$

puts X in separating duality with the space $V := \mathbb{R}^d \times C$ of continuous functions on $\{-1\} \cup [0, T]$. The weak topology on X will be denoted by $\sigma(X, V)$. We will make repeated use of the integration by parts formula

$$\int v_t dx_t = x_{T+} \cdot v_T - x_0 \cdot v_0 - \int x_t dv_t,$$

which is valid for any $x \in X$ and any $v \in C$ of bounded variation; this can be deduced, e.g., from [3, Theorem VI.90] or Folland [5, Theorem 3.36].

The *Lagrangian* associated with (P_u) is the convex-concave function on $X \times Y$ defined by

$$L(x, y) = \inf_{u \in M} \{ f(x, u) - \langle u, y \rangle \}.$$

The conjugate of φ can be expressed as

$$\begin{aligned} \varphi^*(y) &= \sup_{u \in M} \{ \langle u, y \rangle - \varphi(u) \} \\ &= \sup_{u \in M, x \in X} \{ \langle u, y \rangle - f(x, u) \} \\ &= -g(y), \end{aligned}$$

where

$$g(y) := \inf_{x \in X} L(x, y).$$

If φ is closed (i.e. either proper and lower semicontinuous or a constant function), the biconjugate theorem (see e.g. [18, Theorem 5]) gives the dual representation

$$\varphi(u) = \sup_{y \in C} \{ \langle u, y \rangle + g(y) \}.$$

Clearly, $g(y) = -f^*(0, y)$, where f^* is the conjugate of f. We always have

$$f^*(v, y) = \sup_{x \in X} \{ \langle x, v \rangle - L(x, y) \},$$

and, as soon as f is closed in u,

$$f(x, u) = \sup_{y \in Y} \{ \langle u, y \rangle + L(x, y) \}.$$

Our first goal is to derive a more concrete expression for L. This will involve the *Hamiltonian* $H : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \overline{\mathbb{R}}$ defined by

$$H_t(x, y) = \inf_{u \in \mathbb{R}^d} \{ K_t(x, u) - u \cdot y \}.$$

The Hamiltonian is convex in x and concave in y. The function $t \mapsto H_t(x_t, y_t)$ is measurable for every $x \in X$ and $y \in C$. Indeed, by [23, Proposition 14.45 and Theorem 14.50], $(y, t) \mapsto -H_t(x_t, y)$ is a normal integrand for every $x \in X$, so the measurability follows from that of y. The integral functional

$$I_H(x, y) = \int H_t(x_t, y_t) d\mu$$

is thus well defined on $X \times C$. Again, we set $I_H(x, y) = +\infty$ unless the positive part of the integrand is integrable. The function I_H is convex in x and concave in y. The set $\operatorname{dom} H_t := \operatorname{dom}_1 H_t \times \operatorname{dom}_2 H_t$ where

$$\operatorname{dom}_1 H_t = \{x \in \mathbb{R}^d \mid H_t(x, y) < +\infty \; \forall y \in \mathbb{R}^d\},$$
$$\operatorname{dom}_2 H_t = \{y \in \mathbb{R}^d \mid H_t(x, y) > -\infty \; \forall x \in \mathbb{R}^d\}$$

is known as the *domain* of H_t. This set is nonempty for all t, because $K_t(\cdot, \cdot)$ is proper [13, Theorem 34.2]. The domain of I_H is defined similarly.

Recall that a set-valued mapping S from $[0, T]$ to $\mathbb{R}^d$ is *inner semicontinuous* (isc) if the preimage of every open set is open; see [23, Chapter 5]. Following [10], we define

$$(\mu\text{-}\liminf S)_t = \{y \in \mathbb{R}^d \mid \forall A \in \mathcal{H}_y^o \; \exists O \in \mathcal{H}_t : \; \mu(S^{-1}(A) \cap O) = \mu(O)\},$$

where $\mathcal{H}_y^o$ is the collection of open neighborhoods of $y \in \mathbb{R}^d$ and $\mathcal{H}_t$ is the collection of all neighborhoods of $t \in [0, T]$. A mapping S is *outer μ-regular* if $(\mu\text{-}\liminf S)_t \subseteq \operatorname{cl} S_t$. If S is outer μ-regular, then we have that $y_t \in \operatorname{cl} S_t$ for all t whenever $y \in C$ is such that $y_t \in \operatorname{cl} S_t$ μ-almost everywhere; see [10, Theorem 1]. By [10, Theorem 2], the converse implication holds when S is isc convex-valued with $\operatorname{int} S_t \neq \emptyset$ for all t. Outer μ-regularity together with inner semicontinuity generalize the *full lower semicontinuity* condition used in [17, 20]. We denote the relative interior of a set A by $\operatorname{rint} A$.

THEOREM 2.1. *Assume that*

1. $t \mapsto \operatorname{dom}_2 H_t$ is isc and outer μ-regular,

2. $\{y \in C \mid y_t \in \operatorname{rint} \operatorname{dom}_2 H_t \; \forall t\} \subset \operatorname{dom}_2 I_H$,

3. For every $x \in \operatorname{dom}_1 I_H$ there exist $\alpha \in L^1$ and $w \in L^1$ with

$$H_t(x_t, y) \leq -y \cdot w_t + \alpha_t \; \mu\text{-}a.e.,$$

Then f is closed in u, and the Lagrangian can be expressed as

$$L(x, y) = \begin{cases} I_H(x, y) + \langle Dx, y \rangle + k(x_0, x_{T+}) & \text{if } x \in \operatorname{dom}_1 I_H, \\ +\infty & \text{otherwise.} \end{cases}$$

In particular f is proper whenever $\operatorname{dom}_1 I_H \neq \emptyset$. Moreover, if

4. For every $y \in \operatorname{dom}_2 I_H$ there exist $\beta \in L^1$ and $z \in L^1$ with

$$H_t(x, y_t) \geq x \cdot z_t - \beta_t \; \mu\text{-}a.e.,$$

then $f : X \times U \to \overline{\overline{\mathbb{R}}}$ is closed.

Proof. By definition,

$$L(x, y) = \inf_{u \in M} \{J_K(x, Dx + u) + k(x_0, x_{T+}) - \langle u, y \rangle\}$$
$$= \inf_{u \in M} \{J_K(x, u) - \langle u, y \rangle\} + \langle Dx, y \rangle + k(x_0, x_{T+}).$$

Assume first that $x \notin \mathrm{dom}_1 I_H$ so that there is a $\tilde{y} \in C$ such that $I_H(x, \tilde{y}) = +\infty$. Since

$$K_t(x, u) \geq H_t(x, \tilde{y}_t) + u \cdot \tilde{y}_t \quad \forall x, u \in \mathbb{R}^d,$$

we have $J_K(x, Dx + u) = +\infty$ for all $u \in M$, so $L(x, y) = +\infty$ and the given expression for the Lagrangian is valid.

Assume now that $x \in \mathrm{dom}_1 I_H$. We may redefine x_0 and x_{T+} so that $k(x_0, x_{T+}) < +\infty$. To justify the expression for L and that f is proper and closed in u, it suffices to show that the functions $J_K(x, \cdot)$ and $-I_H(x, \cdot)$ are proper and conjugate to each other. By condition 3, there is a Borel μ-null set N with $\{t \mid x_t \notin \mathrm{dom}_1 H_t\} \subseteq N$. Since $-H_t(x_t, \cdot)$ is a normal integrand and $t \mapsto \mathrm{dom}_2 H_t$ is measurable (see [23, Exercise 14.9]), it follows that

$$h_t(y) = \begin{cases} -H_t(x_t, y) & \text{if } t \notin N, \\ \delta_{\mathrm{cl\,dom}_2 H_t}(y) & \text{if } t \in N, \end{cases}$$

is a normal integrand and $I_h = -I_H(x, \cdot)$. Clearly, $h_t^*(u) = K_t(x_t, u)$ for all $t \notin N$. Since, by [13, Theorem 34.2], $\mathrm{dom}_2 H_t = \{y \mid \exists v : (v, y) \in \mathrm{dom} K_t^*\}$ we have, by [13, Theorem 13.3], that $(h_t^*)^\infty(u) = K_t^\infty(0, u)$ for all t and thus,

$$J_K(x, \theta) = \int h_t^*((d\theta/d\mu)_t) d\mu_t + \int (h_t^*)^\infty((d\theta^s/d|\theta^s|)_t) d|\theta^s|_t.$$

By [13, Theorem 34.3], $\mathrm{cl\,dom}\, h_t = \mathrm{cl\,dom}_2 H_t$ for all t, so $t \mapsto \mathrm{dom}\, h_t$ is isc and outer μ-regular. The mapping $t \mapsto \mathrm{rint\,dom}\, h_t$ is also isc and convex-valued, so, by [8, Theorem 3.1''], there is a $\bar{y} \in C$ with $\bar{y}_t \in \mathrm{rint\,dom}\, h_t$ for all t. Thus, condition 2 implies that $\mathrm{dom}\, I_h \neq \emptyset$. By condition 3, $K_t(x_t, w_t) \leq \alpha_t$ μ-a.e., so, by choosing $d\theta/d\mu = w$ and $\theta^s = 0$, we see that $J_K(x, \theta) < \infty$. Hence all the assumptions of [10, Theorem 3] are met, so $J_K(x, \cdot)$ is a proper closed convex function with the conjugate $-I_H(x, \cdot)$.

Assume now that condition 4 holds. It remains to show that f is lower semicontinuous on $X \times M$. By the above,

$$f(x, u) = \sup_{y \in C} \{\langle Dx + u, y \rangle + I_H(x, y)\} + k(x_0, x_{T+}).$$

We start by showing that the supremum can be restricted to $y \in C$ with $y_t \in \mathrm{rint\,dom}_2 H_t$ for all t. If x_t does not belong to $\mathrm{dom}_1 H_t$ almost everywhere, then, by [13, Theorem 34.3], $H_t(x_t, \bar{y}_t) = +\infty$ on a set of positive measure, so $I_H(x, \bar{y}) = +\infty$. On the other hand, if $x_t \in \mathrm{dom}_1 H_t$ μ-a.e. and if $I_H(x, y) > -\infty$, then, by [13, Theorem 34.3], $y_t \in \mathrm{cl\,dom}_2 H_t$ μ-a.e., so outer μ-regularity implies that $y_t \in \mathrm{cl\,dom}_2 H_t$ for all t. Defining $y^\nu = \frac{1}{\nu}\bar{y} + (1 - \frac{1}{\nu})y$, we have $y_t^\nu \in \mathrm{rint\,dom}_2 H_t$ for all ν and, by concavity, $I_H(x, y^\nu) \geq \frac{1}{\nu} I_H(x, \bar{y}) + (1 - \frac{1}{\nu}) I_H(x, y)$.

When $y \in C$ with $y_t \in \mathrm{rint\,dom}_2 H_t$ for all t, the function $H_t(\cdot, y_t)$ is lsc for all t by [13, Theorem 34.2], so, by [23, Proposition 14.47], $(x, t) \mapsto H_t(x, y_t)$ is a normal integrand. By [21, Theorem 3C], condition 4 implies that $I_H(\cdot, y)$ is lsc on $(L^\infty, \sigma(L^\infty, L^1))$.

To finish the proof, it suffices to show that the embedding $(X, \sigma(X, V)) \hookrightarrow (L^\infty, \sigma(L^\infty, L^1))$ is continuous. Let $\bar{w} \in L^1$, $\varepsilon > 0$ and $\bar{z}_t = \int_{[0,t]} \bar{w}_t d\mu_t$. Integra-

tion by parts gives

$$\{x \in X \mid |\int x_t \cdot \bar{w}_t d\mu_t| < \varepsilon\} = \{x \in X \mid |\int x_t d\bar{z}_t| < \varepsilon\}$$

$$= \{x \in X \mid |\bar{z}_T \cdot x_{T+} - \int \bar{z}_t dx_t| < \varepsilon\}$$

$$= \{x \in X \mid |\bar{z}_T \cdot x_0 + \int (\bar{z}_T - \bar{z}_t) dx_t| < \varepsilon\}.$$

Thus, since $\bar{z}$ is continuous, $\sigma(L^\infty, L^1)$-open sets are $\sigma(X, V)$-open. $\square$

Conditions 1 and 2 in Theorem 2.1 are needed to apply the results of [10] on convex conjugates of integral functionals. If $t \mapsto \mathrm{dom}_2\, H_t$ is isc with $\mathrm{int}\, \mathrm{dom}_2\, H_t \neq \emptyset$ for all t, then, under conditions 2 and 3, outer μ-regularity of $t \mapsto \mathrm{dom}_2\, H_t$ is necessary for the conclusion of the theorem to hold. This follows by applying [10, Theorem 3] to I_h in the proof above.

We will next derive a more explicit expression for the conjugate of f. By [23, Theorem 14.50], the function

$$\tilde{K}_t(y, v) = K_t^*(v, y) = \sup_{x, u \in \mathbb{R}^d} \{x \cdot v + u \cdot y - K_t(x, u)\}.$$

is a proper convex normal integrand. The functional

$$J_{\tilde{K}}(y, \theta) = \int \tilde{K}_t(y_t, (d\theta^a/d\mu)_t) d\mu + \int \tilde{K}_t^\infty(0, (d\theta^s/d|\theta^s|)_t) d|\theta^s|$$

is thus well defined on $C \times M$. We also define

$$\tilde{k}(\tilde{a}, \tilde{b}) = k^*(\tilde{a}, -\tilde{b}) = \sup_{a, b \in \mathbb{R}^d} \{a \cdot \tilde{a} - b \cdot \tilde{b} - k(a, b)\}.$$

A function $x : \mathbb{R} \to \mathbb{R}^d$ is left-continuous if and only if it is continuous with respect to the topology τ_l generated by sets of the form $\{(s, t] \mid s < t\}$. We will say that a set-valued mapping S is *left-inner semicontinuous* (or left-isc) if it is isc with respect to τ_l. Similarly, S is said to be *left-outer μ-regular* if it is outer μ-regular with respect to τ_l. By [10, Theorem 2], a left-isc convex-valued mapping S with $\mathrm{int}\, \mathrm{dom}\, S_t \neq \emptyset$ for all t is left-outer μ-regular if and only if $x_t \in \mathrm{cl}\, S_t$ for all t whenever x is a left-continuous function with $x_t \in \mathrm{cl}\, S_t$ μ-almost everywhere. We denote by $\mathbb{B}(x, r)$ the open ball with center $x \in \mathbb{R}^d$ and radius $r > 0$.

THEOREM 2.2. *In addition to hypotheses of Theorem 2.1, assume that*
1. $t \mapsto \mathrm{dom}_1\, H_t$ *is left-isc and left-outer μ-regular,*
2. $\emptyset \neq \{x \in X \mid \exists r > 0 : \mathbb{B}(x_t, r) \subset \mathrm{dom}_1\, H_t \,\forall t\} \subset \mathrm{dom}_1\, I_H.$

Then

$$f^*(v, y) = \begin{cases} J_{\tilde{K}}(y, D\tilde{v}) + \tilde{k}(v_{-1} + \tilde{v}_0, \tilde{v}_T) & \text{if } \tilde{v} \in C \cap X, \\ +\infty & \text{otherwise}, \end{cases}$$

where $\tilde{v}_t = y_t - v_t$ for $t \in [0, T]$.

Proof. By Theorem 2.1,

$$f^*(v, y) = \sup_{x \in X} \{\langle x, v \rangle - L(x, y)\}$$

$$= \sup_{x \in X} \{\langle x, v \rangle - I_H(x, y) - \langle Dx, y \rangle - k(x_0, x_{T+})\}$$

$$= \sup_{x \in X} \{x_0 \cdot v_{-1} - I_H(x, y) - \langle Dx, \tilde{v} \rangle - k(x_0, x_{T+})\}.$$

Assume first that $y \notin \text{dom}_2 I_H$ so that there is an $\tilde{x} \in X$ such that $I_H(\tilde{x}, y) = -\infty$. Since $I_H(\tilde{x}, y)$ is independent of the end points of $\tilde{x}$, we get $f^*(v, y) = +\infty$. The expression for f^* then clearly holds if $\tilde{v} \notin C \cap X$. Since $\tilde{K}_t(y, v) \geq \tilde{x}_t \cdot v - H_t(\tilde{x}_t, y)$, we have $J_{\tilde{K}}(y, D\tilde{v}) = +\infty$, so the expression is valid also for $\tilde{v} \in C \cap X$. We may thus assume that $y \in \text{dom}_2 I_H$.

Let $\bar{x}$ belong to the set in condition 2. Redefining $\bar{x}_0$ and $\bar{x}_{T+}$, we may assume that $k(\bar{x}_0, \bar{x}_{T+}) < \infty$. Since $y \in \text{dom}_2 I_H$, we have that $I_H(\cdot, y)$ is proper on X. In view of condition 2, [16, Theorem 2] implies that there is an $r > 0$ and an $\alpha \in \mathbb{R}$ such that $I_H(\bar{x} + x, y) \leq \alpha$ whenever $x \in X$ with $x_t \in \mathbb{B}(0, r)$ for all t. Therefore,

$$f^*(v, y) \geq \sup_{x \in C_c^1} \{\bar{x}_0 \cdot v_{-1} - I_H(\bar{x} + x, y) - \langle D(\bar{x} + x), \tilde{v} \rangle - k(\bar{x}_0, \bar{x}_{T+}) \mid x_t \in \mathbb{B}(0, r) \, \forall t\}$$

$$\geq \bar{x}_0 \cdot v_{-1} - \alpha - \langle D\bar{x}, \tilde{v} \rangle - k(\bar{x}_0, \bar{x}_{T+}) + \sup_{x \in C_c^1} \{-\langle Dx, \tilde{v} \rangle \mid x_t \in \mathbb{B}(0, r) \, \forall t\},$$

where C_c^1 is the set of continuously differentiable $\mathbb{R}^d$-valued functions with compact support in $(0, T)$. By [1, Proposition 3.6], the last supremum equals the total variation of $\tilde{v}$ on $(0, T)$ and consequently $f^*(v, y) = +\infty$ unless $\tilde{v}$ is of bounded variation on $[0, T]$. When $\tilde{v}$ is of bounded variation, integration by parts gives

$$f^*(v, y) = \sup_{x \in X} \{x_0 \cdot v_{-1} - I_H(x, y) - \int \tilde{v}_t dx_t - k(x_0, x_{T+})\}$$

$$= \sup_{x \in X} \{\int x_t d\tilde{v}_t - I_H(x, y) + x_0 \cdot (v_{-1} + \tilde{v}_0) - x_{T+} \cdot \tilde{v}_T - k(x_0, x_{T+})\}$$

$$= \sup_{x \in X} \{\int x_t d\tilde{v}_t - I_H(x, y)\} + \sup_{x \in X} \{x_0 \cdot (v_{-1} + \tilde{v}_0) - x_{T+} \cdot \tilde{v}_T - k(x_0, x_{T+})\}$$

$$= \sup_{x \in X} \{\int x_t d\tilde{v}_t - I_H(x, y)\} + \tilde{k}(v_{-1} + \tilde{v}_0, \tilde{v}_T).$$

Analogously to the proof of Theorem 2.1, we can restrict the supremum to the set $\{x \in X \mid x_t \in \text{int} \, \text{dom}_1 H_t \, \forall t\}$. Thus, by [13, Corollary 34.2.1],

$$f^*(v, y) = \sup_{x \in X} \{\int x_t d\tilde{v}_t - I_{\underline{H}}(x, y)\} + \tilde{k}(v_{-1} + \tilde{v}_0, \tilde{v}_T),$$

where $\underline{H}_t$ denotes the closure of H_t with respect to x. The rest of the proof is analogous to the proof of Theorem 2.1 except that instead of [10, Theorem 3] we use [10, Theorem 4] on integral functionals of left-continuous functions of bounded variation. $\square$

3. A closedness criterion. This section gives sufficient conditions for the closedness of φ by applying general results on the conjugate duality framework derived in the appendix. To this end, we write φ as

$$\varphi(u) = \inf_{a \in \mathbb{R}^d} \varphi_0(a, u),$$

where

$$\varphi_0(a, u) = \inf_{x \in X} \{f(x, u) \mid x_0 = a\}.$$

We will proceed in two steps by first giving conditions for closedness of φ_0. The function φ_0 describes the dependence of the optimal value on $u \in M$ as well as on the initial state much like the cost-to-go function in the Hamilton-Jacobi theory of optimal control; see e.g. [24, 6]. Once the closedness of φ_0 has been established, we can apply the classical recession criterion from finite-dimensional convex analysis to verify the closedness of φ.

The bilinear form

$$\langle (a, u), (\tilde{a}, y) \rangle = a \cdot \tilde{a} + \langle u, y \rangle$$

puts the space $\mathbb{R}^d \times M$ in separating duality with $\mathbb{R}^d \times C$. The following result establishes the lower semicontinuity of φ_0 with respect to the corresponding weak topology. The proof relies on regularity properties of differential equations much like the proof of [20, Theorem 3']. We use the same interiority condition but we relax the continuity assumptions on the domain of the Hamiltonian.

THEOREM 3.1. *In addition to the hypotheses of Theorem 2.2, assume that there exists $\bar{y} \in \mathrm{dom}\, g \cap AC$ with $\bar{y}_t \in \mathrm{int}\, \mathrm{dom}_2\, H_t$ for all t. Then φ_0 is closed, the infimum in the definition of φ_0 is attained for every (a, u) and*

$$\varphi_0^\infty(a, u) = \inf_{x \in X} \{ f^\infty(x, u) \mid x_0 = a \}.$$

Proof. Note that φ_0 is the value function associated with $f_0 : X \times (\mathbb{R}^d \times M) \to \overline{\mathbb{R}}$ defined by

$$f_0(x, (a, u)) = f(x, u) + \delta_0(x_0 - a).$$

By Theorem 5.1 below, it suffices to show that

$$v \mapsto \inf_{(\tilde{a}, y)} \{ f_0^*(v, (\tilde{a}, y)) - \langle (a, u), (\tilde{a}, y) \rangle \} \tag{3.1}$$

is bounded above in a neighborhood of the origin for all (a, u).

The Lagrangian L_0 associated with f_0 can be written as

$$L_0(x, (\tilde{a}, y)) = L(x, y) - x_0 \cdot \tilde{a},$$

so $f_0^*(v, (\tilde{a}, y)) = f^*((v_{-1} + \tilde{a}, v|_{[0,T]}), y)$. By Theorem 2.2,

$$f_0^*(v, (\tilde{a}, y)) = J_{\tilde{K}}(y, D(y - v|_{[0,T]})) + \tilde{k}(v_{-1} + \tilde{a} + y_0 - v_0, y_T - v_T).$$

It suffices to establish the existence of a continuous function $v \mapsto (\tilde{a}^v, y^v)$ from V to $\mathbb{R}^d \times C$ such that $y^v \in AC$, $y^0 = \bar{y}$, $y_T^v = \bar{y}_T$, $y_0^v + \tilde{a}^v = \bar{y}_0$, and such that the function

$$v \mapsto \int \tilde{K}_t(y^v + v_t, \dot{y}_t^v) d\mu_t \tag{3.2}$$

is bounded above in a neighborhood of the origin. Indeed, we will then have

$$\inf_{(\tilde{a}, y)} \{ f_0^*(v, (\tilde{a}, y)) - \langle (a, u), (\tilde{a}, y) \rangle \}$$
$$= \inf_{\tilde{a}, y \in C \cap X} \{ J_{\tilde{K}}(y + v, Dy) + \tilde{k}(y_0 + \tilde{a}, y_T) - \langle (a, u), (\tilde{a} - v_{-1}, y + v|_{[0,T]}) \rangle \}$$
$$\leq J_{\tilde{K}}(y^v + v, Dy^v) + \tilde{k}(y_0^v + \tilde{a}^v, y_T^v) - \langle (a, u), (\tilde{a}^v - v_{-1}, y^v + v|_{[0,T]}) \rangle$$

so that (3.1) is bounded from above on a neighborhood of the origin.

By [16, Lemma 2], there is an $\bar{r} > 0$ such that $\mathbb{B}(\bar{y}_t, \bar{r}) \subset \text{dom}_2 H_t$ for all t; see [16, p. 460]. We can then choose $v^i \in \mathbb{R}^d$, $i = 0, \dots, d$ and an $r > 0$ such that $|v^i| < \bar{r}$ and $\mathbb{B}(0, r)$ belongs to the interior of the convex hull of $\{v^i \mid i = 0, \dots, d\}$. Having assumed the hypotheses of Theorem 2.2, conditions 2 and 4 of Theorem 2.1 give the existence of functions $z^i \in L^1$ and nonnegative $\beta^i \in L^1$ such that

$$H_t(x, \bar{y}_t + v^i) - x \cdot z^i_t \geq -\beta^i_t.$$

Taking infimum over $x \in \mathbb{R}^d$ gives

$$\tilde{K}_t(\bar{y}_t + v^i, z^i_t) d\mu_t \leq \beta^i_t. \tag{3.3}$$

Let $Z_t = [z^1_t - z^0_t \ \dots \ z^d_t - z^0_t]$ and $W = [v^1 - v^0 \ \dots \ v^d - v^0]$. Then W is nonsingular and

$$z^i_t = A_t(\bar{y}_t + v^i) + b_t, \quad i = 0, \dots, d, \tag{3.4}$$

where $A_t = Z_t W^{-1}$ and $b_t = z^0_t - Z_t W^{-1}(\bar{y}_t + v^0)$. Moreover, the integrability of z^i and boundedness of $\bar{y}$ imply that $t \mapsto A_t$ and $t \mapsto b_t$ belong to L^1. By Lemma 5.3 below, there is a $y^v \in AC$ such that

$$dy^v_t = F_t(y^v_t + v_t) d\mu_t, \quad y^v_T = \bar{y}_T,$$

where

$$F_t(y) = \begin{cases} \dot{\bar{y}}_t & \text{if } y = \bar{y}_t \\ (1 - \frac{|y - \bar{y}_t|}{r})\dot{\bar{y}}_t + \frac{|y - \bar{y}_t|}{r}[A_t(\bar{y}_t + r\frac{y - \bar{y}_t}{|y - \bar{y}_t|}) + b_t] & \text{otherwise}; \end{cases}$$

moreover, $v \mapsto (\tilde{a}^v, y^v)$, where $\tilde{a}^v = \bar{y}_0 - y^v_0$, is a continuous transformation from V to $\mathbb{R}^d \times C$. We have that $y^0 = \bar{y}$, $y^v_T = \bar{y}_T$ and $y^v_0 + \tilde{a}^v = \bar{y}_0$ for all v. Next we establish that (3.2) is bounded above in a neighborhood of the origin which will finish the proof.

Since $y^0 = \bar{y}$, there is a $\delta > 0$ such that $\|y^v + v|_{[0,T]} - \bar{y}\| < r$ whenever $\|v\| < \delta$. Denoting $\alpha_t = \frac{|y^v_t + v_t - \bar{y}_t|}{r}$ and $w_t = \alpha_t^{-1}(y^v_t + v_t - \bar{y}_t)$, we have that $y^v_t + v_t = (1 - \alpha_t)\bar{y}_t + \alpha_t(\bar{y}_t + w_t)$ and consequently, by the definition of F,

$$\tilde{K}_t(y^v_t + v_t, \dot{y}^v_t) = \tilde{K}_t(y^v_t + v_t, F_t(y^v_t + v_t))$$
$$\leq (1 - \alpha_t)\tilde{K}_t(\bar{y}_t, \dot{\bar{y}}_t) + \alpha_t \tilde{K}_t(\bar{y}_t + w_t, A_t(\bar{y}_t + w_t) + b_t).$$

The function w can be expressed as $w_t = \sum_{i=0}^d \alpha^i_t v^i_t$, where $\alpha^i : [0, T] \to \mathbb{R}$ are measurable with $\sum_{i=0}^d \alpha^i_t = 1$. Since $|w_t| < r$ for all t, we have $0 \leq \alpha^i_t \leq 1$ for all t, so, by (3.4) and (3.3),

$$\tilde{K}_t(\bar{y}_t + w_t, A_t(\bar{y}_t + w_t) + b_t) \leq \sum_{i=0}^d \alpha^i_t \tilde{K}_t(\bar{y}_t + v^i, z^i_t) \leq \sum_{i=0}^d \alpha^i_t \beta^i_t.$$

We define $\beta_t = \max\{0, \tilde{K}_t(\bar{y}_t, \dot{\bar{y}}_t)\} + \sum_{i=0}^d \alpha^i_t \beta^i_t$ so that $\beta \in L^1$ and

$$\int \tilde{K}_t(y^v_t + v_t, \dot{y}^v_t) d\mu_t \leq \int \beta_t d\mu_t$$

whenever $\|v\| < \delta$.   □

Combining Theorem 3.1 with the classical recession condition gives sufficient conditions for the closedness of φ.

THEOREM 3.2. *In addition to the hypotheses of Theorem 3.1, assume that*

$$\{x \in X \mid f^\infty(x, 0) \le 0\}$$

is a linear space. Then φ is closed and the infimum in (P_u) is attained for every $u \in M$.

Proof. By Theorem 3.1,

$$\{a \in \mathbb{R}^d \mid \varphi_0^\infty(a, 0) \le 0\} = \{a \in \mathbb{R}^d \mid \exists x \in X : f^\infty(x, 0) \le 0, \, x_0 = a\},$$

which is linear when $\{x \in X \mid f^\infty(x, 0) \le 0\}$ is linear. Since

$$\varphi(u) = \inf_{a \in \mathbb{R}^d} \varphi_0(a, u),$$

the claim follows from Theorem 3.1 and Corollary 5.2 below.   □

The linearity condition in Theorem 3.2 is analogous to the condition

$$\{y \in AC \mid \int K_t^\infty(y_t, \dot{y}_t) d\mu_t + k^\infty(y_0, y_T) \le 0\}$$

in [17, Theorem 3]. Indeed, the recession function f^∞ can be expressed in terms of K^∞ and k^∞ as follows.

LEMMA 3.3. *Assume that f is proper and closed and that there exist $z \in L^1$, $y \in L^\infty$ and $\beta \in L^1$ such that*

$$K_t(x, u) \ge x \cdot z_t + u \cdot y_t - \beta_t \quad \mu\text{-a.e.}$$

Then

$$f^\infty(x, u) = J_{K^\infty}(x, Dx + u) + k^\infty(x_0, x_{T+}).$$

Proof. We may assume without a loss of generality that $K_t(0, 0) = 0$. By monotone convergence theorem,

$$\begin{aligned}
J_K^\infty(x, \theta) &= \lim_{\alpha \nearrow \infty} \frac{1}{\alpha} J_K(\alpha x, \alpha \theta) \\
&= \lim_{\alpha \nearrow \infty} \int \frac{1}{\alpha} K_t(\alpha x_t, \alpha (d\theta^a/d\mu)_t) d\mu_t + \int K_t^\infty(0, d\theta^s)/d|\theta^s|)_t) d|\theta^s|)|_t \\
&= \int K_t^\infty(x_t, (d\theta^a/d\mu)_t) d\mu_t + \int K_t^\infty(0, d\theta^s)/d|\theta^s|)_t) d|\theta^s|)|_t \\
&= J_{K^\infty}(x, \theta).
\end{aligned}$$

The expression then follows from the general fact that if f_1, f_2 are closed convex functions and A is a continuous linear mapping such that $f_1 \circ A + f_2$ is proper, then $(f_1 \circ A + f_2)^\infty = f_1^\infty \circ A + f_2^\infty$.   □

The assumptions in Lemma 3.3 are satisfied under the assumptions of Theorem 2.1 whenever $\text{dom}_1 I_H \ne \emptyset$. Indeed, then we have that $\text{dom}_2 I_H \ne \emptyset$ (see the proof of

Theorem 2.1), so, by the definition of Hamiltonian, there exist $y \in C$, $z \in L^1$ and $\beta \in L^1$ such that

$$K_t(x, u) \geq x \cdot z_t + u \cdot y_t - \beta_t.$$

Combining the previous results with the biconjugate theorem gives a dual representation for the value function.

THEOREM 3.4. *Assume that*
  1. $t \mapsto \mathrm{dom}_1 H_t$ *is left-isc and left outer μ-regular*
  2. $\emptyset \neq \{x \in X \mid \exists r > 0 : \mathbb{B}(x_t, r) \subset \mathrm{dom}_1 H_t \; \forall t\} \subset \mathrm{dom}_1 I_H$,
  3. $t \mapsto \mathrm{dom}_2 H_t$ *is isc and outer μ-regular*
  4. $\{y \in C \mid y_t \in \mathrm{int}\, \mathrm{dom}_2 H_t \; \forall t\} \subset \mathrm{dom}_2 I_H$,
  5. *there exists a $\bar{y} \in \mathrm{dom}\, g \cap AC$ with $\bar{y}_t \in \mathrm{int}\, \mathrm{dom}_2 H_t$ for all t,*
  6. $\{x \mid J_{K^\infty}(x, Dx + u) + k^\infty(x_0, x_{T+}) \leq 0\}$ *is a linear space.*
Then the infimum in (P_u) *is attained for every u and*

$$\varphi(u) = \sup_{y \in C \cap X} \{\langle u, y \rangle - J_{\tilde{K}}(y, Dy) - \tilde{k}(y_0, y_T)\}.$$

Proof. In view of Lemma 3.3 and Theorem 3.2, it suffices to show that conditions 3 and 4 in Theorem 2.1 are satisfied.

Assume that $x \in \mathrm{dom}_1 I_H$ and let $h_t(y) = -H_t(x_t, y)$ so that $h_t^*(u) = K_t(x_t, u)$. By [16, Lemma 2], there is an $r > 0$ such that $\mathbb{B}(\bar{y}_t, r) \subset \mathrm{dom}_2 H_t$ for all t; see [16, p. 460]. Therefore, by condition 2, $t \mapsto h_t(\bar{y}_t + y)$ is summable whenever $|y| < r$, so, by [21, Proposition 3G], there is a $w \in L^1$ such that $I_K(x, w) < \infty$. This implies that

$$H_t(x_t, y) \leq -y \cdot w_t + K_t(x_t, w_t) \quad \mu\text{-a.e.}$$

Thus condition 3 in Theorem 2.1 holds. Condition 4 in Theorem 2.1 is verified similarly. □

4. Optimality conditions. This section derives optimality conditions for problem (P_u) when $u = 0$. That is, we will be looking at the problem

$$\text{minimize} \int K_t(x_t, \dot{x}_t^a) d\mu_t + \int K_t^\infty(0, \dot{x}_t^s) d|Dx^s|_t + k(x_0, x_{T+}) \quad \text{over } x \in X, \quad (P)$$

where $\dot{x}^a = d(Dx^a)/d\mu$ and $\dot{x}^s = d(Dx^s)/d|Dx^s|$. We associate with (P) the problem

$$\text{minimize} \int \tilde{K}_t(y_t, \dot{y}_t^a) d\mu_t + \int \tilde{K}_t^\infty(0, \dot{y}_t^s) d|Dy^s|_t + \tilde{k}(y_0, y_T) \quad \text{over } y \in C \cap X. \quad (D)$$

For a mapping $S_t : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ with $t \mapsto \mathrm{gph}\, S_t$ closed-valued and measurable, and for a function $z \in X$ of bounded variation, we write $Dz \in S(z)$ if

$$\dot{z}_t^a \in S_t(z_t) \quad \mu\text{-a.e.,}$$
$$\dot{z}_t^s \in S_t^s(z_t) \quad |Dz^s|\text{-a.e.,}$$

where the mapping $S_t^s : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is defined for each t as the *graphical inner limit* (see [23, Chapter 5]) of the mappings $(\alpha S_t)(z) := \alpha S_t(z)$ as $\alpha \searrow 0$. Here $t \mapsto \mathrm{gph}\, S_t^s$ is closed-valued and measurable; see [23, Theorem 14.20]. In particular, $\{t \mid \dot{z}_t^a \in S_t(z_t)\}$ and $\{t \mid \dot{z}_t^s \in S_t^s(z_t)\}$ are measurable sets (see [23, Section 14.B]), so $Dz \in S(z)$ is indeed well-defined. This definition is inspired by [22, Section 14] where

the Hamiltonian conditions were extended from absolutely continuous trajectories to trajectories of bounded variation. Indeed, when S_t is maximal monotone, [23, Theorem 12.37] says that S_t^s is the normal cone mapping of the closure of $\mathrm{dom}\, S_t$ and $S_t^s(z)$ is the recession cone of $S_t(z)$ for all $z \in \mathrm{dom}\, S_t$.

We say that $x \in X$ and $y \in C \cap X$ satisfy the *generalized Hamiltonian equation* if

$$D(x, y) \in \Pi \tilde{\partial} H(x, y),$$

where $\Pi(v, u) = (u, v)$ and

$$\tilde{\partial} H_t(x, y) = \partial_x H_t(x, y) \times \partial_y[-H_t](x, y).$$

Since, by [23, Example 12.27], $\tilde{\partial} H_t$ is maximal monotone, $(\tilde{\partial} H)_t^s$ equals the normal cone mapping $N_{\mathrm{cl\, dom}\, H_t}$ of $\mathrm{cl\, dom}\, H_t$; see [23, Theorem 12.37] and [13, Theorem 37.4]. Moreover, $t \mapsto \mathrm{gph}\, \tilde{\partial} H_t$ is closed-valued and measurable [23, Example 12.8 and Theorem 14.56] and consequently the generalized Hamiltonian equation is well-defined.

When $\mathrm{dom}\, H_t = \mathbb{R}^d \times \mathbb{R}^d$ (no state constraints), we have $N_{\mathrm{cl\, dom}\, H_t}(x, y) = \{0\}$, so feasible trajectories are necessarily absolutely continuous and the generalized Hamiltonian equation reduces to that studied e.g. in [14]. When $\mathrm{dom}_1 H_t = \mathbb{R}^d$ (no state constraints in (P)), we recover the optimality conditions of [17] for optimal control problems; see [17, Lemma 4].

As usual $x \in X$ and $y \in C \cap X$ are said to satisfy the *transversality condition* if

$$(y_0, -y_T) \in \partial k(x_0, x_{T+}).$$

THEOREM 4.1. *Assume that $t \mapsto \mathrm{dom}_1 H_t$ is left-outer μ-regular and that $t \mapsto \mathrm{dom}_2 H_t$ is outer μ-regular. Then $\inf(P) \geq -\inf(D)$. For $\inf(P) = -\inf(D)$ to hold with attainment at feasible x and y respectively, it is necessary and sufficient that x and y satisfy the generalized Hamiltonian equation and the transversality condition.*

Proof. We have $x_t \in \mathrm{dom}_1 H_t$ and $y_t \in \mathrm{dom}_2 H_t$ μ-a.e. so, by [10, Theorem 1], $x_t \in \mathrm{cl\, dom}_1 H_t$ and $y_t \in \mathrm{cl\, dom}_2 H_t$ for all t. Consequently, $K_t^\infty(0, x) \geq x \cdot y_t$ and $\tilde{K}_t^\infty(0, v) \geq v \cdot x_t$ for all t. We get

$$\begin{aligned}
K_t(x_t, \dot{x}_t^a) + \tilde{K}_t(y_t, \dot{y}_t^a) &\geq x_t \cdot \dot{y}_t^a + y_t \cdot \dot{x}_t^a \qquad \mu\text{-a.e.,} \\
K_t^\infty(0, \dot{x}_t^s) &\geq \dot{x}_t^s \cdot y_t \qquad |Dx^s|\text{-a.e.,} \\
\tilde{K}_t^\infty(0, \dot{y}_t^s) &\geq \dot{y}_t^s \cdot x_t \qquad |Dy^s|\text{-a.e.,} \\
k(x_0, x_{T+}) + \tilde{k}(y_0, y_T) &\geq x_0 \cdot y_0 - x_{T+} \cdot y_T.
\end{aligned} \tag{4.1}$$

Integration by parts gives

$$\begin{aligned}
&J_K(x, Dx) + k(x_0, x_{T+}) + J_{\tilde{K}}(y, Dy) + \tilde{k}(y_0, y_T) \\
&\geq \int y_t dx_t + \int x_t dy_t + x_0 \cdot y_0 - y_T \cdot x_{T+} = 0,
\end{aligned} \tag{4.2}$$

where the inequality holds as equality if and only if the inequalities in (4.1) hold as equalities almost everywhere. In particular, we get $\inf(P) \geq -\inf(D)$.

Since $\partial \delta_{\mathrm{cl\, dom}_1 H_t}(x) = N_{\mathrm{cl\, dom}_1 H_t}(x)$ and since $\tilde{K}_t^\infty(0, \cdot)$ is the support function of $\mathrm{cl\, dom}_1 H_t$, we have $\tilde{K}_t(0, v) = x \cdot v$ if and only if $v \in N_{\mathrm{cl\, dom}_1 H_t}(x)$. Similarly $K_t(0, u) = u \cdot y$ if and only if $u \in N_{\mathrm{cl\, dom}_2 H_t}(y)$. By [13, Theorem 37.5], we have

$K_t(x, u) + \tilde{K}_t(y, v) = x \cdot v + u \cdot y$ if and only if $(v, u) \in \tilde{\partial} H_t(x, y)$. Therefore, (4.2) holds as an equality if and only if the generalized Hamiltonian equation and the transversality condition hold. □

The conditions of Theorem 4.1 generalize those in [20, Theorem 2]. Indeed, outer semicontinuous mappings are both left-outer μ-regular and outer μ-regular; see (3) in [10]. On the other hand, in [20, Theorem 2] both trajectories are allowed to be discontinuous.

Combining Theorem 4.1 with [15, Theorem 1(b)] we obtain the following, where the problem (P_{AC}) is defined in the introduction.

THEOREM 4.2. *Assume that μ is the Lebesgue measure and that*

1. $t \mapsto \mathrm{dom}_1 H_t$ is left outer μ-regular,

2. $t \mapsto \mathrm{dom}_2 H_t$ is outer μ-regular,

3. for all $x \in \mathbb{R}^d$ there exist $w \in L^1$ and $\alpha \in L^1$ such that

$$H_t(x, y) \leq -y \cdot w_t + \alpha_t \quad \mu\text{-a.e.,}$$

4. there exist $z \in L^1$, $y \in L^\infty$ and $\beta \in L^1$ such that

$$H_t(x, y_t) \geq x \cdot z_t - \beta_t \quad \mu\text{-a.e.}$$

5. $\{y \in AC \mid \int \tilde{K}_t^\infty(y_t, \dot{y}_t) d\mu_t + \tilde{k}^\infty(y_0, y_T) \leq 0\}$ is a linear space.
Then $\inf(P_{AC}) = \inf(P) = -\inf(D)$, the optimal values are finite and the infimum in (D) is attained by some $y \in AC$. In particular, $x \in X$ attains the infimum in (P) if and only if it satisfies the generalized Hamiltonian equation and the transversality condition with some $y \in AC$.

Proof. Condition 3 implies that $\mathrm{dom}_1 H_t = \mathbb{R}^d$ μ-a.e. which together with condition 1 gives that $\mathrm{dom}_1 H_t = \mathbb{R}^d$ for all t. Hence we have that $\tilde{K}_t^\infty(0, v) = \delta_0(v)$ and consequently $J_{\tilde{K}}(y, Dy) = +\infty$ unless $y \in AC$.

By condition 3 and by the definition of the Hamiltonian, for every $x \in \mathbb{R}^d$ there exist functions $w \in L^1$ and $\alpha \in L^1$ such that

$$K_t(x, w_t) \leq \alpha_t \quad \mu\text{-a.e.}$$

Similarly condition 4 implies that there exist functions $z \in L^1$, $y \in L^\infty$ and $\beta \in L^1$ such that

$$K_t(x, u) \geq x \cdot z_t + u \cdot y_t - \beta_t \quad \mu\text{-a.e.}$$

Therefore, the conditions (A)–(C) and D_0 in [15] hold. Consequently, we get from condition 4 and [15, Theorem 3] that the assumptions of [15, Theorem 1.(b)] are satisfied, so $\inf(P_{AC}) = -\inf(D)$, these optimal values are finite and the infimum in (D) is attained by some y. Combining these facts with Theorem 4.1 gives the rest of the claims. □

Conditions 3–5 of Theorem 4.2 are just reformulations of the assumptions of [15, Theorem 1(b)] so that they are readily comparable with the other assumptions made in this article.

5. Appendix. The first part of this appendix is concerned with the general conjugate duality framework of Rockafellar [18]. Accordingly, X and U denote arbitrary locally convex topological vector spaces in separating duality with V and Y, respectively. We fix a proper closed convex function $f : X \times U \to \overline{\overline{\mathbb{R}}}$ and denote the associated value function by

$$\varphi(u) = \inf_{x \in X} f(x, u).$$

Given $u \in U$, we define the extended real-valued function γ_u on V by

$$\gamma_u(v) = \inf_{y \in Y}\{f^*(v, y) - \langle u, y \rangle\}.$$

Note that the domain of γ_u equals

$$\Gamma := \{v \mid \exists y : f^*(v, y) < \infty\}$$

for every u.

The *recession function* h^∞ of a closed proper convex $h : U \to \overline{\mathbb{R}}$ is defined by

$$h^\infty(u) = \sup_{\alpha > 0} \frac{h(\alpha u + \bar{u}) - h(\bar{u})}{\alpha},$$

where the supremum is independent of the choice of $\bar{u} \in \operatorname{dom} h$; see [13, Theorem 8.5] for a proof in the finite-dimensional case. The recession function is sublinear and closed whenever h is closed; see [11].

THEOREM 5.1. *Assume that, for every u, the function γ_u is bounded from above on a neighborhood of the origin relative to $\operatorname{aff}\Gamma$. Then φ is closed and proper, the infimum in the definition of φ is attained for every $u \in U$ and*

$$\varphi^\infty(u) = \inf_{x \in X} f^\infty(x, u).$$

Proof. Assume first that $\operatorname{aff}\Gamma = V$. Since γ_u is convex (see e.g. [18, Theorem 1]) and bounded from above on a neighborhood of the origin, we have that $\gamma_u^* = f(\cdot, u)$ is inf-compact and $\gamma_u^{**}(0) = \gamma_u(0)$ (see e.g. [18, Theorem 10]). Therefore

$$\varphi(u) = \inf_{x \in X} f(x, u) = -\gamma_u^{**}(0) = -\gamma_u(0) = \sup_{y \in Y}\{\langle u, y \rangle - f^*(0, y)\},$$

where the infimum is attained and the last expression is closed in u. This implies together with the properness of f that φ is closed and proper.

Let $\bar{u} \in \operatorname{dom}\varphi$ and $\bar{x} \in X$ be such that $\varphi(\bar{u}) = f(\bar{x}, \bar{u})$. We have that

$$\begin{aligned}
\varphi^\infty(u) &= \sup_{\alpha > 0} \frac{\varphi(\bar{u} + \alpha u) - \varphi(\bar{u})}{\alpha} \\
&= \sup_{\alpha > 0} \inf_{x \in X} f_\alpha(x, u),
\end{aligned}$$

where

$$f_\alpha(x, u) = \frac{f(\bar{x} + \alpha x, \bar{u} + \alpha u) - f(\bar{x}, \bar{u})}{\alpha}.$$

Clearly,

$$\varphi^\infty(u) \le \inf_{x \in X} \sup_{\alpha > 0} f_\alpha(x, u) = \inf_{x \in X} f^\infty(x, u).$$

To prove the converse, let $\beta > \sup_{\alpha > 0} \inf_{x \in X} f_\alpha(x, u)$ and let

$$B_\alpha = \{x \in X \mid f_\alpha(x, u) \le \beta\}.$$

The functions f_α are non-decreasing in α, so the sets B_α are non-increasing in α. Since the functions $x \mapsto f_\alpha(x, u)$ inherit inf-compactness from $x \mapsto f(x, u)$, we get,

by the finite intersection property, that there is an $x' \in X$ with $x' \in B_\alpha$ for every $\alpha > 0$. Thus,

$$\sup_{\alpha > 0} f_\alpha(x', u) \le \beta$$

or in other words, $f^\infty(x', u) \le \beta$. Since $\beta > \sup_{\alpha > 0} \inf_{x \in X} f_\alpha(x, u)$ was arbitrary, we have

$$\inf_{x \in X} f^\infty(x, u) \le \sup_{\alpha > 0} \inf_{x \in X} f_\alpha(x, u) = \varphi^\infty(u),$$

which completes the proof for the case aff $\Gamma = V$.

We now turn to the general case $\tilde{V} := \text{aff } \Gamma \subseteq V$. Let $N = \{x \in X \mid \langle x, \tilde{v} \rangle = 0 \; \forall \tilde{v} \in \tilde{V}\}$, $[x] = x + N$ and $X/N = \{[x] \mid x \in X\}$. By Hahn-Banach theorem, every continuous linear functional on $\tilde{V}$ extends to an element of X. On the other hand, $x' \in X$ and $x \in X$ define the same continuous linear functional on $\tilde{V}$ if and only if $x' \in [x]$. Thus X/N can be identified with the continuous dual of $\tilde{V}$ with the pairing $\langle [x], \tilde{v} \rangle = \langle x, \tilde{v} \rangle$.

Defining $\tilde{f} : X/N \times U \to \overline{\mathbb{R}}$ by $\tilde{f} = (f^*|_{\tilde{V} \times Y})^*$, we have

$$\tilde{f}([x], u) = \sup\{\langle x, \tilde{v} \rangle + \langle u, y \rangle - f^*(\tilde{v}, y)\} = f(x, u)$$

and

$$\varphi(u) = \inf_{[x] \in X/N} \tilde{f}([x], u).$$

Since $\tilde{f}^* = f^*|_{\tilde{V} \times Y}$, we can apply the first part of the proof to the conjugate duality framework corresponding to $\tilde{f}$. Thus φ is closed, the infimum in the definition of φ is attained and since $\tilde{f}^\infty([x], u) = f^\infty(x, u)$, we get

$$\varphi^\infty(u) = \inf_{[x] \in X/N} \tilde{f}^\infty([x], u) = \inf_{[x] \in X/N} f^\infty(x, u) = \inf_{x \in X} f^\infty(x, u),$$

which finishes the proof.   □

The following corollary was used in the proof of Theorem 3.2.

COROLLARY 5.2. *Assume that $X = R^d$ and that*

$$\{x \mid f^\infty(x, 0) \le 0\}$$

is a linear space. Then φ is closed and proper, the infimum in the definition of φ is attained for every $u \in U$ and

$$\varphi^\infty(u) = \inf_{x \in X} f^\infty(x, u).$$

Proof. Since γ_u is now a convex function on $\mathbb{R}^d$, it suffices to show that the origin belongs to the relative interior of $\text{dom } \gamma_u = \text{dom } \gamma_0$; see [13, Theorem 10.1]. By [13, Theorem 7.4.1], we have $\text{rint dom } \gamma_0 = \text{rint dom cl } \gamma_0$ while, by [13, Corollary 13.3.4(b)], $0 \in \text{rint dom cl } \gamma_0$ if and only if

$$\mathcal{L} = \{x \mid (\gamma_0^*)^\infty(x) \le 0\}$$

is a linear space. By [13, Theorem 8.7],

$$\mathcal{L} = \{x \mid \gamma_0^*(x) \le 0\}^\infty = \{x \mid f(x,0) \le 0\}^\infty = \{x \mid f^\infty(x,0) \le 0\},$$

where we have used the fact that $\gamma_u^*(x) = f(x,u)$, by definition. □

The following lemma was used in the proof of Theorem 3.1. Its proof is rather standard in the case when μ is the Lebesque measure.

LEMMA 5.3. *Let* $F : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ *be jointly measurable. Assume that there exists a* $c \in L^1$ *such that* $|F_t(y^1) - F_t(y^2)| \le |y^1 - y^2|c_t$ *and* $|F_t(y^1)| \le (1 + |y^1|)c_t$ *for all* t *and* $y^1, y^2 \in \mathbb{R}^d$. *Then for every* $a \in \mathbb{R}^d$ *and* $v \in C$ *there exists a unique* $y^v \in AC$ *such that*

$$dy_t^v = F_t(y_t^v + v_t)d\mu_t, \quad y_0 = a. \tag{5.1}$$

Moreover, the mapping $v \mapsto y^v$ *is continuous.*

Proof. Define $\mathcal{T}_v : C \to C$ by

$$(\mathcal{T}_v y)_t = a + \int_{[0,t]} F_s(y_s + v_s)d\mu_s.$$

Let $\gamma_t = \int_{[0,t]} c_s d\mu_s$. For any $y^1, y^2 \in C$ we have

$$|(\mathcal{T}_v y^1)_t - (\mathcal{T}_v y^2)_t| \le \int_{[0,t]} |y_s^1 - y_s^2|d\gamma_s,$$

so, by induction, $|(\mathcal{T}_v^\nu y^1)_t - (\mathcal{T}_v^\nu y^2)_t| \le \|y^1 - y^2\|\frac{(\gamma_t)^\nu}{\nu!}$. For ν large enough, $\mathcal{T}_v^\nu$ is a contraction and $\mathcal{T}_v$ has a unique fixed point, i.e., there is a unique $y^v \in C$ satisfying (5.1).

Let $r > 0$. For every $v \in C$ with $\|v\| < r$, we have

$$|y_t^v| \le |a| + \int_{[0,t]} (1 + r + |y_s^v|)d\gamma_s$$

$$\le |a| + (1+r)\gamma_T + \int_{[0,t]} |y_s^v|d\gamma_s,$$

so, by Gronwall's inequality ([4, p. 498]), $\|y^v\| \le (|a| + (1+r)\gamma_T)e^{\gamma_T}$. Therefore, for every $v \in C$ with $\|v\| < r$, there is a $\beta \in \mathbb{R}$ such that

$$|y_t^v - y_{t'}^v| \le \int_{[t,t']} (1 + |y_s^v|)d\gamma_s \le \beta(\gamma_{t'} - \gamma_t).$$

Thus the set $\{y^v \mid \|v\| < r\}$ is uniformly bounded and equicontinuous. Assume that $v \mapsto y^v$ is not continuous. Then there is a sequence $(v^\nu)_{\nu=1}^\infty$ converging to v such that $y^{v^\nu} \to \hat{y}$ and $\hat{y} \ne y^v$. By dominated convergence and (5.1),

$$\hat{y}_t = a + \int_{[0,t]} F_s(\hat{y}_s + v_s)d\mu_s \quad \forall t$$

so that, by the uniqueness of the fixed point of $\mathcal{T}_v$, we get $\hat{y} = y^v$, which is a contradiction. □

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