Convex duality in optimal investment under illiquidity

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August 16, 2013

Abstract

We study the problem of optimal investment by embedding it in the general conjugate duality framework of convex analysis. This allows for various extensions to classical models of liquid markets. In particular, we obtain a dual representation for the optimum value function in the presence of portfolio constraints and nonlinear trading costs that are encountered e.g. in modern limit order markets. The optimization problem is parameterized by a sequence of financial claims. Such a parameterization is essential in markets without a numeraire asset when pricing swap contracts and other financial products with multiple payout dates. In the special case of perfectly liquid markets or markets with proportional transaction costs, we recover well-known dual expressions in terms of martingale measures.

Key words: Optimal investment, illiquidity, convex duality

1 Introduction

Convex duality has long been an integral part of mathematical finance. Classical references include Harrison and Kreps [15], Harrison and Pliska [16], Kreps [25] and Dalang et al. [8] where the no-arbitrage principle behind the Black–Scholes formula was related to the existence of certain dual variables; see Delbaen and Schachermayer [10] for a detailed discussion of the topic. In problems of optimal investment, convex duality has become an important tool in the analysis of optimal solutions; see e.g. Kramkov and Schachermayer [23], Delbaen et al. [9], Karatzas and Žitković [21] and Rogers [40] and their references.

This paper studies convex duality in problems of optimal investment in illiquid financial markets where one may encounter frictions or restrictions when transferring wealth through time or between assets. Our model extends the classical linear model by allowing for portfolio constraints and nonlinear transaction costs that are encountered e.g. in limit order markets. In particular, we

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do not assume, a priori, the existence of a numeraire that allows for free transfer of (both positive as well as negative amounts of) wealth through time. This is a significant departure from the classical model where a numeraire is assumed in order to express wealth processes in terms of stochastic integrals. Such models are at odds with real markets where much of trading consists of exchanging sequences of cash-flows. Such trades would be unnecessary if one could postpone payments to a single date by shorting the numeraire.

While in classical models of perfectly liquid markets, convex analysis can often be reduced to an application of basic separation theorem, in the presence of nonlinear illiquidity effects, more sophisticated convex analysis is needed; see for example Cvitanić and Karatzas [6], Jouini and Kallal [18, 17], Guasoni [14], Schachermayer [43], Rokhlin [41], Bielecki, Cialenco and Rodriguez [4] as well as the books of Kabanov and Safarian [19] and Rogers [40]. The present paper builds on the market model introduced in Pennanen [30] where trading costs and portfolio constraints are described by general convex normal integrands and measurable closed convex sets, respectively. This discrete-time model provides a unifying framework for modeling transaction costs and portfolio constraints as well as nonlinear illiquidity effects.

The classical superhedging principle was extended to this model in [33, 32]. The present paper makes similar extensions to the duality theory of optimal investment. More precisely, we derive a dual representation for the optimal value function of the optimal investment problem studied in [34] as a basis for valuation of financial contracts. The dual representation is obtained by applying the stochastic optimization duality framework developed in [31] as an instance of the conjugate duality framework of Rockafellar [38]. This allows for a unified treatment of many well known models of mathematical finance where dual correspondences have often been derived case by case. In particular, we illustrate how the dual variables in the general case are connected to martingale measures and shadow prices that have been extensively studied in models with a numeraire.

2 Optimal investment

This section recalls the optimal investment problem from [34]. The problem is parameterized by financial liabilities characterized by a sequence of cash-flows the agent has to deliver over time. The dependence of the optimum value on the liabilities is important in valuation of swap contracts and other financial products that provide payments at multiple points in time. Section 3 derives dual expressions for the optimum value function.

Consider a financial market where a finite set J of assets can be traded over finite discrete time $t = 0, \dots, T$. We will model uncertainty by a probability space $(\Omega, \mathcal{F}, P)$ and the information by a nondecreasing sequence $(\mathcal{F}_t)_{t=0}^T$ of sub-sigma algebras of $\mathcal{F}$. At time t one does not know which scenario $\omega \in \Omega$ will realize but only to which element of $\mathcal{F}_t$ it belongs to. The filtration property $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ means that information increases over time.

We describe trading strategies by $(\mathcal{F}_t)_{t=0}^T$-adapted sequences $x = (x_t)_{t=0}^T$ of $\mathbb{R}^J$-valued stochastic processes. The random vector x_t describes the portfolio of assets held over $(t, t+1]$. The adaptedness means that x_t is $\mathcal{F}_t$-measurable, i.e. the portfolio chosen at time t only depends on information available at time t. The linear space of adapted trading strategies will be denoted by $\mathcal{N}$. Unless $\mathcal{F}_T$ has only a finite number of elements with positive probability, $\mathcal{N}$ is an infinite-dimensional space. We will assume that $\mathcal{F}_0 = \{\Omega, \emptyset\}$ so that x_0 is independent of ω. Implementing a portfolio process $x \in \mathcal{N}$ requires buying a portfolio $\Delta x_t := x_t - x_{t-1}$ of assets at time t. Negative purchases are interpreted as sales.

Trading costs will be described by an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence $S = (S_t)_{t=0}^T$ of convex *normal integrands* on $\mathbb{R}^J \times \mathcal{F}$. This means that for each t, the set-valued mapping $\omega \mapsto \{(x, \alpha) \in \mathbb{R}^J \times \mathbb{R} \,|\, S_t(x, \omega) \le \alpha\}$ is closed convex-valued and $\mathcal{F}_t$-measurable[1]. The value of $S_t(x, \omega)$ is interpreted as the cost of buying a portfolio $x \in \mathbb{R}^J$ at time t in state ω. Accordingly, we assume that $S_t(0, \omega) = 0$. Implementing a trading strategy $x \in \mathcal{N}$ requires investing $S_t(\Delta x_t)$ units of cash at time t. The measurability condition on S_t implies that $\omega \mapsto S_t(\Delta x_t(\omega), \omega)$ is $\mathcal{F}_t$-measurable, i.e. the cost of a portfolio is known at the time of purchase. Indeed, the cost is the composition of the measurable functions $\omega \mapsto (\Delta x_t(\omega), \omega)$ and S_t; see e.g. [39, Corollary 14.34].

The classical model of perfectly liquid markets corresponds to $S_t(x, \omega) = s_t(\omega) \cdot x$ where $s = (s_t)_{t=0}$ is an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence of price vectors independent of the traded amounts. Markets with proportional transaction costs and/or bid-ask-spreads can be modeled with sublinear cost functions

$$S_t(x, \omega) = \sup\{s \cdot x \,|\, s \in [s_t^-(\omega), s_t^+(\omega)]\},$$

where the $(\mathcal{F}_t)_{t=0}^T$-adapted $\mathbb{R}^J$-valued processes s^- and s^+ give the bid- and ask-prices, respectively; see [18]. Convex trading costs arise naturally also in modern limit order markets, where the cost of a "market order" is nonlinear and convex in the traded amount. Parametric convex market models have been proposed e.g. in Çetin and Rogers [5] and Malo and Pennanen [26].

We model *portfolio constraints* by requiring that the portfolio x_t chosen at time t has to belong to a closed convex set D_t much as e.g. in [28, 41, 11]. We allow D_t to be random but assume that it is $\mathcal{F}_t$-measurable. We assume that $0 \in D_t$ almost surely, i.e. that the zero portfolio is feasible. The classical unconstrained model corresponds to $D \equiv \mathbb{R}^J$ while short selling constraints, studied e.g. in Cvitanic and Karatzas [6] and Jouini and Kallal [17] can be described by $D \equiv \mathbb{R}^J_+$. As observed there, short selling constraints can also be used to model different interest rates for lending and borrowing. Indeed, this can be done by introducing lending and borrowing accounts whose unit prices appreciate at lending and borrowing rates, respectively, and by restricting the investments in these assets to be nonnegative and nonpositive, respectively.

[1] A set-valued mapping $T : \Omega \rightrightarrows \mathbb{R}^n$ is $\mathcal{F}_t$-measurable if $\{\omega \in \Omega \,|\, T(\omega) \cap U \neq \emptyset\} \in \mathcal{F}_t$ for every open $U \subset \mathbb{R}^n$.

Consider now an agent whose financial position is described by an $(\mathcal{F}_t)_{t=0}^T$-adapted sequence of cash-flows $c = (c_t)_{t=0}^T$ in the sense that the agent has to deliver a random amount c_t of cash at time t. We allow c_t to take both positive and negative values so it may describe both endowments as well as liabilities. In particular, $-c_0$ may be interpreted as an initial endowment while the subsequent payments c_t, $t = 1, \dots, T$ may be interpreted as the cash-flows associated with financial liabilities. We will denote the linear space of $(\mathcal{F}_t)_{t=0}^T$-adapted sequences of cash-flows by

$$\mathcal{M} = \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P)\}.$$

Here and in what follows, $L^0(\Omega, \mathcal{F}_t, P)$ stands for the linear space of equivalence classes of $\mathcal{F}_t$-measurable real-valued functions. As usual, two measurable functions are equivalent if they coincide P-almost surely. Given $c \in \mathcal{M}$, and nondecreasing[2] convex functions $\mathcal{V}_t$ on $L^0(\Omega, \mathcal{F}_t, P)$, we will study the problem

$$\text{minimize} \quad \sum_{t=0}^T \mathcal{V}_t(S_t(\Delta x_t) + c_t) \quad \text{over} \quad x \in \mathcal{N}_D, \tag{ALM}$$

where $\mathcal{N}_D := \{x \in \mathcal{N} \,|\, x_t \in D_t,\ t = 0, \dots, T-1,\ x_T = 0\}$ and $\Delta x_t := x_t - x_{t-1}$. We always define $x_{-1} = 0$ so the elements of $\mathcal{N}_D$ describe trading strategies that start and end at liquidated positions. The functions $\mathcal{V}_t$ measure the *disutility* (regret, loss, ...) caused by the net expenditure $S_t(\Delta x_t) + c_t$ of updating the portfolio and paying out the claim c_t at time t. We allow $\mathcal{V}_t$ to be extended real-valued but assume that $\mathcal{V}_t(0) = 0$. We interpret the value of $\mathcal{V}_t(S_t(\Delta x_t) + c_t)$ as $+\infty$ unless $S_t(\Delta x_t) \in L^0(\Omega, \mathcal{F}_t, P)$. In other words, the formulation of (ALM) includes the implicit constraint that $\Delta x_t \in \operatorname{dom} S_t$ almost surely.

Problem (ALM) can be interpreted as an *asset-liability management* problem where one looks for a trading strategy whose proceeds fit the liabilities $c \in \mathcal{M}$ optimally as measured by the "disutility functions" $\mathcal{V}_t$. Despite the simple appearance, (ALM) covers many more familiar instances of portfolio optimization problems. In particular, when[3] $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$, we can write it as

$$\begin{array}{lll} \text{minimize} & \mathcal{V}_T(S_T(\Delta x_T) + c_T) \quad \text{over} \quad x \in \mathcal{N}_D \\ \text{subject to} & S_t(\Delta x_t) + c_t \le 0, \quad t = 0, \dots, T-1. \end{array} \tag{1}$$

This is an illiquid version of the classical utility maximization problem.

Example 1 (Numeraire and stochastic integration) *Assume, as e.g. in Çetin and Rogers [5] and Czichowsky et al. [7], that there is a perfectly liquid asset (numeraire), say* $0 \in J$*, such that*

$$S_t(x) = x^0 + \tilde{S}_t(\tilde{x}) \quad \textit{and} \quad D_t = \mathbb{R} \times \tilde{D}_t,$$

[2]A function $\mathcal{V}_t : L^0(\Omega, \mathcal{F}_t, P) \to \overline{\mathbb{R}}$ is nondecreasing if $\mathcal{V}_t(c_t^1) \le \mathcal{V}_t(c_t^2)$ whenever $c_t^1 \le c_t^2$ almost surely.

[3]Here and in what follows, δ_C denotes the *indicator function* of a set C: $\delta_C(x)$ equals 0 or $+\infty$ depending on whether $x \in C$ or not.

where $\tilde{S}$ and $\tilde{D}$ are the cost process and the constraints for the remaining risky assets $\tilde{J} = J \setminus \{0\}$. We can then use the budget constraint in (1) *to substitute out the numeraire from the problem. Indeed, defining*

$$x_t^0 = x_{t-1}^0 - \tilde{S}_t(\Delta\tilde{x}_t) - c_t \quad t = 0, \dots, T-1,$$

the budget constraint holds as an equality for $t = 1, \dots, T-1$ and

$$x_{T-1}^0 = -\sum_{t=0}^{T-1} \tilde{S}_t(\Delta\tilde{x}_t) - \sum_{t=0}^{T-1} c_t.$$

Substituting this in the objective (and recalling that $x_T = 0$ for $x \in \mathcal{N}_D$), problem (1) *becomes*

$$\text{minimize} \qquad \mathcal{V}_T\left(\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) + \sum_{t=0}^{T} c_t\right) \quad \text{over} \quad x \in \mathcal{N}_D,$$

which shows that in the presence of a numeraire, the timing of payments is irrelevant. This is the problem studied in [5] in the case of strictly convex $\tilde{S}_t$ and in [7] in the case of sublinear $\tilde{S}_t$. In linear market models with $\tilde{S}_t(\tilde{x}, \omega) = \tilde{s}_t(\omega) \cdot \tilde{x}_t$, we can express the accumulated trading costs as a stochastic integral,

$$\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) = \sum_{t=0}^{T} \tilde{s}_t \cdot \Delta\tilde{x}_t = -\sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1}.$$

We then recover constrained discrete-time versions of the utility maximization problems studied e.g. in [23, 9, 24, 2, 3]. In [23, 24, 2], the financial position of the agent was described solely in terms of an initial endowment $w \in \mathbb{R}$ without future liabilities. This corresponds to $c_0 = -w$ and $c_t = 0$ for $t > 0$.

We will denote the optimal value of problem (ALM) by

$$\varphi(c) := \inf_{x \in \mathcal{N}_D} \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t).$$

It is easily verified that φ is a *convex* function on $\mathcal{M}$. The value function φ has a central role in the valuation of financial contracts with multiple payout dates; see [34]. In the completely risk averse case where $\mathcal{V}_t = \delta_{L^0_-}$ for all $t = 0, \dots, T$, we get $\varphi = \delta_{\mathcal{C}}$, where

$$\mathcal{C} = \{c \in \mathcal{M} \mid \exists x \in \mathcal{N}_D : \; S_t(\Delta x_t) + c_t \le 0 \quad \forall t\}$$

is the set of claim processes that can be *superhedged* without a cost; see [33] for further discussion and references on superhedging. On the other hand, since the functions $\mathcal{V}_t$ are nondecreasing, we can write φ as the infimal convolution

$$\varphi(c) = \inf_{d \in \mathcal{M}} \left\{ \sum_{t=0}^{T} \mathcal{V}_t(d_t) \;\middle|\; c - d \in \mathcal{C} \right\} \tag{2}$$
$$= \inf_{d \in \mathcal{C}} \sum_{t=0}^{T} \mathcal{V}_t(c_t - d_t),$$

of $\delta_{\mathcal{C}}$ and the function

$$\mathcal{V}(d) = \sum_{t=0}^{T} \mathcal{V}_t(d_t)$$

as is seen by first writing (ALM) in the form

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{t=0}^{T} \mathcal{V}_t(d_t) \quad \text{over} \quad x \in \mathcal{N},\ d \in \mathcal{M} \\ \text{subject to} & S_t(\Delta x_t) + c_t \leq d_t, \quad x_t \in D_t. \end{array} \tag{3}$$

The variable d may be interpreted as investments the agent makes to his portfolio over time. Alternatively, one may interpret $-c$ and $-d$ as endowment and consumption, respectively, in the spirit of the optimal consumption-investment problem of Karatzas and Žitković [21] in liquid markets in continuous time.

3 Duality

Prices of financial products are often expressed in terms of dual variables of one kind or another. Prices of bonds can be expressed in terms of *zero curves*, which represent the time-value of money. In models with a cash-account, on the other hand, prices of random cash-flows are often expressed in terms of *martingale measures*. In illiquid markets without a cash-account, one needs more general dual variables that encompass both the time value of money as well as the randomness. This section derives a dual representation for the value function φ of (ALM) in terms of such variables. When specialized to more traditional market models, we recover (discrete-time versions of) some well-known duality results for portfolio optimization. Pricing formulas for contingent claims can then be derived from the general relationships between the optimal investment and contingent claim valuation; see [34].

Our strategy is to embed (ALM) in the general stochastic optimization duality framework of [31] which is essentially an instance of the classical conjugate duality framework of Rockafellar [38]. The bilinear form

$$\langle c, y \rangle := E \sum_{t=0}^{T} c_t y_t$$

puts the space

$$\mathcal{M}^1 := \{(c_t)_{t=0}^T \,|\, c_t \in L^1(\Omega, \mathcal{F}_t, P)\}$$

of integrable sequences of cash-flows in separating duality with the space

$$\mathcal{M}^\infty := \{(y_t)_{t=0}^T \,|\, y_t \in L^\infty(\Omega, \mathcal{F}_t, P)\}$$

of essentially bounded adapted processes. Given a convex function f on $\mathcal{M}^1$, its *conjugate* on $\mathcal{M}^\infty$ is defined by

$$f^*(y) = \sup_{c \in \mathcal{M}^1} \{\langle c, y \rangle - f(c)\}.$$

Being the pointwise supremum of continuous linear functions, f^* is convex and lower semicontinuous with respect to the weak topology $\sigma(\mathcal{M}^\infty, \mathcal{M}^1)$. The classical biconjugate theorem says that if f is proper and lower semicontinuous with respect to the L^1-norm, then it has the dual representation

$$f(c) = \sup_{y \in \mathcal{M}^\infty} \{\langle c, y\rangle - f^*(y)\}; \tag{4}$$

see Moreau [27]. This abstract formula is behind many fundamental duality results in mathematical finance. In order to apply it to (ALM), we will first establish the lower semicontinuity of the value function φ.

We will assume from now on that

$$\mathcal{V}_t(c_t) = Ev_t(c_t) := \int v_t(c_t(\omega), \omega) dP(\omega),$$

where v_t is an $\mathcal{F}_t$-measurable normal integrand on $\mathbb{R} \times \Omega$ such that $v_t(\cdot, \omega)$ is proper, nondecreasing and convex with $v_t(0, \omega) = 0$ for every $\omega \in \Omega$. As usual, we define the integral of a measurable function as $+\infty$ unless the positive part of the function is integrable.

Given a market model (S, D), we obtain another market model (S^∞, D^∞) by defining $S_t^\infty(\cdot, \omega)$ and $D_t^\infty(\omega)$ pointwise as the recession function and recession cone of $S_t(\cdot, \omega)$ and $D_t(\omega)$, respectively. When S is sublinear and D is conical, we simply have $(S, D) = (S^\infty, D^\infty)$. By [37, Corollary 8.3.2 and Theorem 8.5],

$$S_t^\infty(x, \omega) = \sup_{\alpha > 0} \frac{S_t(\alpha x, \omega)}{\alpha},$$

$$D_t^\infty(\omega) = \bigcap_{\alpha > 0} \alpha D_t(\omega).$$

The required measurability properties hold by [39, Exercises 14.54 and 14.21] while the convexity and topological properties come directly from the definitions. The following is derived in [34] from a more general result of [35] on stochastic optimization.

Theorem 2 *Assume that* $\{x \in \mathcal{N}_{D^\infty} \,|\, S_t^\infty(\Delta x_t) \leq 0\}$ *is a linear space and that* $v_t(\cdot, \omega)$ *are nonconstant functions with* $v_t \geq m$ *for some integrable function* $m \in L^1$. *Then the value function* φ *is a proper lower semicontinuous convex function on* $\mathcal{M}^1$ *and the infimum in* (ALM) *is attained for every* $c \in \mathcal{M}^1$.

The linearity condition in Theorem 2 is a generalization of the *no-arbitrage* condition in classical perfectly liquid markets; see [32, Section 4]. In particular, when $S_t(x) = s_t \cdot x$ and $D_t \equiv \mathbb{R}^J$, the linearity condition means that any $x \in \mathcal{N}_D$ with $s_t \cdot \Delta x \leq 0$ almost surely for all t has $s_t \cdot \Delta x = 0$ almost surely for all t, i.e. there are no self-financing trading strategies that generate nonzero revenue. This is exactly the no-arbitrage condition. In nonlinear market models, the linearity condition is implied by the so called "robust no-arbitrage" condition; see [32, Section 4]. In the most risk averse case where $v_t = \delta_{\mathbb{R}_-}$ for every t, Theorem 2 says that the set $\mathcal{C}$ is closed. In the classical linear model of Example 1, we thus recover the key closedness result of Schachermayer [42, Lemma 2.1]. The linearity condition holds also if $D_t^\infty(\omega) \cap \{x \in \mathbb{R}^J \,|\, S_t^\infty(x,\omega) \leq 0\} = \{0\}$ almost surely for every t. Indeed, since $x_{-1} = 0$, by definition, this condition implies $\{x \in \mathcal{N}_{D^\infty} \,|\, S_t^\infty(\Delta x_t) \leq 0\} = \{0\}$. This certainly holds if $D_t^\infty \subseteq \mathbb{R}_+^J$ and $\{x \in \mathbb{R}^J \,|\, S_t^\infty(x,\omega) \leq 0\} \cap \mathbb{R}_+^J = \{0\}$. Here, the first condition means that infinite short positions are prohibited while the second means that there are no completely worthless assets.

Nonconstancy of $v_t(\cdot,\omega)$ in Theorem 2 simply means that the investor always prefers more money to less. The lower bound is a more significant restriction since it excludes e.g. the logarithmic utility. In Kramkov and Schachermayer [23] and Rasonyi and Stettner [36] such bounds were avoided but, at present, it is unclear if the lower bound can be relaxed in illiquid markets. However, Guasoni [13, Theorem 5.2] gives the existence of optimal solutions in a continuous time model with a cash-account and proportional transaction with similar conditions on the utility function as in [23] and [36]. Much like in [36] and the proof of Theorem 2, his approach was based on the "direct method" rather than duality arguments as e.g. in [23].

While Theorem 2 establishes the validity of the dual representation for the value function φ, its conjugate φ^* can be expressed in terms of the support function

$$\sigma_{\mathcal{C}}(y) = \sup_{c \in \mathcal{M}^1} \{\langle c, y\rangle \,|\, c \in \mathcal{C}\}$$

of $\mathcal{C}$ and the conjugates

$$v_t^*(y,\omega) = \sup_{c \in \mathbb{R}} \{cy - v_t(c,\omega)\}$$

of the disutility functions v_t as follows.

Lemma 3 *The conjugate of φ can be expressed as*

$$\varphi^*(y) = \sigma_{\mathcal{C}}(y) + E\sum_{t=0}^{T} v_t^*(y_t).$$

Proof. Problem (ALM) can be written as

$$\text{minimize} \quad Ef(x,d,c) \quad \text{over} \quad x \in \mathcal{N},\ d \in \mathcal{M},$$

where $f : (\mathbb{R}^J)^{T+1} \times \mathbb{R}^{T+1} \times \mathbb{R}^{T+1} \times \Omega \to \overline{\mathbb{R}}$ is defined

$$f(x,d,c,\omega) = \begin{cases} \sum_{t=0}^{T} v_t(d_t,\omega) & \text{if } S_t(\Delta x_t,\omega) + c_t \leq d_t,\ x_t \in D_t(\omega),\ x_T = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Since v_t and S_t are normal integrands and D_t are measurable, it follows from Theorem 14.36 and Proposition 14.44 of [39] that f is a normal integrand. We are thus in the general stochastic optimization framework of [31] so, by [31, Theorem 2.2],

$$-\varphi^*(y) = \inf_{x\in\mathcal{N},d\in\mathcal{M}} El(x,d,y), \tag{5}$$

where $l(x,d,y,\omega) = \inf_{c\in\mathbb{R}^{T+1}}\{f(x,d,c,\omega) - \sum_{t=0}^T c_t y_t\}$. We can write l as

$$l(x,d,y,\omega) = \begin{cases} +\infty & \text{unless } x \in X(\omega), \\ \sum_{t=0}^T [v_t(d_t,\omega) + y_t S_t(\Delta x_t,\omega) - d_t y_t] & \text{if } x \in X(\omega) \text{ and } y \geq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

where $X(\omega) = \{x \in (\mathbb{R}^J)^{T+1} \,|\, \Delta x_t \in \operatorname{dom} S_t(\cdot,\omega),\ x_t \in D_t(\omega)\ \forall t\}$. Unless $y \geq 0$ almost surely, the infimum in (5) equals $-\infty$ (it is attained e.g. by $(x,d) = (0,0)$). For $y \geq 0$,

$$\begin{aligned} -\varphi^*(y) &= \inf_{x\in\mathcal{N},d\in\mathcal{M}} \left\{ E\sum_{t=0}^T [y_t S_t(\Delta x_t) + v_t(d_t) - d_t y_t] \,\middle|\, x \in X \right\} \\ &= \inf_{x\in\mathcal{N}} \left\{ E\sum_{t=0}^T y_t S_t(\Delta x_t) \,\middle|\, x \in X \right\} + \inf_{d\in\mathcal{M}} E\sum_{t=0}^T [v_t(d_t) - d_t y_t] \\ &= \inf_{x\in\mathcal{N}} \left\{ E\sum_{t=0}^T y_t S_t(\Delta x_t) \,\middle|\, x \in X \right\} - E\sum_{t=0}^T v_t^*(y_t), \end{aligned}$$

where the second equality holds since we can restrict the minimization, without affecting the infimum, to those $x \in \mathcal{N}$ and $d \in \mathcal{M}$ for which both integrands have integrable positive parts. The last equality comes from the interchange rule for normal integrands; see e.g. [39, Theorem 14.60]. A similar argument in the case $v_t \equiv \delta_{\mathbb{R}_-}$, shows that

$$-\sigma_{\mathcal{C}}(y) = \inf_{x\in\mathcal{N}} \left\{ E\sum_{t=0}^T y_t S_t(\Delta x_t) \,\middle|\, x \in X \right\}$$

for $y \geq 0$ while $\sigma_{\mathcal{C}}(y) = \infty$ for $y \not\geq 0$. □

The form of the conjugate in Lemma 3 is not surprising given the expression (2) of φ as the infimal convolution of $\delta_{\mathcal{C}}$ and $\mathcal{V}$. However, the infimal convolution is taken in the space $\mathcal{M}$ of all adapted claim processes, not over $\mathcal{M}^1$. One could, of course, concentrate on integrable claim processes from the beginning and study the function

$$\tilde{\varphi}(c) = \inf_{d\in\mathcal{M}^1} \left\{ E\sum_{t=0}^T v_t(d_t) \,\middle|\, c - d \in \mathcal{C} \right\}$$

which has the same conjugate and essentially the same economic interpretation as φ. However, the proof of Theorem 2 breaks down if $d \in \mathcal{M}$ is restricted to $\mathcal{M}^1$

and we do not know whether $\tilde{\varphi}$ is lower semicontinuous under the assumptions of Theorem 2.

Combining Theorem 2 and Lemma 3 with the biconjugate theorem gives the following.

Theorem 4 *Under assumptions of Theorem 2, the value function of* (ALM) *has the dual representation*

$$\varphi(c) = \sup_{y \in \mathcal{M}^\infty} \left\{ \langle c, y \rangle - \sigma_{\mathcal{C}}(y) - E \sum_{t=0}^{T} v_t^*(y_t) \right\}.$$

In general, the supremum in the above expression need not be attained; see Remark 7 below. When $\mathcal{C}$ is a cone, the dual representation can be written

$$\varphi(c) = \sup_{y \in \mathcal{C}^*} \left\{ \langle c, y \rangle - E \sum_{t=0}^{T} v_t^*(y_t) \right\},$$

where $\mathcal{C}^* := \{y \in \mathcal{M}^\infty \,|\, \langle c, y \rangle \leq 0 \ \forall c \in \mathcal{C} \cap \mathcal{M}^1\}$ is the polar cone of $\mathcal{C}$. This is similar in form with the primal formulation

$$\varphi(c) = \inf_{d \in \mathcal{C}} \left\{ E \sum_{t=0}^{T} v_t(c_t - d_t) \right\}.$$

Example 5 *Much of duality theory in optimal investment has studied the optimum value as a function of the initial endowment only; see e.g. Kramkov and Schachermayer [23] or Klein and Rogers [22]. The function $V(c_0) = \varphi(c_0, 0, \dots, 0)$ gives the optimum value of* (ALM) *for an agent with initial capital $-c_0$ and no future liabilities/endowments. Using* (2)*, we can write this as*

$$V(c_0) = \inf_{d \in \mathcal{C}(c_0)} E \sum_{t=0}^{T} v_t(d_t), \tag{6}$$

where the set $\mathcal{C}(c_0) = \{d \in \mathcal{M} \,|\, (c_0, 0, \dots, 0) - d \in \mathcal{C}\}$ consists of sequences of investments needed to finance a riskless trading strategy when starting with initial capital $-c_0$. If φ is proper and lower semicontinuous (see Theorem 2), the biconjugate theorem gives

$$V(c_0) = \sup_{y \in \mathcal{M}^\infty} \{c_0 y_0 - \varphi^*(y)\} = \sup_{y_0 \in \mathbb{R}} \{c_0 y_0 - U(y_0)\}$$

where

$$U(y_0) = \inf_{z \in \mathcal{M}^\infty} \{\varphi^*(z) \,|\, z_0 = y_0\}.$$

If $\mathcal{C}$ is conical, we can use Lemma 3 to write U analogously to (6) *as*

$$U(y_0) = \inf_{z \in \mathcal{Y}(y_0)} E \sum_{t=0}^{T} v_t^*(z_t),$$

where $\mathcal{Y}(y_0) = \{z \in \mathcal{C}^* \,|\, z_0 = y_0\}$. In general, there is no reason to believe that U is lower semicontinuous nor that the infimum in its definition is attained. In some cases, however, it is possible to enlarge the set $\mathcal{Y}(y_0)$ so that the function U becomes lower semicontinuous and the infimum is attained; see [7, Section 4] for details.

In order to relate the above to more familiar duality results in optimal investment, we need an explicit expression for the support function $\sigma_{\mathcal{C}}$. A cost process S is said to be *integrable* if $S_t(x,\cdot)$ is integrable for every $x \in \mathbb{R}^J$ and $t = 0,\dots,T$. Integrability implies that $\operatorname{dom} S_t(\cdot,\omega) = \mathbb{R}^J$ almost surely[4]. A linear cost process $S_t(x,\omega) = s_t(\omega)\cdot x$ is integrable if and only if s_t are integrable. The following result, where $\mathcal{N}^1$ denotes the space of integrable $\mathbb{R}^J$-valued adapted processes and E_t the conditional expectation with respect to $\mathcal{F}_t$, is from [30, Lemma A1]. Its proof is based on the classical Fenchel–Moreau–Rockafellar duality theorem.

Lemma 6 *If S is integrable, then*

$$\sigma_{\mathcal{C}}(y) = \inf_{z\in\mathcal{N}^1}\left\{\sum_{t=0}^{T} E(y_tS_t)^*(z_t) + \sum_{t=0}^{T-1} E\sigma_{D_t}(E_t\Delta z_{t+1})\right\}$$

for every $y \in \mathcal{M}^\infty_+$ while $\sigma_{\mathcal{C}}(y) = +\infty$ for $y \notin \mathcal{M}^\infty_+$. Moreover, the infimum is attained for every $y \in \mathcal{M}^\infty_+$.

If S is sublinear and integrable (so that $\operatorname{dom} S_t = \mathbb{R}^J$ and S_t^* is the indicator of its domain), we get $(y_tS_t)^*(z) = \delta(z \,|\, y_t \operatorname{dom} S_t^*)$. If in addition, D is conical, we have $\sigma_{D_t} = \delta_{D_t^*}$ and

$$\sigma_{\mathcal{C}} = \delta_{\mathcal{C}^*},$$

where, by Lemma 6,

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty_+ \,|\, \exists z \in \mathcal{N}^1 :\ z_t \in y_t \operatorname{dom} S_t^*,\ E_t[\Delta z_{t+1}] \in D_t^*\}.$$

The polar cone of $\mathcal{C}$ can also be written as[5]

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty_+ \,|\, \exists s \in \mathcal{N} :\ s_t \in \operatorname{dom} S_t^*,\ E_t[\Delta(y_{t+1}s_{t+1})] \in D_t^*\}.$$

In unconstrained linear models with $S_t(x,\omega) = s_t(\omega)\cdot x$ and $D_t(\omega) = \mathbb{R}^J$, we get $\operatorname{dom} S_t^* = \{s_t\}$ and $D_t^*(\omega) = \{0\}$ so that

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty_+ \,|\, ys \text{ is a martingale}\},$$

while in models with bid-ask spreads, we get $\operatorname{dom} S_t^* = [s_t^-, s_t^+]$ and

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty_+ \,|\, \exists s \in \mathcal{N} :\ s_t^- \le s_t \le s_t^+,\ ys \text{ is a martingale}\}.$$

When short selling is prohibited, i.e. when $D_t = \mathbb{R}^J_+$, the condition $E_t[\Delta(y_{t+1}s_{t+1})] \in D_t^*$ means that ys is a supermartingale.

[4]Integrability implies $P(S_t(x,\cdot) \in \mathbb{R}) = 1$ for every $x \in \mathbb{R}^J$ so $P(S_t(x,\cdot) \in \mathbb{R}\ \forall x \in \mathbb{Q}^J) = 1$, where $\mathbb{Q}$ denotes the rational numbers. Since $S_t(\cdot,\omega)$ are convex, this implies $P(S_t(x,\omega) \in \mathbb{R}\ \forall x \in \mathbb{R}^J) = 1$.

[5]Here we use the fact that the integrability of S implies that every selector of $\operatorname{dom} S_t^*$ is integrable.

Remark 7 (Shadow prices) *In unconstrained models with bid-ask spreads, the dual representation of the value function can be written*

$$\varphi(c) = \sup_{y\in\mathcal{M}^\infty}\left\{\langle c,y\rangle - E\sum_{t=0}^{T} v_t^*(y_t)\,\middle|\, \exists s\in\mathcal{N}:\ s_t^- \le s_t \le s_t^+,\ ys \text{ is a martingale}\right\}.$$

If the supremum is attained, the corresponding $s\in\mathcal{N}$ has the property that the optimum value of (ALM) *is not affected if trading costs are reduced by replacing S by the linear cost functions*

$$\tilde{S}_t(x,\omega) = s_t(\omega)\cdot x.$$

Indeed, the corresponding value function $\tilde{\varphi}$ satisfies

$$\begin{aligned}\varphi(c) &\ge \tilde{\varphi}(c)\\ &\ge \sup_{y\in\mathcal{M}^\infty}\{\langle c,y\rangle - \tilde{\varphi}^*(y)\}\\ &= \sup_{y\in\mathcal{M}^\infty}\left\{\langle c,y\rangle - E\sum_{t=0}^{T} v_t^*(y_t)\,\middle|\, ys \text{ is a martingale}\right\}.\end{aligned}$$

where the first inequality holds because $S\ge\tilde{S}$ and the equality comes from Lemma 3. If φ is proper and lower semicontinuous and if the supremum in its dual representation is attained by a $y\in\mathcal{M}^\infty$ and s, then the last supremum equals $\varphi(c)$. In [7], price processes s such that $\varphi(c)=\tilde{\varphi}(c)$ are called shadow prices. *If the disutility functions v_t are strictly increasing, then in the presence of a shadow price s, the optimal solution $x\in\mathcal{N}_D$ of* (ALM)*, which exists under the assumptions of Theorem 2, must satisfy the complementarity conditions*

$$\Delta x_t > 0 \implies s_t = s_t^+ \quad \text{and} \quad \Delta x_t < 0 \implies s_t = s_t^-$$

since otherwise x would achieve strictly lower trading costs and thus, a strictly lower objective value in the model with the linear cost functions $\tilde{S}_t$. Section 3 of [7,] gives an example where shadow prices do not exist and thus, the supremum in the dual representation of φ is not attained.

In the presence of a numeraire (see Example 1), the elements of $\mathcal{C}^*$ can be expressed in terms of probability measures.

Corollary 8 (Numeraire and martingale measures) *Assume that S is integrable and that*

$$S_t(x,\omega) = x^0 + \tilde{S}_t(\tilde{x},\omega) \quad \text{and} \quad D_t(\omega) = \mathbb{R}\times\tilde{D}_t(\omega),$$

where $\tilde{S}$ is sublinear and $\tilde{D}$ is conical. Then[6]

$$\mathcal{C}^* = \operatorname{pos}\{y\in\mathcal{M}_+^\infty \,|\, \exists Q\in\mathcal{P}: y_t = E_t\frac{dQ}{dP}\}$$

where $\mathcal{P} = \{Q \ll P \,|\, \exists\tilde{s}\in\tilde{\mathcal{N}}:\ \tilde{s}_t\in\operatorname{dom}\tilde{S}_t^*,\ E_t^Q\Delta\tilde{s}_{t+1}\in\tilde{D}_t^*\ Q\text{-a.s.}\}$ and $\tilde{\mathcal{N}}$ denotes the set of adapted $\mathbb{R}^{J\setminus\{0\}}$-valued processes.

[6]For a subset C of a vector space, $\operatorname{pos} C := \{\alpha x \,|\, \alpha\ge 0,\ x\in C\}$.

Proof. We get

$$\operatorname{dom} S_t = \{1\} \times \operatorname{dom} \tilde{S}_t^* \quad \text{and} \quad D_t^* = \{0\} \times \tilde{D}_t^*$$

so, by Lemma 6, $y \in \mathcal{C}^*$ iff y is a nonnegative martingale and $E_t[\Delta(y_{t+1}\tilde{s}_{t+1})] \in \tilde{D}_t^*$ for some $(\mathcal{F}_t)_{t=0}^T$-adapted selector $\tilde{s}$ of $\operatorname{dom} \tilde{S}^*$ such that $y\tilde{s}$ is integrable. Clearly, $\mathcal{C}^* = \operatorname{pos}\{y \in \mathcal{C}^* \,|\, Ey_T = 1\}$. If $y \in \mathcal{C}^*$ with $Ey_T = 1$, then y_T is the density of a probability measure $Q \ll P$ and, by [12, Proposition A.12],

$$E_t^Q[\Delta\tilde{s}_{t+1}] = \frac{E_t[y_T \Delta\tilde{s}_{t+1}]}{E_t y_T} = \frac{E_t[\Delta(y_{t+1}\tilde{s}_{t+1})]}{E_t y_T} \quad Q\text{-a.s.}$$

Since $\tilde{D}_t^*$ is a cone, this implies $E_t^Q[\Delta\tilde{s}_{t+1}] \in \tilde{D}_t^*$ Q-almost surely. Thus, $\{y \in \mathcal{C}^* \,|\, Ey_T = 1\} \subseteq \{y \in \mathcal{M}_+^\infty \,|\, \exists Q \in \mathcal{P} : y_t = E_t \frac{dQ}{dP}\}$. Conversely, if $y_t = E_t[dQ/dP]$ for some $Q \in \mathcal{P}$, we get similarly that

$$E_t[y_T \Delta\tilde{s}_{t+1}] \in \tilde{D}_t^* \quad Q\text{-a.s..}$$

Since $E_t[y_T \Delta\tilde{s}_{t+1}] = 0$ P-almost surely on any $A \in \mathcal{F}_t$ with $Q(A) = 0$ and since $0 \in \tilde{D}_t^*$, we get

$$E_t[y_T \Delta\tilde{s}_{t+1}] \in \tilde{D}_t^* \quad P\text{-a.s.}$$

and thus, $y \in \mathcal{C}^*$. We thus have

$$\{y \in \mathcal{C}^* \,|\, Ey_T = 1\} = \{y \in \mathcal{M}_+^\infty \,|\, \exists Q \in \mathcal{P} : y_t = E_t \frac{dQ}{dP}\}$$

which completes the proof. □

In the classical linear model with $S_t(x,\omega) = s_t(\omega) \cdot x$ and $D_t = \mathbb{R}^J$, we have $\operatorname{dom} S_t^* = \{s_t\}$ and $D_t^* = \{0\}$ so the set $\mathcal{P}$ in Corollary 8 becomes the set of absolutely continuous martingale measures. In unconstrained models with bid-ask spreads, $\mathcal{P}$ is the set of absolutely continuous probability measures Q such that there exists a Q-martingale s with $s_t^- \leq s_t \leq s_t^+$ for all t.

Example 9 *In the setting of Corollary 8, the dual representation of φ in Theorem 4 can be written as*

$$\begin{aligned}
\varphi(c) &= \sup_{y \in \mathcal{C}^*} E \sum_{t=0}^T [c_t y_t - v_t^*(y_t)] \\
&= \sup_{\lambda \geq 0} \sup_{Q \in \mathcal{P}} E \sum_{t=0}^T \left\{ c_t \lambda E_t \frac{dQ}{dP} - \sum_{t=0}^T v_t^*(\lambda E_t \frac{dQ}{dP}) \right\} \\
&= \sup_{\lambda \geq 0} \sup_{Q \in \mathcal{P}} \left\{ \lambda E^Q \sum_{t=0}^T c_t - E \sum_{t=0}^T v_t^*(\lambda E_t \frac{dQ}{dP}) \right\}.
\end{aligned}$$

This is an illiquid discrete-time version of the dual problem of the optimal consumption problem from Karatzas and Žitković [21] which extends the duality *framework of Kramkov and Schachermayer [23] by allowing consumption of wealth over time. When $v_t = \delta_{\mathbb{R}_-}$ for $t < T$, we obtain a discrete-time version with illiquidity effects of the duality results in Owen and Žitković [29] and Biagini, Frittelli and Graselli [3]. In the exponential case*

$$v_T(c) = \frac{1}{\alpha}(e^{\alpha c} - 1),$$

we get $v_T^*(y) = (y \ln y - y + 1)/\alpha$ and the supremum over λ is attained at

$$\lambda = \exp(E(\alpha \sum_{t=0}^{T} c_t y_t - y_T \ln y_T)).$$

This gives

$$\begin{aligned}\varphi(c) &= \sup_{Q \in \mathcal{P}} \left\{ \frac{1}{\alpha} \exp[\alpha E^Q \sum_{t=0}^{T} c_t - H(Q|P)] - \frac{1}{\alpha} \right\} \\ &= \frac{1}{\alpha} \exp\left[\sup_{Q \in \mathcal{P}} \{\alpha E^Q \sum_{t=0}^{T} c_t - H(Q|P)\} \right] - \frac{1}{\alpha}.\end{aligned}$$

In the linear case $\tilde{S}_t(x) = \tilde{s}_t \cdot x$, this reduces to a discrete-time version of the duality framework of [9]; see also [20] and [1].

4 Conclusions

This paper studied optimal investment in the general conjugate duality framework of convex analysis. This has the benefit of allowing for various generalizations to the classical market models based on the theory of stochastic integration. In particular, the introduction of portfolio constraints and nonlinear illiquidity effects poses no particular problems compared to the classical model of perfectly liquid markets. One could also include dividend payments as proposed in [4]. Separation of dividend payments from "total returns" is important in the presence of transaction costs.

The optimization problem studied in this paper has important applications in accounting and in the valuation of swap contracts and other financial products with multiple payout dates. This has been studied in [34]. The related duality theory will be developed elsewhere.

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