The Trick which Became a Theory
A Brief History of the Replica Method

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The Replica Trick for Disordered Systems

Dirty Tricks Should Not Work ...

Alternative Versions & Real Replicas

Physical Meaning of Replica Dimension

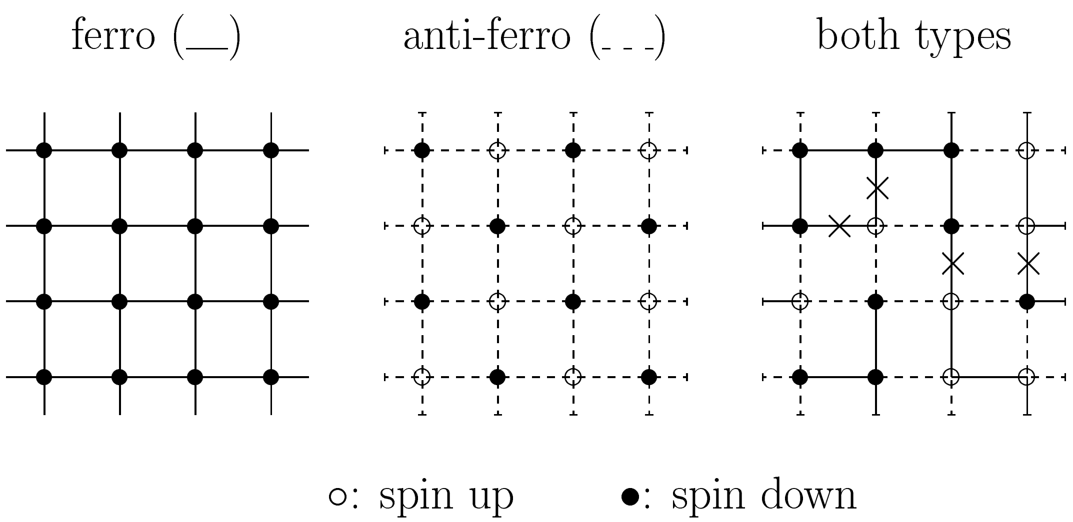
Physics Behind Parisi's RSB Scheme

New Avenues in Replica Theory

I: THE REPLICA TRICK FOR DISORDERED SYSTEMS

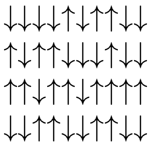
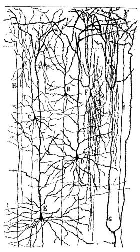
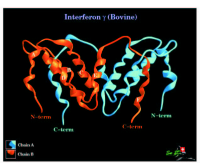
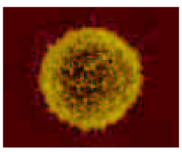
Disordered many-particle systems

- (pseudo-) randomness in microscopic parameters, e.g. interaction energies
- no microscopic periodicity
- high degree of frustration, incompatible forces



- many relevant time-scales, difficult to equilibrate
- many techniques no longer applicable, e.g. transfer matrices, renormalization, ...
- even mean-field (range-free) models are non-trivial

Examples of Disordered Systems

	area	*variables*	*disorder*	*size*
	spin-glasses (impure metals)	spin orientations	RKKY interactions	10^{20}
	neural networks	neuronal firing rates	synapses	10^{10}
	machine learning	programme code	example data	10^{4}
	protein folding	amino-acid orientations	chemical composition	10^{3}
	immunology	lymphocyte concentrations	surface shapes	10^{8}
	market models	trading actions	trading strategies	10^{3}

‘Harmonic Oscillator’ of Disordered Systems Theory

Sherrington-**K**irkpatrick model (1975):

- N Ising spins $\sigma_i \in \{-1, 1\}$,
  indep. random exchange energies J_{ij}

$$H = -\sum_{i<j} J_{ij}\sigma_i\sigma_j, \qquad \overline{J_{ij}} = \frac{J_0}{N}, \qquad \overline{J_{ij}^2} - \overline{J_{ij}}^2 = \frac{J^2}{N}$$

- Let $\boldsymbol{\sigma} = \{\sigma_i\}$, $\boldsymbol{J} = \{J_{ij}\}$, $\beta = 1/kT$,
  free energy per spin:

$$f[\boldsymbol{J}] = -\frac{1}{\beta N}\log Z[\boldsymbol{J}] \qquad Z[\boldsymbol{J}] = \sum_{\boldsymbol{\sigma}} e^{-\beta H}$$

- To be calculated: disorder-averaged free energy

$$\begin{aligned}\overline{f} &= \overline{f[\boldsymbol{J}]} = -\frac{1}{\beta N}\overline{\log Z[\boldsymbol{J}]} \\ &= -\frac{1}{\beta N}\int \left[\prod_{i<j}\frac{dz_{ij}e^{-\frac{1}{2}z_{ij}^2}}{\sqrt{2\pi}}\right]\log\left[\sum_{\boldsymbol{\sigma}} e^{\frac{\beta J_0}{2N}(\sum_i \sigma_i)^2 + \frac{\beta J}{\sqrt{N}}\sum_{i<j} z_{ij}\sigma_i\sigma_j} + \mathcal{O}(\frac{1}{N})\right]\end{aligned}$$

simple ∞-range model, but still trouble !
How to carry out the disorder average ?

The Replica Trick
(Marc Kac, 1968)

- $x^n = 1 + n \log x + \mathcal{O}(n^2)$

$$\overline{\log Z[\boldsymbol{J}]} = \lim_{n \to 0} \frac{1}{n} \log \overline{Z^n[\boldsymbol{J}]}$$

- n integer:

$$Z^n[\boldsymbol{J}] = \left[\sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})} \right]^n = \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}$$

partition function of n *independent* replicas of system

- disorder average:

$$\overline{f} = - \lim_{n \to 0} \frac{1}{\beta N n} \log \left[\sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} e^{-\beta H_{\text{eff}}(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^n)} \right]$$

$$H_{\text{eff}}(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^n) = -\frac{1}{\beta} \log \overline{e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}}$$

n *interacting* replicas, *no* disorder

- exchange limits
  $N \to \infty$ versus $n \to 0$

Application to the SK Model

$$\lim_{N\to\infty} \overline{f} \;=\; \lim_{n\to 0}\; \min\left\{-\frac{1}{\beta}\log 2 + \frac{\beta J^2}{4n}\sum_{\alpha\gamma} q_{\alpha\gamma}^2 + \frac{J_0}{2n}\sum_{\alpha} m_\alpha^2 \right.$$
$$\left. - \frac{1}{\beta n}\log \sum_{\sigma_1,\ldots,\sigma_n} e^{\frac{1}{2}\beta^2 J^2 \sum_{\alpha\gamma} q_{\alpha\gamma}\sigma_\alpha\sigma_\gamma + \beta J_0 \sum_\alpha m_\alpha \sigma_\alpha}\right\}$$

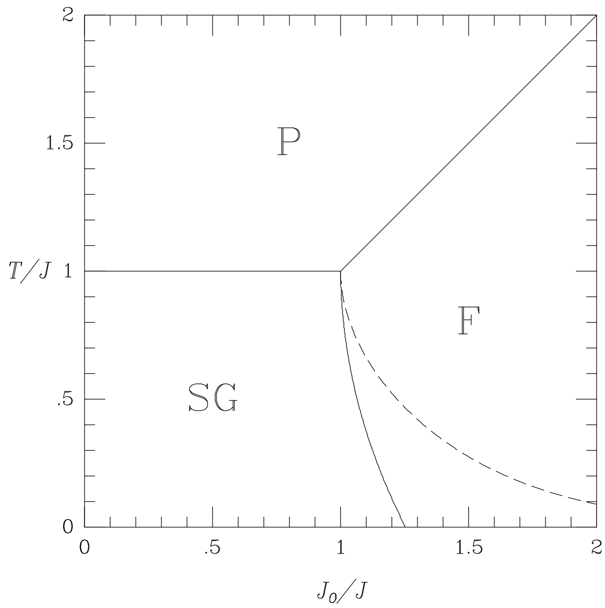
order parameters

$$m_\alpha = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle \sigma_i^\alpha\rangle} \qquad q_{\alpha\beta} = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle \sigma_i^\alpha \sigma_i^\beta\rangle}$$

$$\alpha,\beta = 1,\ldots,n$$

replica-symmetric (RS) ansatz:

$$m_\alpha = m \qquad q_{\alpha\beta} = \delta_{\alpha\beta} + q\,[1-\delta_{\alpha\beta}]$$



$$m = \lim_{N\to\infty} N^{-1}\sum_i \overline{\langle\sigma_i\rangle}$$
$$q = \lim_{N\to\infty} N^{-1}\sum_i \overline{\langle\sigma_i\rangle^2}$$

P : $m = q = 0$

F : $m \neq 0,\; q > 0$

SG : $m = 0,\; q > 0$

II: DIRTY TRICKS SHOULD NOT WORK ...

Problems

- negative entropy at low temperatures
- limits commute in $\overline{f} = \lim_{N\to\infty} \lim_{n\to 0} \frac{-1}{\beta N n} \log \overline{Z^n}$?
- simple identities: $\frac{d}{dn}[n^{-1} \log \overline{Z^n}] \geq 0$, $\frac{d^2}{dn^2} \log \overline{Z^n} \geq 0$
  violated at low temperatures
- saddle-point *not* minimum of free energy,
  $\overline{f} = \max_q \min_m f(m, q)$
- AT instability: Hessian of $f(\{q_{\alpha\beta}, m_\alpha\})$ develops negative eigenvalues at low temperatures

Some answers

- f self-averaging, $\lim_{N\to\infty} \frac{-1}{\beta N} \log Z[\boldsymbol{J}]$ exists
- limits commute: $\overline{f} = -\lim_{n\to 0} \lim_{N\to\infty} \frac{1}{\beta n} \overline{\log Z^{n/N}}$
- curvature sign changes:

  e.g. $f(\{q_{\alpha\beta}\}) = \sum_{\alpha\neq\beta=1}^{n} q_{\alpha\beta}^2 \quad \to \quad \text{saddle point}: \quad q_{\alpha\beta} = 0$

$$\text{let } q_{\alpha\beta} = q \;\; \forall \alpha \neq \beta : \qquad f(\ldots) = n(n-1)q^2$$

$$q = 0 : \quad \text{min for } n > 1, \quad \text{max for } n < 1$$

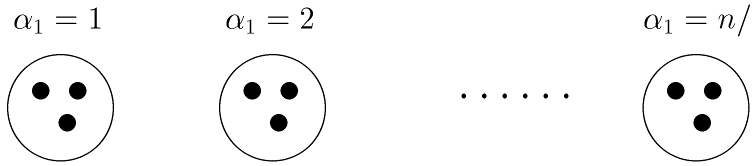
$f(\{q_{\alpha\beta}, m_\alpha\})$ invariant under replica permutations,
saddle-point *not invariant*: Replica Symmetry Breaking

$$q_{\alpha\beta} = \lim_{N\to\infty} N^{-1} \sum_i \overline{\langle \sigma_i^\alpha \sigma_i^\beta \rangle} \qquad \alpha, \beta = 1 \ldots n$$

Parisi scheme

- 1-step RSB: divide n replicas into subsets of equal size m:

$$\alpha \to (\alpha_1, \alpha_2) \qquad \begin{array}{ll} \alpha_1 = 1 \ldots n/m & \text{subset label} \\ \alpha_2 = 1 \ldots m & \text{internal label} \end{array}$$



$$q_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)} = \begin{cases} q_1 & \text{if } \alpha_1 = \beta_1 \quad \text{(same subset)} \\ q_0 & \text{if } \alpha_1 \neq \beta_1 \quad \text{(different subsets)} \end{cases}$$

$$\begin{pmatrix} 1 & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & & q & . & . \\ . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . \\ . & . & q & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & 1 \end{pmatrix} \to \begin{pmatrix} 1 & q_1 & q_1 & . & . & . & . & . & . \\ q_1 & 1 & q_1 & . & . & . & . & q_0 & . \\ q_1 & q_1 & 1 & . & . & . & . & . & . \\ . & . & . & 1 & q_1 & q_1 & . & . & . \\ . & . & . & q_1 & 1 & q_1 & . & . & . \\ . & . & . & q_1 & q_1 & 1 & . & . & . \\ . & . & . & . & . & . & 1 & q_1 & q_1 \\ . & q_0 & . & . & . & . & q_1 & 1 & q_1 \\ . & . & . & . & . & . & q_1 & q_1 & 1 \end{pmatrix}$$

- insert into $f(\{q_{\alpha\beta}, m_\alpha\})$, let $m \in [0, 1]$

$$\overline{f}_{\text{1RSB}} = \text{extr}_{m\in[0,1],q_0,q_1} f(m, q_0, q_1)$$

- Repeat breaking up of diagonal blocks *ad infinitum*

$$\overline{f}_{\infty\text{RSB}} = \text{extr}_{m_0,m_1,\ldots\in[0,1];q_0,q_1,\ldots} f(m_0, m_1, \ldots, q_0, q_1, \ldots)$$

The case for replicas á la Parisi

1. no negative entropy at low temperatures
2. Hessian of $f(\{q_{\alpha\beta}, m_\alpha\})$ has no negative eigenvalues
3. agreement with special cases where alternatives work: spherical models, random energy model, TAP equations, generating functional analysis (RS = FDT)
4. agreement with simulations in many areas (spin-glasses, optimization problems, neural networks, immune networks, market models, Boolean satisfiability problems, error-correcting codes, ...)

$$\begin{array}{ccccccccccc} 0 & \leq & m_0 & \leq & m_1 & \leq & m_2 & \leq & \ldots & \leq & 1 \\ 0 & \leq & q_0 & \leq & q_1 & \leq & q_2 & \leq & \ldots & \leq & 1 \end{array} \quad \rightarrow \quad x \in [0,1],\ \ q(x)$$

$$\begin{array}{rl} \text{RS}: & q(x) = q_0 \\[1ex] \text{1RSB}: & q(x) = q_0 \ \ \text{for } x \in [0, x_1] \\ & q(x) = q_1 \ \ \text{for } x \in [x_1, 1] \\[1ex] \text{2RSB}: & q(x) = q_0 \ \ \text{for } x \in [0, x_1] \\ & q(x) = q_1 \ \ \text{for } x \in [x_1, x_2] \\ & q(x) = q_2 \ \ \text{for } x \in [x_2, 1] \\ \vdots & \qquad \vdots \\ \infty\text{RSB}: & \text{generally continuous pieces} \end{array}$$

The case against replicas á la Parisi

1. physical meaning of function $q(x)$?
2. block sizes: $m_i \in \{1, \ldots, n\}$ replaced by $m_i \in [0, 1]$ ...
3. why monotonicity $m_0 \leq m_1 \leq m_2 \leq \ldots$?
4. physical meaning of replica dimension n ?
5. one takes limit $n \to 0$ by first sending $n \to \infty$ ...
6. $\{q_{\alpha\beta}\} \in \mathbb{R}^{\frac{1}{2}n(n-1)}$, analysis in space with dimension < 0
7. uniqueness of the scheme ?

language barriers ...

'the space of 0×0 *matrices is a very large space'*

(G. Parisi, 1979)

III: ALTERNATIVE VERSION, REAL REPLICAS

$$\frac{\int dx\; e^{-\beta H(x)} g(x)}{\int dx\; e^{-\beta H(x)}} = \lim_{n\to 0} \int \prod_{\alpha=1}^{n} [dx_\alpha e^{-\beta H(x_\alpha)}] g(x_1)$$

Meaning of $q(x)$

Imagine two real system copies $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$

$$\begin{aligned} P(q) &= \frac{\sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \delta\left[q - \frac{1}{N}\sum_i \sigma_i \sigma_i'\right] e^{-\beta H(\boldsymbol{\sigma}) - \beta H(\boldsymbol{\sigma}')}}{\sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} e^{-\beta H(\boldsymbol{\sigma}) - \beta H(\boldsymbol{\sigma}')}} \\ &= \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{\alpha\neq\beta} \sum_{\boldsymbol{\sigma}^1 \ldots \boldsymbol{\sigma}^n} \delta\left[q - \frac{1}{N}\sum_i \sigma_i^\alpha \sigma_i^\beta\right] \prod_{\gamma=1}^{n} e^{-\beta H(\boldsymbol{\sigma}^\gamma)} \end{aligned}$$

$$\overline{P(q)} = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{\alpha\neq\beta} \delta\left[q - q_{\alpha\beta}\right]$$

Consequences

- $\overline{P(q)} = dx/dq$
- interpretation of $m_i \in [0,1]$: consider 1RSB

$$\begin{pmatrix} 1 & q_1 & q_1 & . & . & . & . & . & . \\ q_1 & 1 & q_1 & . & . & . & . & q_0 & . \\ q_1 & q_1 & 1 & . & . & . & . & . & . \\ . & . & . & 1 & q_1 & q_1 & . & . & . \\ . & . & . & q_1 & 1 & q_1 & . & . & . \\ . & . & . & q_1 & q_1 & 1 & . & . & . \\ . & . & . & . & . & . & 1 & q_1 & q_1 \\ . & q_0 & . & . & . & . & q_1 & 1 & q_1 \\ . & . & . & . & . & . & q_1 & q_1 & 1 \end{pmatrix}$$

$\overline{P(q)} = \lim_{n\to 0}$
$\frac{n-m}{n-1}\delta[q-q_0] + \frac{m-1}{n-1}\delta[q-q_1]$

$$\frac{n-m}{n-1} \geq 0 \quad \rightarrow \quad m \geq 0 \qquad\qquad \frac{m-1}{n-1} \geq 0 \quad \rightarrow \quad m \leq 1$$

- replicas and ergodicity:

  assume L ergodic sectors,
  with measures $p_\ell(\boldsymbol{\sigma})$ and probabilities w_ℓ

$$
\begin{aligned}
P(q) &= \lim_{N\to\infty} \langle\langle \delta[q - \frac{1}{N}\sum_i \sigma_i\sigma_i'] \rangle\rangle \\
&= \lim_{N\to\infty} \sum_{\ell\ell'=1}^{L} w_\ell w_{\ell'} \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} p_\ell(\boldsymbol{\sigma}) p_{\ell'}(\boldsymbol{\sigma}')\ \delta[q - \frac{1}{N}\sum_i \sigma_i\sigma_i']
\end{aligned}
$$

  mean field model:

  $\frac{1}{N}\sum_i \sigma_i\sigma_i' = Q_{\ell\ell'} + \mathcal{O}(N^{-\frac{1}{2}})$

$$
\overline{P(q)} = \sum_{\ell\ell'=1}^{L} \overline{w_\ell w_{\ell'}\ \delta[q - Q_{\ell\ell'}]}
$$

  Hence:

  ergodicity = replica symmetry
  finite nr of ergodic sectors: $\overline{P(q)}$ sum of δ-peaks
  infin nr of ergodic sectors: $\overline{P(q)}$ with continuous pieces

IV: PHYSICAL MEANING OF REPLICA DIMENSION n

Adiabatically slow bond dynamics

$$\frac{d}{dt}J_{ij} \;=\; \frac{1}{N}\langle\sigma_i\sigma_j\rangle_{\boldsymbol{J}} + \frac{K}{N} - \mu J_{ij} + \xi_{ij}$$

$$\langle\xi_{ij}(t)\rangle = 0, \quad \langle\xi_{ij}(t)\xi_{k\ell}(t')\rangle = \frac{2}{\tilde{\beta}N}\delta_{(i,j),(k,\ell)}\delta(t-t')$$

each bond J_{ij}:

stochastic motion in harmonic potential
+ frustration removing force $\langle\sigma_i\sigma_j\rangle_{\boldsymbol{J}}$

Note:

$$\frac{d}{dt}J_{ij} \;=\; -\frac{\partial}{\partial J_{ij}}\mathcal{H}[\boldsymbol{J}] + \xi_{ij}$$

$$\mathcal{H}[\boldsymbol{J}] = \frac{1}{N\beta}\log Z[\boldsymbol{J}] - \frac{K}{N}\sum_{i<j} J_{ij} + \frac{1}{2}\mu\sum_{i<j} J_{ij}^2$$

Bond equilibrium:

$$\tilde{f} = -\lim_{N\to\infty}\frac{1}{\tilde{\beta}N}\log\int d\boldsymbol{J}\; e^{-\tilde{\beta}\mathcal{H}[\boldsymbol{J}]}$$

transform parameters:

$$K \to \mu J_0,\ \mu \to 1/\tilde{\beta} J^2,\ \tilde{\beta} \to n\beta$$

$$P(J_{ij}) = (2\pi J^2/N)^{-\frac{1}{2}} e^{-\frac{1}{2}N(J_{ij}-J_0/N)^2/J^2}$$

slowly evolving bonds:

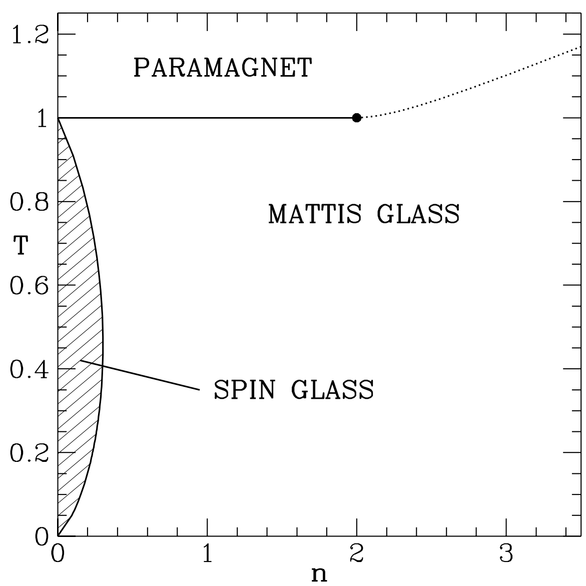
$$\overline{f} \ = \ -\lim_{N\to\infty} \frac{1}{nN\beta} \log \int \prod_{i<j} [dJ_{ij} P(J_{ij})]\, Z^n[\boldsymbol{J}] \ + \ \text{const}$$

random bonds (SK):

$$\tilde{f} \ = \ -\lim_{n\to 0} \lim_{N\to\infty} \frac{1}{nN\beta} \log \int \prod_{i<j} [dJ_{ij} P(J_{ij})]\, Z^n[\boldsymbol{J}] \ + \ \text{const}$$

$n = T/\tilde{T}$, ratio of temperatures

(i.e. ratio of energies, or ratio of time-scales)



phase diagram for $J = 1,\ J_0 = 0$

IV: PHYSICS BEHIND PARISI'S RSB SCHEME

Boltzmann's $\mathcal{H}$-function

$$\mathcal{H}(t) = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \left\{ H(\boldsymbol{\sigma}) + T \log p_t(\boldsymbol{\sigma}) \right\} \qquad \frac{d\mathcal{H}}{dt} \leq 0, \quad \mathcal{H} \geq F$$

Fast and slow spins

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{\rm f}, \boldsymbol{\sigma}_{\rm s}) \qquad p(\boldsymbol{\sigma}_{\rm f}, \boldsymbol{\sigma}_{\rm s}) = p(\boldsymbol{\sigma}_{\rm f}|\boldsymbol{\sigma}_{\rm s}) p(\boldsymbol{\sigma}_{\rm s})$$

$$\begin{aligned} \mathcal{H}(t) &= \sum_{\boldsymbol{\sigma}_{\rm s}} p_t(\boldsymbol{\sigma}_{\rm s}) \left\{ H_{\rm eff}(\boldsymbol{\sigma}_{\rm s}) + T \log p_t(\boldsymbol{\sigma}_{\rm s}) \right\} \\ H_{\rm eff}(\boldsymbol{\sigma}_{\rm s}) &= \sum_{\boldsymbol{\sigma}_{\rm f}} p_t(\boldsymbol{\sigma}_{\rm f}|\boldsymbol{\sigma}_{\rm s}) \left\{ H(\boldsymbol{\sigma}_{\rm f}, \boldsymbol{\sigma}_{\rm s}) + T \log p_t(\boldsymbol{\sigma}_{\rm f}|\boldsymbol{\sigma}_{\rm s}) \right\} \end{aligned}$$

disparate time-scales:

- First, $p(\boldsymbol{\sigma}_{\rm f}|\boldsymbol{\sigma}_{\rm s})$ will evolve to minimize $H_{\rm eff}(\boldsymbol{\sigma}_{\rm s})$:
  $$p(\boldsymbol{\sigma}_{\rm f}|\boldsymbol{\sigma}_{\rm s}) \;\to\; Z_{\rm f}^{-1}(\boldsymbol{\sigma}_{\rm s})\, e^{-\beta H(\boldsymbol{\sigma}_{\rm f}, \boldsymbol{\sigma}_{\rm s})} \qquad Z_{\rm f}(\boldsymbol{\sigma}_{\rm s}) = \sum_{\boldsymbol{\sigma}_{\rm f}} e^{-\beta H(\boldsymbol{\sigma}_{\rm f}, \boldsymbol{\sigma}_{\rm s})}$$
- Second: postulate stationarity of entropy of slow vars, miminize $\mathcal{H}(t)$ with $S_{\rm s} = -\sum_{\boldsymbol{\sigma}_{\rm s}} p(\boldsymbol{\sigma}_{\rm s}) \log p(\boldsymbol{\sigma}_{\rm s})$ constant
  $$p(\boldsymbol{\sigma}_{\rm s}) \;\to\; Z_{\rm s}^{-1} e^{-\tilde{\beta} H_{\rm eff}(\boldsymbol{\sigma}_{\rm s})} \qquad Z_{\rm s} = \sum_{\boldsymbol{\sigma}_{\rm s}} e^{-\tilde{\beta} H_{\rm eff}(\boldsymbol{\sigma}_{\rm s})}$$
- $\tilde{\beta} = \tilde{m}\beta$:
  $$Z_{\rm s} = \sum_{\boldsymbol{\sigma}_{\rm s}} [Z_{\rm f}(\boldsymbol{\sigma}_{\rm s})]^{\tilde{m}} \qquad \tilde{m} : \quad \tilde{m}^2 (\partial F_{\rm s} / \partial \tilde{m}) = S_{\rm s}$$

Generalization to hierarchy of time-scales

$$\begin{aligned} \text{time scales :} \quad & \infty > \tau_0 \gg \ldots \gg \tau_\ell \gg \ldots \gg \tau_L > 0 \\ \text{temperatures :} \quad & \infty > T_0 \geq \ldots \geq \ T_\ell \geq \ldots \geq \ T_L > 0 \end{aligned}$$

ordering of $\{T_\ell\}$: thermodynamic stability
(if not: fast levels become heat baths for slow ones)

- ergodic equilibration at each scale, constrained S_ℓ:

$$Z_L = \sum_{\boldsymbol{\sigma}_L} e^{-\beta_L H(\{\boldsymbol{\sigma}\})} \qquad \ell < L: \quad Z_\ell = \sum_{\boldsymbol{\sigma}_\ell} [Z_{\ell+1}]^{\tilde{m}_{\ell+1}}$$

- $\beta_L = \beta$, $\tilde{m}_\ell = T_\ell / T_{\ell-1}$,

$$\tilde{m}_\ell: \qquad \beta_{\ell+1} \tilde{m}_{\ell+1}^2 \frac{\partial F_\ell}{\partial \tilde{m}_{\ell+1}} = S_\ell, \qquad F_\ell = -T_\ell \log Z_\ell$$

- F_0: generator of observables

$$H(\{\boldsymbol{\sigma}\}) \to H(\{\boldsymbol{\sigma}\}) + \lambda \psi(\{\boldsymbol{\sigma}\}): \qquad \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} F_0 = \langle \psi(\{\boldsymbol{\sigma}\}) \rangle$$

$\tilde{m}_\ell < 1: \ S_\ell > S_{\mathrm{equil}}$,
standard Boltzmann state: $\tilde{m}_\ell = 1$ for all ℓ

Application to SK model

- divide spins in $L+1$ mobility groups of size $|I_\ell| = \epsilon_\ell N$:

$$\{1,\ldots,N\} = \bigcup_{\ell=0}^{L} I_\ell$$

  system can choose $\{\epsilon_\ell\}$

- disorder-averaged slowest free energy:

$$\overline{f_0} = -\lim_{N\to\infty} \frac{1}{\beta_0 N}\overline{\log Z_0} = -\lim_{N\to\infty}\lim_{\tilde{n}\to 0} \frac{1}{N\tilde{n}\beta_0} \log \overline{Z_0^{\tilde{n}}}$$

- nested set of $\tilde{n}\prod_{\ell=1}^{L} \tilde{m}_\ell$ replicas
  spin at level ℓ: carries replica indices $\{a_0, .., a_\ell\}$

  $a_0 \in \{1, .., \tilde{n}\}$: from disorder average
  $a_k \in \{1, .., \tilde{m}_k\}$: from constrained equilibrations

- minimize $\overline{f_0}$ wrt $\{\epsilon_\ell\}$:

  $\epsilon_L = 1, \quad \epsilon_\ell = 0 \ \forall \ell < L$ (vanishing fraction of slow spins)

Result:

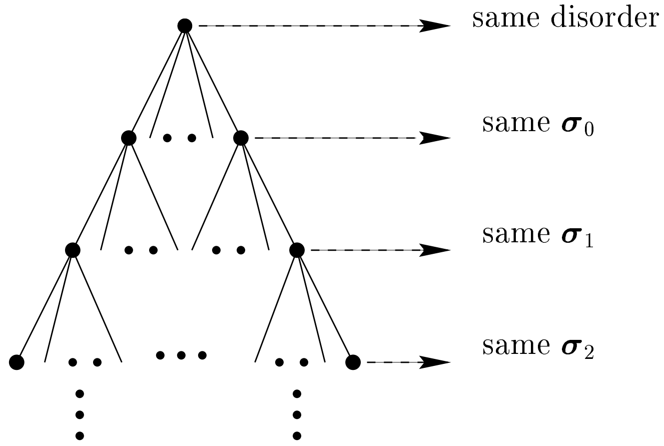
Parisi's L-level RSB scheme,

$$n = \tilde{n}\prod_{\ell=1}^{L}\tilde{m}_\ell, \qquad m_\ell = \prod_{k=\ell}^{L}\tilde{m}_k = T/T_{\ell-1}$$

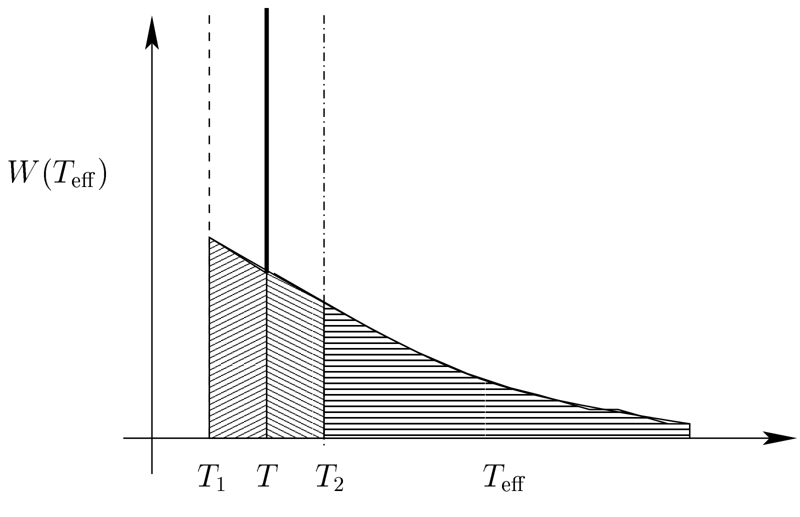
$$q_\ell = \lim_{N\to\infty}\frac{1}{N}\sum_j \overline{\langle \ldots \langle\langle\langle \ldots \langle \sigma_j\rangle_L \ldots\rangle_{\ell+1}\rangle_\ell^2\rangle_{\ell-1}\ldots\rangle_0}$$

$$\epsilon_\ell = 0 \ \rightarrow \ S_\ell/N = 0 \ \rightarrow \ \partial\overline{f_0}/\partial m_\ell = 0$$

Explanations:



- Ultra-metric organization of states in RSB: simple consequence of hierarchy of spin clusters
- $n = \tilde{n} \prod_{\ell=1}^{L} \tilde{m}_\ell$: explains why $n \to 0$ was taken via $n \to \infty$
- $m_\ell = T/T_{\ell-1}$: explains $0 \le m_0 \le m_1 \le m_2 \le \ldots \le 1$ (thermodynamic stability)
- memory effects, thermo-cycling experiments:



(a) cool to $T_1 < T$, then heat back to T:
states of spins with $T_{\rm eff} > T$ unchanged (memory effects)

(b) heat to $T_2 > T$, then cool back to T:
states of spins with $T \le T_{\rm eff} \le T_2$ erased

V: NEW AVENUES IN REPLICA THEORY

- Replicas in dynamics: based on

$$\frac{\int dx\ w_t(x)g(x)}{\int dx\ w_t(x)} = \lim_{n\to 0}\int\prod_{\alpha=1}^{n}[dx_\alpha w_t(x_\alpha)]g(x_1)$$

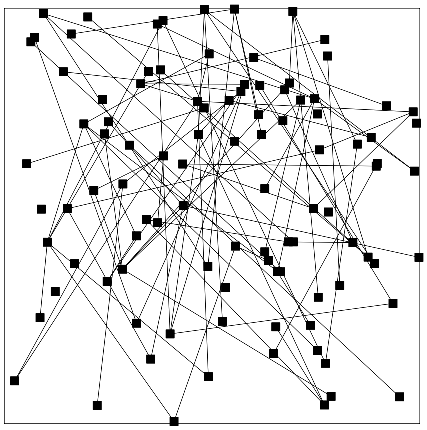
- replica theory of disordered quantum systems
(replicated quantum operators, Trotter formula, etc)
- Finite dimensional disordered systems:
replica field theory (not for the faint-hearted ...)
- Systems on finitely connected random graphs

Order parameter types

Dependence on connectivity c
(average number of bonds/spin)

connectivity	*variables*	*order param*	*RS ansatz*
$c = N$	discrete	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$c = N$	continuous	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$1 \ll c \ll N$	discrete	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$1 \ll c \ll N$	continuous	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$c = \mathcal{O}(1)$	discrete	$P(\sigma_1, \ldots, \sigma_n)$	functions, $P(h)$
$c = \mathcal{O}(1)$	continuous	$P(\sigma_1, \ldots, \sigma_n)$	functionals, $W[\{P\}]$

Soft spin
finite connectivity problems



$N=100,\ c=2$

N spins on random graph, $c_{ij}\in\{0,1\}$

$$H=-\sum_{i<j}c_{ij}J_{ij}\sigma_i\sigma_j+\sum_i V(\sigma_i)$$

- $P(c_{ij})=\frac{c}{N}\delta_{c_{ij},1}+(1-\frac{c}{N})\delta_{c_{ij},0},\quad c=\mathcal{O}(N^0)$
- indep random interaction energies distributed according to $P[J]$
- disorder: $\{c_{ij},J_{ij}\}$

$$\begin{aligned}\overline{f} &= -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta N n}\log\overline{Z^n}\\ &= \ldots\ldots\ldots\\ &= \lim_{n\to 0}\frac{1}{\beta n}\,\mathrm{extr}\Big\{\frac{1}{2}c\int d\boldsymbol{\sigma}d\boldsymbol{\sigma}'\ P(\boldsymbol{\sigma})P(\boldsymbol{\sigma}')[\int dJ\ P(J)e^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}-1]\\ &\qquad -\log\int d\boldsymbol{\sigma}\ e^{c\int d\boldsymbol{\sigma}'\ P(\boldsymbol{\sigma}')[\int dJ\ P(J)e^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}-1]-\beta\sum_\alpha V(\sigma_\alpha)}\Big\}\end{aligned}$$

order parameter:

$$P(\sigma_1,\ldots,\sigma_n)=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\prod_\alpha\delta[\sigma_\alpha-\sigma_i^\alpha]}$$

note:

$$m_\alpha=\int d\boldsymbol{\sigma}\ P(\boldsymbol{\sigma})\sigma_\alpha\qquad q_{\alpha\beta}=\int d\boldsymbol{\sigma}\ P(\boldsymbol{\sigma})\sigma_\alpha\sigma_\beta$$

saddle-point equation:

$$P(\boldsymbol{\sigma}) = \frac{e^{c\int d\boldsymbol{\sigma}'\ P(\boldsymbol{\sigma}')[\int dJ\ P(J)e^{\beta J \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'}-1]-\beta\sum_\alpha V(\sigma_\alpha)}}{\int d\boldsymbol{\sigma}'\ e^{c\int d\boldsymbol{\sigma}''\ P(\boldsymbol{\sigma}'')[\int dJ\ P(J)e^{\beta J \boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}''}-1]-\beta\sum_\alpha V(\sigma'_\alpha)}}$$

Replica symmetry

$$P_{\rm RS}(\boldsymbol{\sigma}) = \int\{dP\}\ W[\{P\}]\ \prod_\alpha P(\sigma_\alpha)$$

RS order parameter: functional $W[\{P\}]$

insert into eqns, take $n \to 0$:

RS saddle-point eqn:

$$\begin{aligned} W[\{P\}] \ &= \ \sum_{\ell\geq 0} \frac{c^\ell}{\ell!} e^{-c} \int \prod_{k\leq \ell} [\{dP_k\} W[\{P_k\}] dJ_k P(J_k)] \\ &\times \prod_\sigma \delta \left[P(\sigma) - \frac{e^{-\beta V(\sigma)} \prod_{k=1}^\ell \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma\sigma'}}{\int d\sigma'' e^{-\beta V(\sigma'')} \prod_{k=1}^\ell \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma''\sigma'}} \right] \end{aligned}$$

RS free energy:

$$\begin{aligned} \overline{f}_{\rm RS} \ &= \ \frac{c}{2\beta} \int \{dP_1 dP_2\}\ W[\{P_1\}] W[\{P_2\}] \int dJ\ P(J) \\ &\qquad \times \log\left[\int d\sigma d\sigma'\ P_1(\sigma) P_2(\sigma') e^{\beta J \sigma\sigma'} \right] \\ &\quad -\frac{1}{\beta} \sum_{\ell\geq 0} \frac{c^\ell}{\ell!} e^{-c} \int \prod_{k=1}^\ell [\{dP_k\} W[\{P_k\}] dJ_k P(J_k)] \\ &\qquad \times \log\left[\int d\sigma\ e^{-\beta V(\sigma)} \prod_{k=1}^\ell \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma\sigma'} \right] \end{aligned}$$

Textbooks

- replica theory a la Parisi and its applications, with many reprints of original papers:
  M Mezard, G Parisi and MA Virasoro 1987 *Spin Glass Theory and Beyond* (World Scientific)
- spin glass theory, including replica theory and generating functional analysis:
  Fisher KH and Hertz J 1991 *Spin Glasses* (Cambridge UP)
- interdisciplinary applications of replica theory:
  H Nishimori 2001 *Statistical Physics of Spin Glasses and Information Processing* (Oxford UP)

More recent papers

- physical meaning of replica dimension:
  ACC Coolen, RW Penney and D Sherrington 1993 *Phys. Rev. B* **48** 16116
- physics behind Parisi scheme:
  J Van Mourik and ACC Coolen 2001 *J. Phys. A* **34** L111
- dynamics with replicas:
  ACC Coolen, SN Laughton and D Sherrington 1996 *Phys. Rev. B* **53** 8184
- finite connectivity replica theory:
  I Kanter I and H Sompolinsky 1987 *Phys. Rev. Lett.* **58** 164