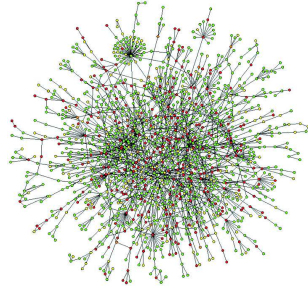


Spin models on random graphs with controlled topologies beyond degree constraints

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(work in progress)



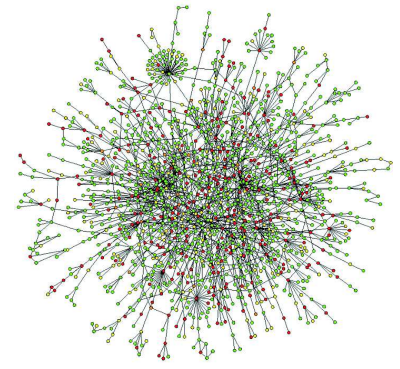
Motivation - processes on random graphs
Deformed random graph ensembles
Replica analysis
Phase diagrams for specific deformations
Summary and outlook

1. MOTIVATION

Model processes on complex real graphs
by *solvable* processes on suitable *random* graphs

e.g. Ising systems,

graph $\mathbf{c}^* = \{c_{ij}^*\}$ $H = -J \sum_{i<j} c_{ij}^* \sigma_i \sigma_j$ $c_{ij}^* \in \{0, 1\}$



level 0 :

measure average connectivity $\langle k \rangle = N^{-1} \sum_{ij} c_{ij}^*$
draw random \mathbf{c} from Erdős-Rényi ensemble

$$\text{Prob}(\mathbf{c}) = \prod_{i<j} \left[\frac{\langle k \rangle}{N} \delta_{c_{ij},1} + \left(1 - \frac{\langle k \rangle}{N}\right) \delta_{c_{ij},0} \right]$$

level 1 :

measure degrees $\{k_1^*, \dots, k_N^*\}$, $k_i^* = \sum_j c_{ij}^*$
draw random \mathbf{c} from degree-constrained ensemble

$$\text{Prob}(\mathbf{c}) = \frac{1}{\mathcal{Z}_N} \prod_{i<j} \left[\frac{\langle k \rangle}{N} \delta_{c_{ij},1} + \left(1 - \frac{\langle k \rangle}{N}\right) \delta_{c_{ij},0} \right] \cdot \prod_i \delta_{k_i^*, \sum_j c_{ij}}$$

Examples:

Erdős-Renyi
(level 0)

degree-constrained
(level 1)

exact solution

1D Ising model:

$$\langle k \rangle = 2$$

$$T_c/J \approx 1.820$$

$$p(k) = \delta_{k,2}$$

$$T_c/J = 0$$

$$T_c/J = 0$$

2D Ising model:

$$\langle k \rangle = 4$$

$$T_c/J \approx 3.915$$

$$p(k) = \delta_{k,4}$$

$$T_c/J \approx 2.885$$

$$T_c/J \approx 2.269$$

3D Ising model:

$$\langle k \rangle = 6$$

$$T_c/J \approx 5.944$$

$$p(k) = \delta_{k,6}$$

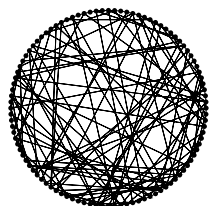
$$T_c/J \approx 4.933$$

$$T_c/J \approx 4.51?$$

small world model:

$$\langle k \rangle = 2 + c$$

$$p(k \geq 2) = \frac{e^{-c} c^{k-2}}{(k-2)!}$$



$$c = 0 : T_c/J \approx 1.820$$

$$c = 1 : T_c/J \approx 2.885$$

$$c = 2 : T_c/J \approx 3.915$$

$$T_c/J = 0$$

$$T_c/J \approx 2.183$$

$$T_c/J \approx 3.403$$

$$T_c/J = 0$$

$$T_c/J \approx 2.269$$

$$T_c/J \approx 3.466$$

models with the same $p(k)$ can behave quite differently ...

Level 2 random graph ensembles?

- include topological information on \mathbf{c}^* beyond degrees
- keep model solvable
- additional information must be relevant (phase diagram)

Deformed degree-constrained ensembles

$$\text{level 0 :} \quad \text{Prob}(\mathbf{c}) = \prod_{i < j} \left[\frac{\langle k \rangle}{N} \delta_{c_{ij},1} + \left(1 - \frac{\langle k \rangle}{N}\right) \delta_{c_{ij},0} \right]$$

$$\text{level 1 :} \quad \text{Prob}(\mathbf{c}) = \frac{1}{\mathcal{Z}_N} \prod_{i < j} \left[\frac{\langle k \rangle}{N} \delta_{c_{ij},1} + \left(1 - \frac{\langle k \rangle}{N}\right) \delta_{c_{ij},0} \right] \cdot \prod_i \delta_{k_i^*, \sum_j c_{ij}}$$

$$\text{level 2 :} \quad \text{Prob}(\mathbf{c}) = \frac{1}{\mathcal{Z}_N} \prod_{i < j} \left[\frac{\langle k \rangle}{N} Q(k_i, k_j) \delta_{c_{ij},1} + \left(1 - \frac{\langle k \rangle}{N}\right) Q(k_i, k_j) \delta_{c_{ij},0} \right] \cdot \prod_i \delta_{k_i^*, \sum_j c_{ij}}$$

$$\text{with :} \quad Q(k, k') \geq 0 \quad \forall k, k' \quad \sum_{k, k' \geq 0} p(k)p(k')Q(k, k') = 1$$

Ising models on random graphs
drawn from deformed degree-constrained ensembles

$$H(\boldsymbol{\sigma}) = - \sum_{i < j} c_{ij} \sigma_i J_{ij} \sigma_j$$

$$\text{Prob}(\mathbf{c}) = \frac{1}{\mathcal{Z}_N} \prod_{i < j} \left[\frac{\langle k \rangle}{N} Q(k_i, k_j) \delta_{c_{ij}, 1} + \left(1 - \frac{\langle k \rangle}{N} Q(k_i, k_j)\right) \delta_{c_{ij}, 0} \right] \cdot \prod_i \delta_{k_i, \sum_j c_{ij}}$$

characteristics:

- $\{k_1, \dots, k_N\}$ drawn randomly from $p(k)$
- $\{J_{ij}\}$ drawn randomly from $P(J)$
- ensemble parametrized by: $p(k)$ and $Q(k, k')$
- graphs locally tree-like, e.g. $\lim_{N \rightarrow \infty} p(k, r) = p(k) \delta_{r, 0}$
 $p(k, r) = N^{-1} \sum_i \delta_{k, \sum_j c_{ij}} \delta_{r, \sum_{jk} c_{ij} c_{jk} c_{ki}}$

These models are *solvable*,

calculate average of free energy per spin over disorder (bonds, graphs)

how do phase diagrams depend on $p(k)$ and $Q(k, k')$?

3. EQUILIBRIUM REPLICA ANALYSIS

In a nutshell ...

exploit $\overline{\log Z} = \lim_{n \rightarrow 0} n^{-1} \log \overline{Z^n}$,
and assume initially that n integer

$$\begin{aligned}\bar{f} &= - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \overline{\log \sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}} = - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{\beta n N} \log \left[\overline{\sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}} \right]^n \\ &= - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{\beta n N} \log \sum_{\boldsymbol{\sigma}^1} \dots \sum_{\boldsymbol{\sigma}^n} \overline{e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}}\end{aligned}$$

- carry out average $\overline{\dots}$ over bonds and graphs *first*
- exchange limits $N \rightarrow \infty$ and $n \rightarrow 0$
- steepest descent integration as $N \rightarrow \infty$, for finite n
- ergodicity ansatz for order parameters of replicated spin system
- take the limit $n \rightarrow 0$
- order parameters: functions (effective & cavity field distributions)
- study bifurcations in order parameter eqns via moment expansions

Stage 1:

order parameter eqns after limit $N \rightarrow \infty$

(steepest descent)

$$\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$$

$$F(k, \boldsymbol{\sigma}) = \sum_{k'} Q(k, k') \sum_{\boldsymbol{\sigma}'} D(k', \boldsymbol{\sigma}') \int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}$$

$$D(k, \boldsymbol{\sigma}) = \frac{p(k)k}{\langle k \rangle} \frac{F^{k-1}(k, \boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}'} F^k(k, \boldsymbol{\sigma}')}$$

$$\bar{f} = - \lim_{n \rightarrow 0} \frac{1}{\beta n} \sum_k p(k) \log \left[\sum_{\boldsymbol{\sigma}} [F(k, \boldsymbol{\sigma}) / F(k)]^k \right]$$

Stage 2:

make ergodic ansatz ('replica symmetry')

$$D(k, \boldsymbol{\sigma}) = \int dh D(k, h) \frac{e^{\beta h \sum_{\alpha} \sigma_{\alpha}}}{[2 \cosh(\beta h)]^n} \quad F(k, \boldsymbol{\sigma}) = \int dh F(k, h) e^{\beta h \sum_{\alpha} \sigma_{\alpha}}$$

$$D(k, h) = D(h|k)D(k), \quad \int dh D(h|k) = 1 \quad F(k, h) = F(h|k)F(k), \quad \int dh F(h|k) = 1$$

Stage 3:

take the limit $n \rightarrow 0$,

eliminate $D(k)$ and $D(h|k)$,

leaves closed eqns for $F(k)$ and $F(h|k)$:

$$\begin{aligned} F(k) &= \langle k \rangle^{-1} \sum_{k'} p(k') k' Q(k, k') F^{-1}(k') \\ F(h|k) &= \sum_{k'} \frac{Q(k, k') p(k') k'}{\langle k \rangle F(k) F(k')} \int dJ P(J) \int \prod_{\ell < k} [dh_\ell F(h_\ell | k')] \\ &\quad \times \delta[h - \beta^{-1} \operatorname{atanh}[\tanh(\beta J) \tanh(\beta \sum_{\ell < k} h_\ell)]] \end{aligned}$$

$$\bar{f}_{\text{RS}} = -\frac{1}{\beta} \sum_k p(k) \int \prod_{\ell \leq k} [dh_\ell F(h_\ell | k)] \log [2 \cosh(\beta \sum_{\ell \leq k} h_\ell)]$$

Stage 4:

Physical meaning of replica-symmetric

(RS) order parameters

$$\begin{aligned} \boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) : \quad P(k, \boldsymbol{\sigma}) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \left[\frac{\sum_{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_N} \delta_{k, k_i} \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} e^{-\beta \sum_{\alpha} H(\boldsymbol{\sigma}^{\alpha})}}{\sum_{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_N} e^{-\beta \sum_{\alpha} H(\boldsymbol{\sigma}^{\alpha})}} \right] \\ &= p(k) \int dh W(h|k) \frac{e^{\beta h \sum_{\alpha=1}^n \sigma_{\alpha}}}{[2 \cosh(\beta h)]^n} \end{aligned}$$

degree-conditioned

effective field distr:

$$W(h|k) = \int \prod_{\ell \leq k} [dh_{\ell} F(h_{\ell}|k)] \delta[h - \sum_{\ell \leq k} h_{\ell}]$$

$W(h) = \sum_k p(k) W(h|k)$:

$$m = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sigma_i \rangle} = \int dh W(h) \tanh(\beta h)$$

$$q = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sigma_i \rangle^2} = \int dh W(h) \tanh^2(\beta h)$$

$$\bar{f} = -\frac{1}{\beta} \int dh W(h) \log[2 \cosh(\beta h)]$$

4. PHASE DIAGRAMS FOR SPECIFIC DEFORMATION FUNCTIONS

choose bond distribution

$$P(J) = \frac{1}{2}(1 + \eta)\delta(J - J_0) + \frac{1}{2}(1 - \eta)\delta(J + J_0)$$

Procedure:

- always the paramagnetic (P) soln $F(h|k) = \delta(h)$, where $m = q = 0$
- it is the *only* solution for $T \rightarrow \infty$
- assume bifurcations away from $F(h|k) = \delta(h)$ are *continuous*,
so expand in moments of $F(h|k)$:
assume $\exists \epsilon$ with $0 < |\epsilon| \ll 1$ such that

$$\int dh h^\ell F(h|k) = \mathcal{O}(\epsilon^\ell)$$

- bifurcating order ϵ : state has $m \neq 0, q > 0 \rightarrow$ ferromagnet (F)
- bifurcating order ϵ^2 : state has $m = 0, q > 0 \rightarrow$ spin-glass (SG)

result of bifurcation analysis:

$$\begin{aligned} \eta > \tanh(\beta J_0) : \quad & \text{P} \rightarrow \text{F}, \quad T_{\text{F}}/J_0 = 2/\log \left[\frac{\eta \lambda_{\max}(Q, p) + 1}{\eta \lambda_{\max}(Q, p) - 1} \right] \\ \eta < \tanh(\beta J_0) : \quad & \text{P} \rightarrow \text{SG}, \quad T_{\text{SG}}/J_0 = 2/\log \left[\frac{\sqrt{\lambda_{\max}(Q, p)} + 1}{\sqrt{\lambda_{\max}(Q, p)} - 1} \right] \end{aligned}$$

$$\begin{aligned} \lambda(Q, p) : \quad \text{eigenvalues of} \quad & M_{kk'} = \frac{Q(k, k')p(k')k'(k'-1)}{\langle k \rangle F(k)F(k')}, \quad k, k' = 0, 1, 2, 3, \dots \\ & F(k) = \langle k \rangle^{-1} \sum_{k'} p(k')k'Q(k, k')F^{-1}(k') \end{aligned}$$

notes:

- one expects RSB solutions (broken replica symmetry), but *at or below* the RS critical temperatures
- the F→SG transition is much harder to find analytically, but could be constructed via Parisi-Toulouse hypothesis

Choices considered

$$\text{Type I: } Q(k, k') = g(k)g(k')/\langle g \rangle^2, \quad g(k) \geq 0 \forall k, \quad \langle g \rangle = \sum_k p(k)g(k) > 0$$

$$\text{Type II: } Q(k, k') = [g(k) + g(k')]/2\langle g \rangle, \quad g(k) \geq 0 \forall k, \quad \langle g \rangle > 0$$

$$\text{Type III: } Q(k, k') = \gamma_0 + \gamma\delta_{kk'}, \quad \gamma_0 = 1 - \gamma \sum_k p^2(k), \quad |\gamma| \leq [\sum_k p^2(k)]^{-1}$$

no deformation: $Q(k, k') = 1$

$$\eta > \tanh(\beta J_0) : \quad \text{P} \rightarrow \text{F}, \quad T_{\text{F}}/J_0 = 2/\log \left[\frac{\eta[\langle k^2 \rangle / \langle k \rangle - 1] + 1}{\eta[\langle k^2 \rangle / \langle k \rangle - 1] - 1} \right]$$

$$\eta < \tanh(\beta J_0) : \quad \text{P} \rightarrow \text{SG}, \quad T_{\text{SG}}/J_0 = 2/\log \left[\frac{\sqrt{\langle k^2 \rangle / \langle k \rangle - 1} + 1}{\sqrt{[\langle k^2 \rangle / \langle k \rangle - 1]} - 1} \right]$$

Type I: $Q(k, k') = g(k)g(k')/\langle g \rangle^2$

trivial: $g(k)$ drops out of transition lines and order parameter eqns, for any $p(k)$, complete solution identical to that of $Q(k, k') = 1 \dots$

Type II: $Q(k, k') = [g(k) + g(k')]/2\langle g \rangle$

$$\lambda_{\max}(Q, p) = \frac{2y}{\langle k \rangle} \left\{ \left\langle \frac{k(k-1)g(k)}{[yg(k)+1]^2} \right\rangle + \sqrt{\left\langle \frac{k(k-1)}{[yg(k)+1]^2} \right\rangle \left\langle \frac{k(k-1)g^2(k)}{[yg(k)+1]^2} \right\rangle} \right\}$$

y to be solved from

$$\left\langle \frac{k}{yg(k)+1} \right\rangle = \frac{1}{2}\langle k \rangle$$

Type III: $Q(k, k') = \gamma_0 + \gamma\delta_{kk'}$

$\lambda_{\max}(Q, p)$: largest soln of

$$1 = [1 - \gamma\langle p(k) \rangle] \left\langle \frac{4k(k-1)}{\lambda\langle k \rangle [y + \sqrt{y^2 + 4\gamma p(k)k/\langle k \rangle}]^2 - 4\gamma p(k)k(k-1)} \right\rangle$$

$$y = \frac{1 - \gamma\langle p(k) \rangle}{\langle k \rangle} \left\langle \frac{2k}{y + \sqrt{y^2 + 4\gamma p(k)k/\langle k \rangle}} \right\rangle$$

$\lim_{\gamma_0 \rightarrow 0} \lambda_{\max}(Q, p) = k^* - 1,$

k^* : largest degree with $p(k) > 0$

Results for ensembles with type II and type III deformations

Example degree distributions:

$$\text{Poissonian : } p(k) = c^k e^{-c} / k!$$

$$\text{power law : } p(k) = \left(1 - \frac{c \zeta(3 + \alpha)}{\zeta(2 + \alpha)}\right) \delta_{k0} + (1 - \delta_{k0}) \frac{c k^{-3-\alpha}}{\zeta(2 + \alpha)}$$

$$\zeta(x) = \sum_{k>0} k^{-x},$$

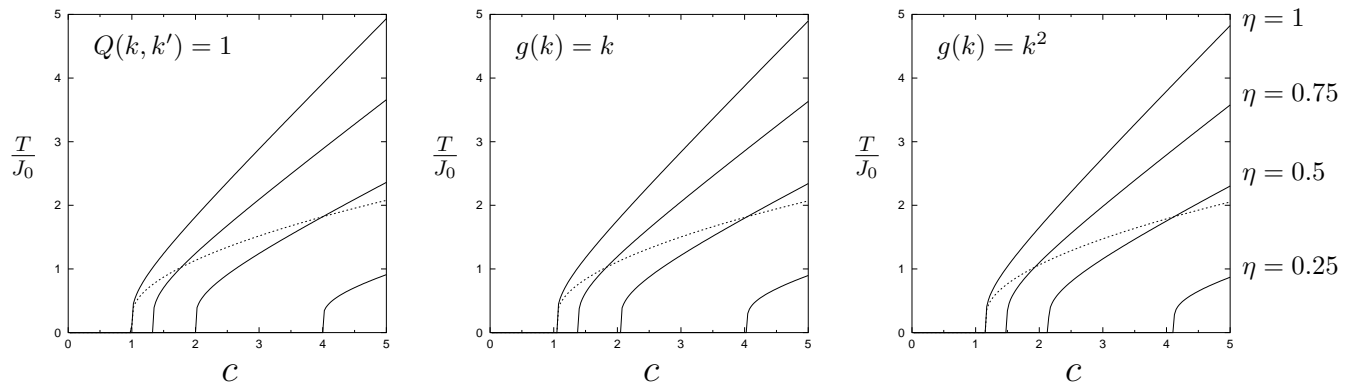
$\alpha \in [0, 1]$, so $\langle k \rangle < \infty$ but $\langle k^2 \rangle \rightarrow \infty$ for $\alpha \downarrow 0$

notes:

- always $\langle k \rangle = c$
- power-law: bifurcation lines type II deformations indep of c
- In practice: $k \leq k_{\max} = 10^8$

Poissonian $p(k)$ with type II deformations

$$Q(k, k') = [k^\ell + (k')^\ell]/2 \langle k^\ell \rangle$$



P \rightarrow SG (dotted) and P \rightarrow F (solid)

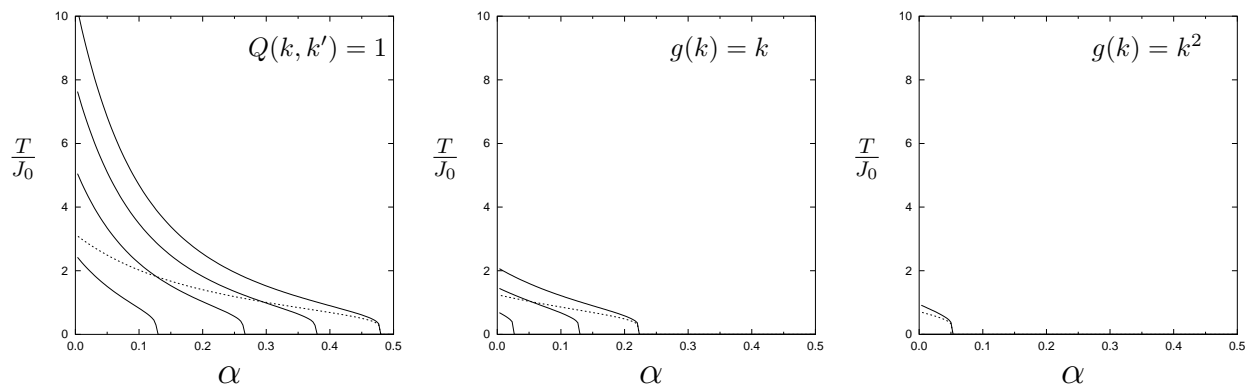
$$P(J) = \frac{1}{2}(1 + \eta)\delta(J - J_0) + \frac{1}{2}(1 - \eta)\delta(J + J_0)$$

impact of deformation:

small reduction of all critical temperatures

Power law $p(k) \sim k^{-3-\alpha}$ with type II deformations

$$Q(k, k') = [k^\ell + (k')^\ell]/2 \langle k^\ell \rangle$$



P \rightarrow SG (dotted) and P \rightarrow F (solid)

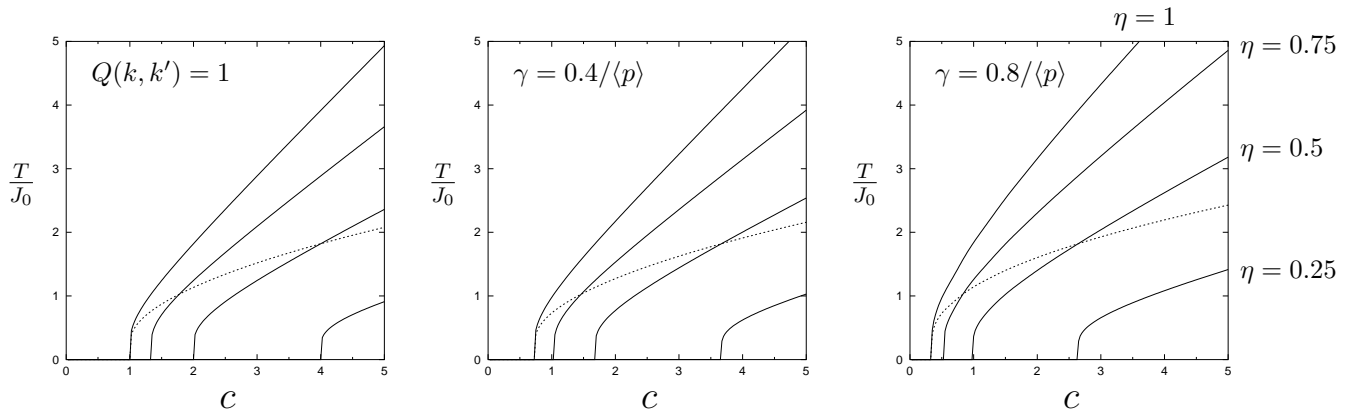
$$P(J) = \frac{1}{2}(1 + \eta)\delta(J - J_0) + \frac{1}{2}(1 - \eta)\delta(J + J_0)$$

impact of deformation:

dramatic reduction of all critical temperatures

Poissonian $p(k)$ with type III deformations

$$Q(k, k') = \gamma_0 + \gamma \delta_{kk'}$$



P→SG (dotted) and P→F (solid)

$$P(J) = \frac{1}{2}(1 + \eta)\delta(J - J_0) + \frac{1}{2}(1 - \eta)\delta(J + J_0)$$

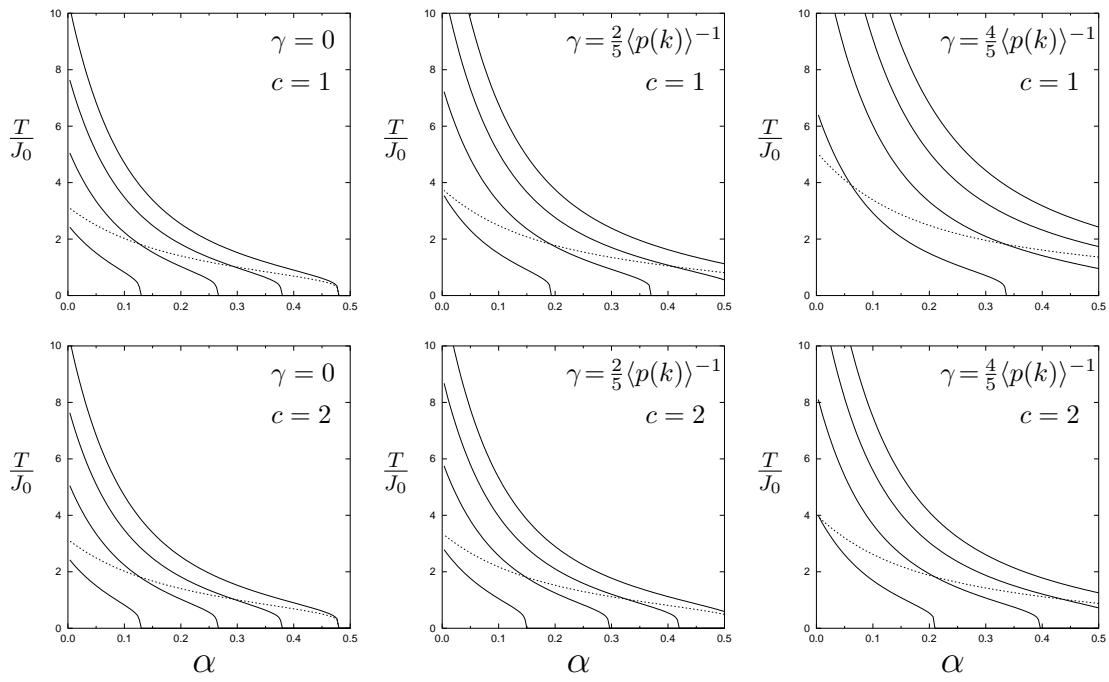
impact of deformation:

significant increase in critical temperatures

$T_F \rightarrow \infty$ and $T_{SG} \rightarrow \infty$ for $\gamma \rightarrow \langle p \rangle^{-1}$

Poissonian $p(k)$ with type III deformations

$$Q(k, k') = \gamma_0 + \gamma \delta_{kk'}$$



5. SUMMARY AND OUTLOOK

- Specifying just the degree distribution $p(k)$ of a connectivity graph for an interacting spin system does not permit reliable predictions on the phase diagram
- Proposed random graph ensembles, characterized by degree distribution $p(k)$ and additional deformation $Q(k, k')$

$$\text{Prob}(\mathbf{c}) = \frac{1}{\mathcal{Z}_N} \prod_{i < j} \left[\frac{\langle k \rangle}{N} Q(k_i, k_j) \delta_{c_{ij}, 1} + \left(1 - \frac{\langle k \rangle}{N} Q(k_i, k_j)\right) \delta_{c_{ij}, 0} \right] \cdot \prod_i \delta_{k_i, \sum_j c_{ij}}$$

- allow us to differentiate between models with same $p(k)$, but different microscopic realizations of these degree statistics
 - spin models with connectivity graphs from these ensembles still solvable
 - impact of deformation via $Q(k, k')$ on phase diagram can be non-negligible
- To be done:
 - physical meaning of the $F(k)$, RSB transition lines
 - application: what is optimal $Q(k, k')$ for a given real graph \mathbf{c}^* ?

Optimal random graph ensemble $\{p(k), Q(k, k')\}$
to serve as ‘solvable proxy’ for a given graph \mathbf{c}^* :

- measure $k_i^* = \sum_j c_{ij}^*$, define $p(k) = N^{-1} \sum_i \delta_{k, k_i^*}$
- maximizing log-likelihood of \mathbf{c}^* for deformed ensemble:
minimize over Q , subject to $\sum_{kk'} p(k)p(k')Q(k, k') = 1$,
the quantity

$$\begin{aligned} \Omega[Q] &= -\frac{1}{N} \log \text{Prob}(\mathbf{c}^*) \\ &= \text{const} + \frac{1}{N} \sum_i k_i^* \log F(k_i^*|Q) - \frac{1}{N} \sum_{i < j} c_{ij}^* \log Q(k_i^*, k_j^*) + \mathcal{O}(N^{-1}) \end{aligned}$$

where $F(k|Q)$ is soln of

$$F(k) = \langle k \rangle^{-1} \sum_{k'} p(k') k' Q(k, k') F^{-1}(k')$$