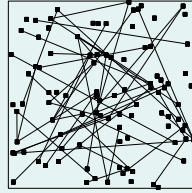


Finitely Connected Vector Spin Systems with Random Matrix Interactions

*ACC Coolen, JPL Hatchett, T Nikolettopoulos, CJ Perez Vicente,
I Pérez Castillo, NS Skantzios, B Wemmenhove*



MOTIVATION

Physical systems

- *finite connectivity:*
more realistic than full connectivity
- *vector spins:*
more realistic than Ising or spherical spins
- *matrix interactions:*
e.g. coupled oscillators, Josephson junctions

Theory

- *how far can we push our present techniques?*
- *more intuition on finitely connected continuous spins*

OVERVIEW

Definitions

*vector spins $\sigma_i \in S_{d-1}$ with random matrix interactions
on finitely connected random graphs*

Replica calculation of free energy

*saddle-point equations
replica-symmetric theory*

d=2: XY spins

*phase transitions
phase diagrams for different chirality distributions
theory versus simulations*

d=3,4,...

*phase transitions
diagrams for $d = 3$, classical Heisenberg spins
theory versus simulations*

Summary

DEFINITIONS

N unit-length vector spins,

$$\boldsymbol{\sigma}_i \in S_{d-1}, \quad \{\boldsymbol{\sigma}\} = (\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_N)$$

$$H(\{\boldsymbol{\sigma}\}) = -J \sum_{i < j} c_{ij} \boldsymbol{\sigma}_i \cdot \mathbf{U}_{ij} \boldsymbol{\sigma}_j + \sum_i V(\boldsymbol{\sigma}_i)$$

quenched disorder:

- random lattice: $c_{ij} \in \{0, 1\}$

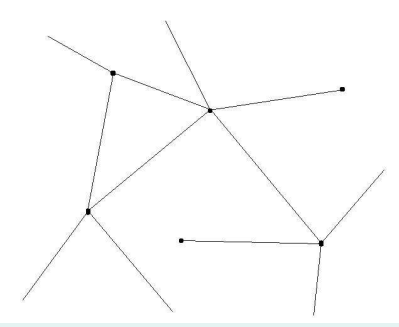
$$\text{Prob}(c_{ij}) = \frac{c}{N} \delta_{c_{ij}, 1} + \left(1 - \frac{c}{N}\right) \delta_{c_{ij}, 0} \quad \text{for all } i < j$$

$c = \mathcal{O}(N^0)$, Erdős-Rényi graph

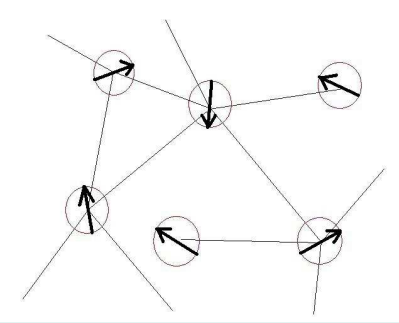
- interactions between spins:

random matrices $\mathbf{U}_{ij} \in \text{SO}(3)$ (rotations in \mathbb{R}^d)

independently drawn from $P(\mathbf{U})$, with $P(\mathbf{U}^\dagger) = P(\mathbf{U})$



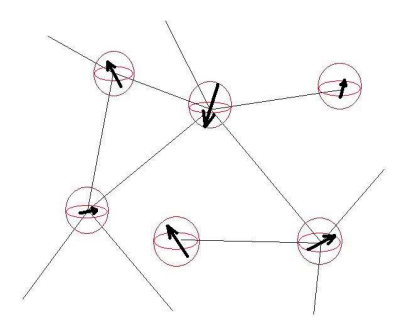
random graph: $\text{Prob}(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1 - \frac{c}{N})\delta_{c_{ij},0}$



$d = 2:$ $\sigma_i = (\cos \phi_i, \sin \phi_i)$

$$H = -J \sum_{i < j} c_{ij} \cos(\phi_i - \phi_j - \omega_{ij}) + \sum_i V(\phi_i)$$

random $\omega_{ij} \in [0, 2\pi]$



$d = 3:$ $\sigma_i = (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i)$

$$H = -J \sum_{i < j} c_{ij} \sigma_i \cdot \mathbf{U}_{ij} \sigma_j + \sum_i V(\sigma_i)$$

random $\mathbf{U}_{ij} \in \text{SO}(3)$

Technicalities

Complicating model ingredients:

- (i) spin variables continuous*
- (ii) spin variables vectorial*
- (iii) spin-interactions represented by random matrices*

- *RS order parameter is a functional*
- *finding phase transitions:*
 - *involves functional moment expansions*
 - *nontrivial eigenvalue problems*
- *finding order parameters:*
 - *population dynamics nontrivial: iterating functionals*
 - *numerically demanding and potentially difficult to converge*
- *numerical simulations:*
 - *generating suitable random matrices*
 - *Langevin dynamics too slow*

REPLICA ANALYSIS

Disorder-averaged free energy per spin:

$$\bar{f} = \lim_{n \rightarrow 0} \frac{1}{n} \left\{ - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \log \int \left[\prod_i \prod_{\alpha=1}^n d\sigma_i^\alpha \right] \overline{e^{-\beta \sum_{\alpha=1}^n H(\{\sigma^\alpha\})}} \right\}$$

Disorder average:

$$\overline{e^{-\beta \sum_{\alpha} H(\{\sigma^\alpha\})}} = \exp \left\{ -\beta \sum_{i\alpha} V(\sigma_i^\alpha) + \mathcal{O}(N^0) \right. \\ \left. + \frac{c}{2N} \sum_{ij} \left[\int d\mathbf{U} P(\mathbf{U}) e^{\beta J \sum_{\alpha} \sigma_i^\alpha \cdot \mathbf{U} \sigma_j^\alpha} - 1 \right] \right\}$$

Order parameters:

$$\{\sigma\} = (\sigma^1, \dots, \sigma^n), \quad \sigma^\alpha, \sigma_i^\alpha \in S_{d-1} \quad P(\{\sigma\}) = \frac{1}{N} \sum_i \prod_{\alpha} \delta[\sigma - \sigma_i^\alpha]$$

minor technicalities

- functional δ -distributions to isolate order parameters $P(\{\boldsymbol{\sigma}\})$
integral representations: conjugate functions $\hat{P}(\{\boldsymbol{\sigma}\})$
- continuum limit for domain S_{d-1} of $\boldsymbol{\sigma}_\alpha$,
gives path integral measure:

$$\prod_{\{\boldsymbol{\sigma}\}} [dP(\{\boldsymbol{\sigma}\})d\hat{P}(\{\boldsymbol{\sigma}\})/2\pi] = \{dPd\hat{P}\}$$

$$\begin{aligned} \bar{f} = & -\lim_{n \rightarrow 0} \frac{1}{\beta n} \text{extr}_{\{P, \hat{P}\}} \left\{ i \int \{d\boldsymbol{\sigma}\} P(\{\boldsymbol{\sigma}\}) \hat{P}(\{\boldsymbol{\sigma}\}) + \log \int \{d\boldsymbol{\sigma}\} e^{-\beta \sum_\alpha V(\boldsymbol{\sigma}_\alpha) - i \hat{P}(\{\boldsymbol{\sigma}\})} \right. \\ & \left. + \frac{1}{2} c \int \{d\boldsymbol{\sigma} d\boldsymbol{\sigma}'\} P(\{\boldsymbol{\sigma}\}) P(\{\boldsymbol{\sigma}'\}) \left[\int d\mathbf{U} P(\mathbf{U}) e^{\beta J \sum_\alpha \boldsymbol{\sigma}_\alpha \cdot \mathbf{U} \boldsymbol{\sigma}'_{\alpha-1}} - 1 \right] \right\} \end{aligned}$$

saddle-point eqns

$$\begin{aligned} P(\{\boldsymbol{\sigma}\}) &= \frac{e^{-\beta \sum_\alpha V(\boldsymbol{\sigma}_\alpha) - i \hat{P}(\{\boldsymbol{\sigma}\})}}{\int \{d\boldsymbol{\sigma}'\} e^{-\beta \sum_\alpha V(\boldsymbol{\sigma}'_\alpha) - i \hat{P}(\{\boldsymbol{\sigma}'\})}} \\ \hat{P}(\{\boldsymbol{\sigma}\}) &= ic \int \{d\boldsymbol{\sigma}'\} P(\{\boldsymbol{\sigma}'\}) \left[\int d\mathbf{U} P(\mathbf{U}) e^{\beta J \sum_\alpha \boldsymbol{\sigma}_\alpha \cdot \mathbf{U} \boldsymbol{\sigma}'_{\alpha-1}} - 1 \right] \end{aligned}$$

Replica symmetric theory

$$P(\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n) = \frac{1}{N} \sum_i \prod_{\alpha=1}^n \delta[\boldsymbol{\sigma}_\alpha - \boldsymbol{\sigma}_i^\alpha]$$

RS ansatz for
continuous variables:

- let $P[\phi|\boldsymbol{\mu}]$ denote a complete parametrized family of functions on S_{d-1}
 $\boldsymbol{\mu} = (\mu_0, \mu_1, \mu_2, \dots)$

$$P_{\text{RS}}(\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n) = \int d\boldsymbol{\mu} w(\boldsymbol{\mu}) \prod_{\alpha} P[\boldsymbol{\sigma}_\alpha|\boldsymbol{\mu}], \quad \int d\boldsymbol{\mu} w(\boldsymbol{\mu}) = 1$$

- representation-independent formulation:

$$P_{\text{RS}}(\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n) = \int \{dP\} W[\{P\}] \prod_{\alpha} P(\boldsymbol{\sigma}_\alpha)$$

RS order parameter:

functional measure $W[\{P\}]$

physical interpretation:

$$\int \{dP\} W[\{P\}] \prod_{\alpha} \left[\int d\boldsymbol{\sigma} P(\boldsymbol{\sigma}) f_{\alpha}(\boldsymbol{\sigma}) \right] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\prod_{\alpha} \langle f_{\alpha}(\boldsymbol{\sigma}_i) \rangle}$$

insert RS ansatz:

$$\begin{aligned} \bar{f}_{\text{RS}} &= \frac{c}{2\beta} \int \{dP_1 dP_2\} W[\{P_1\}] W[\{P_2\}] \int d\mathbf{U} P(\mathbf{U}) \log \left[\int d\boldsymbol{\sigma} d\boldsymbol{\sigma}' P_1(\boldsymbol{\sigma}) P_2(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma} \cdot \mathbf{U} \boldsymbol{\sigma}'} \right] \\ &\quad - \frac{1}{\beta} \sum_{\ell \geq 0} p_\ell \int \prod_{k=1}^{\ell} [\{dP_k\} W[\{P_k\}] d\mathbf{U}_k P(\mathbf{U}_k)] \\ &\quad \times \log \left[\int d\boldsymbol{\sigma} e^{-\beta V(\boldsymbol{\sigma})} \prod_{k=1}^{\ell} \int d\boldsymbol{\sigma}' P_k(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma} \cdot \mathbf{U}_k \boldsymbol{\sigma}'} \right] \end{aligned}$$

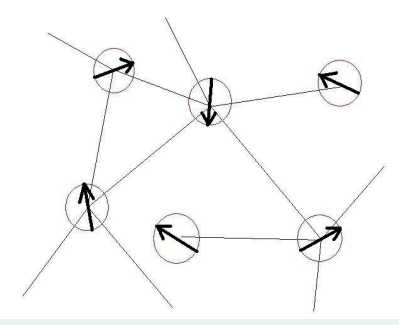
$$p_k = e^{-c} c^k / k!$$

$$\begin{aligned} W[\{P\}] &= \sum_{\ell \geq 0} p_\ell \int \prod_{k \leq \ell} [\{dP_k\} W[\{P_k\}] d\mathbf{U}_k P(\mathbf{U}_k)] \\ &\quad \times \prod_{\boldsymbol{\sigma} \in S_{d-1}} \delta \left[P(\boldsymbol{\sigma}) - \frac{e^{-\beta V(\boldsymbol{\sigma})} \prod_{k=1}^{\ell} \int d\boldsymbol{\sigma}' P_k(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma} \cdot \mathbf{U}_k \boldsymbol{\sigma}'}}{\int d\boldsymbol{\sigma}'' e^{-\beta V(\boldsymbol{\sigma}'')} \prod_{k=1}^{\ell} \int d\boldsymbol{\sigma}' P_k(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma}'' \cdot \mathbf{U}_k \boldsymbol{\sigma}'}} \right] \end{aligned}$$

paramagnetic state:

$$\lim_{\beta \rightarrow 0} W[\{P\}] = \prod_{\boldsymbol{\sigma} \in S_{d-1}} \delta \left[P(\boldsymbol{\sigma}) - \frac{1}{|S_{d-1}|} \right] \quad \lim_{\beta \rightarrow 0} \beta \bar{f}_{\text{RS}} = -\log |S_{d-1}|$$

d=2: XY SPINS WITH CHIRAL INTERACTIONS



$$d = 2: \quad \boldsymbol{\sigma}_i = (\cos \phi_i, \sin \phi_i)$$

$$H = -J \sum_{i < j} c_{ij} \cos(\phi_i - \phi_j - \omega_{ij}) + \sum_i V(\phi_i)$$

random $\omega_{ij} \in [0, 2\pi]$

$V = 0$:

$$W[\{P\}] = \sum_{k \geq 0} p_k \int \prod_{\ell \leq k} [\{dP_\ell\} W[\{P_\ell\}] d\omega_\ell P(\omega_\ell)]$$

$$\times \prod_{\phi \in [0, 2\pi]} \delta \left[P(\phi) - \frac{\prod_{\ell=1}^k \int d\phi' P_\ell(\phi') e^{\beta J \cos(\phi - \phi' - \omega_\ell)}}{\int d\phi'' \prod_{\ell=1}^k \int d\phi' P_\ell(\phi') e^{\beta J \cos(\phi'' - \phi' - \omega_\ell)}} \right]$$

paramagnetic state : $\lim_{\beta \rightarrow 0} W[\{P\}] = \prod_{\phi \in [0, 2\pi]} \delta \left[P(\phi) - \frac{1}{2\pi} \right]$

Phase transitions

*Continuous bifurcations away from paramagnetic state
located by Guzai (i.e. functional moment) expansion*

- *transform:*

$$P(\phi) \rightarrow \frac{1}{2\pi} + \Delta(\phi), \quad W[\{P\}] \rightarrow \tilde{W}[\{\Delta\}]$$

constraint:

$$\tilde{W}[\{\Delta\}] = 0 \quad \text{if} \quad \int_0^{2\pi} d\phi \Delta(\phi) \neq 0$$

- *expand saddle-point eqns
in functional moments*

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\phi_1) \dots \Delta(\phi_r)$$

- *assume: close to continuous bifurcation*
 $\exists \epsilon \ll 1$ such that

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\phi_1) \dots \Delta(\phi_r) = \mathcal{O}(\epsilon^r)$$

Lowest order bifurcation ϵ^1

$$\Psi(\phi) = \frac{c}{2\pi I_0(\beta J)} \int_0^{2\pi} d\phi' \int d\omega P(\omega) e^{\beta J \cos(\phi - \phi' - \omega)} \Psi(\phi') \quad \int_0^{2\pi} d\phi \Psi(\phi) = 0$$

$I_k(z)$:

modified Bessel functions

- soln: Fourier modes $\Psi(\phi) = e^{ik\phi}$

$$c = \min_{k>0} \left\{ \frac{I_k(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega P(\omega) \cos(k\omega) \right\}^{-1}$$

- bifurcating state:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\left\langle \begin{pmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{pmatrix} \right\rangle} = \frac{1}{2} \epsilon \delta_{k1} \begin{pmatrix} \cos(\lambda) \\ \sin(\lambda) \end{pmatrix} + \dots$$

$$\text{P} \rightarrow \text{F} : \quad c = \left\{ \frac{I_1(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega P(\omega) \cos(\omega) \right\}^{-1}$$

$$\text{KT} : \quad c = \min_{k>1} \left\{ \frac{I_k(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega P(\omega) \cos(k\omega) \right\}^{-1}$$

Lowest order bifurcation ϵ^2

$$\Psi(\phi_1, \phi_2) = c \int \frac{d\phi'_1 d\phi'_2}{[2\pi I_0(\beta J)]^2} \left[\int d\omega P(\omega) e^{\beta J \cos(\phi_1 - \phi'_1 - \omega) + \beta J \cos(\phi_2 - \phi'_2 - \omega)} \right] \Psi(\phi'_1, \phi'_2)$$
$$\int d\phi_1 \Psi(\phi_1, \phi_2) = \int d\phi_2 \Psi(\phi_1, \phi_2) = 0$$

- *soln: Fourier modes* $\Psi(\phi_1, \phi_2) = e^{i(k_1\phi_1 + k_2\phi_2)}$
- *bifurcating state:*
no global ferromagn order, yet

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_i [\langle \cos(\phi_i) \rangle^2 + \langle \sin(\phi_i) \rangle^2] > 0$$

$$P \rightarrow \text{SG} : \quad c = I_0^2(\beta J) / I_1^2(\beta J)$$

Phase diagrams

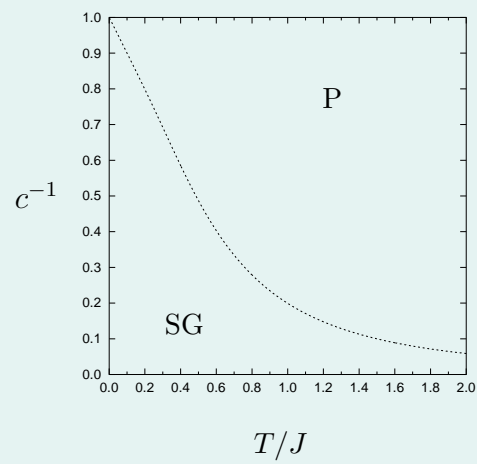
$$P \rightarrow F : \quad c^{-1} = [I_1(\beta J)/I_0(\beta J)] \int_{-\pi}^{\pi} d\omega P(\omega) \cos(\omega)$$

$$P \rightarrow \text{SG} : \quad c^{-1} = [I_1(\beta J)/I_0(\beta J)]^2$$

F \rightarrow SG : cannot yet calculate ...
Parisi – Toulouse hypothesis ?

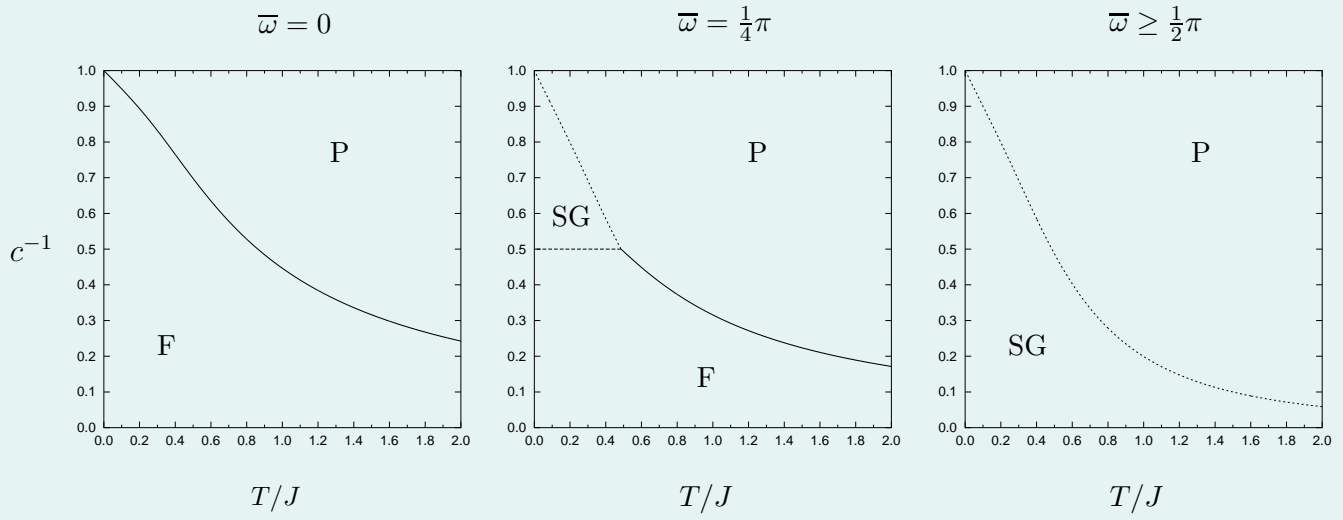
Example:

$$P(\omega) = 1/2\pi$$



Example:

$$P(\omega) = \frac{1}{2}\delta(\omega - \bar{\omega}) + \frac{1}{2}\delta(\omega + \bar{\omega})$$



Theory versus simulations

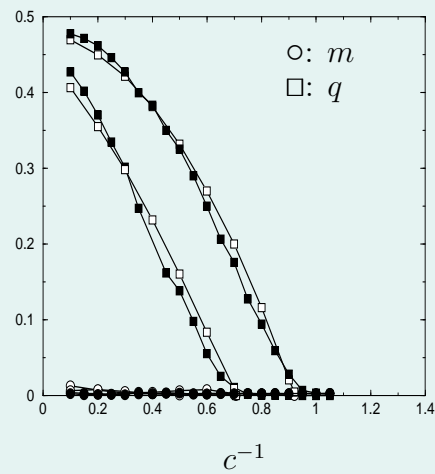
Numerical solution of order parameter equations:

population dynamics with truncated parametrizations $w(\boldsymbol{\mu})$ of functional $W[\{P\}]$

$$m^2 = \left[\frac{1}{N} \sum_i \overline{\langle \cos(\phi_i) \rangle} \right]^2 + \left[\frac{1}{N} \sum_i \overline{\langle \sin(\phi_i) \rangle} \right]^2$$
$$q = \frac{1}{2N} \sum_i \left[\overline{\langle \cos(\phi_i) \rangle^2} + \overline{\langle \sin(\phi_i) \rangle^2} \right]$$

Example:

$$P(\omega) = 1/2\pi$$

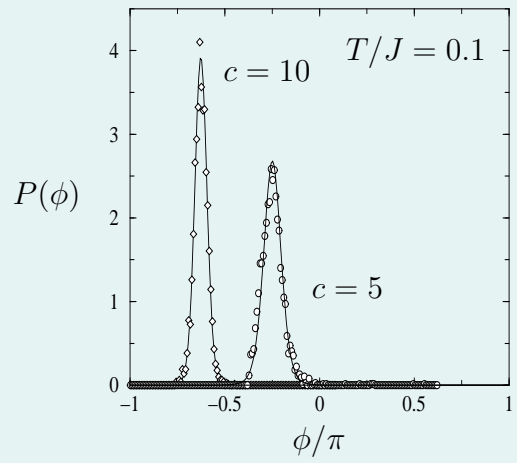
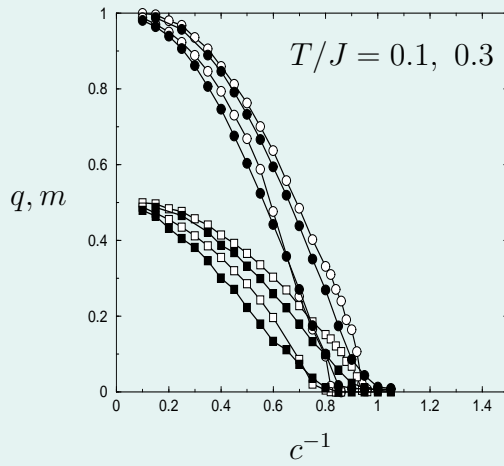


Example:

$$P(\omega) = \delta(\omega)$$

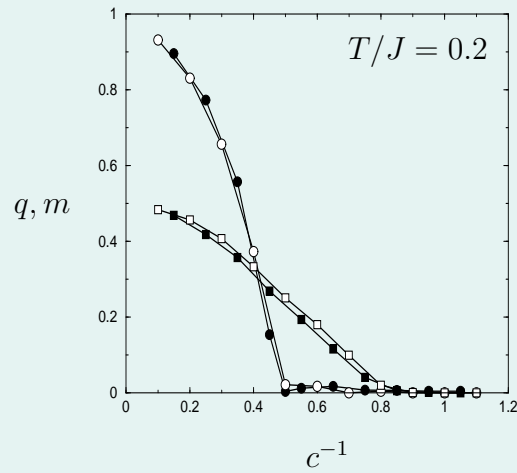
○: m

□: q



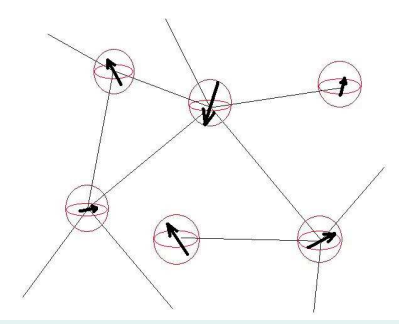
Example:

$$P(\omega) = \frac{1}{2}\delta(\omega - \frac{\pi}{4}) + \frac{1}{2}\delta(\omega + \frac{\pi}{4})$$



confirms
Parisi-Toulouse
for $F \rightarrow SG$ transition

d=3,4,...: HEISENBERG SPINS AND BEYOND



$$\sigma_i \in S_{d-1}$$

$$H = -J \sum_{i < j} c_{ij} \sigma_i \cdot \mathbf{U}_{ij} \sigma_j + \sum_i V(\sigma_i)$$

$$\text{random } \mathbf{U}_{ij} \in \text{SO}(3)$$

$V = 0$:

$$W[\{P\}] = \sum_{\ell \geq 0} p_\ell \int \prod_{k \leq \ell} [\{dP_k\} W[\{P_k\}] d\mathbf{U}_k P(\mathbf{U}_k)] \\ \times \prod_{\sigma \in S_{d-1}} \delta \left[P(\sigma) - \frac{\prod_{k=1}^{\ell} \int d\sigma' P_k(\sigma') e^{\beta J \sigma \cdot \mathbf{U}_k \sigma'}}{\int d\sigma'' \prod_{k=1}^{\ell} \int d\sigma' P_k(\sigma') e^{\beta J \sigma'' \cdot \mathbf{U}_k \sigma'}} \right]$$

$$\text{paramagnetic state : } \lim_{\beta \rightarrow 0} W[\{P\}] = \prod_{\sigma \in S_{d-1}} \delta \left[P(\sigma) - \frac{1}{|S_{d-1}|} \right]$$

Phase transitions

*Continuous bifurcations away from paramagnetic state
located by Guzai (i.e. functional moment) expansion*

$$P(\boldsymbol{\sigma}) \rightarrow |S_{d-1}|^{-1} + \Delta(\boldsymbol{\sigma}), \quad W[\{P\}] \rightarrow \tilde{W}[\{\Delta\}]$$

functional moments

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\boldsymbol{\sigma}_1) \dots \Delta(\boldsymbol{\sigma}_r)$$

assume

$$\exists \epsilon \ll 1 \text{ such that } \int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\boldsymbol{\sigma}_1) \dots \Delta(\boldsymbol{\sigma}_r) = \mathcal{O}(\epsilon^r)$$

generalized

modified Bessel function:

$$I_{0,d}(z) = |S_{d-1}|^{-1} \int_{S_{d-1}} d\boldsymbol{\sigma} e^{z\sigma_1}$$

Lowest order ϵ^1

$$\Psi(\boldsymbol{\sigma}) = \frac{c}{I_{0,d}(\beta J)} \int_{S_{d-1}} \frac{d\boldsymbol{\sigma}'}{|S_{d-1}|} \Psi(\boldsymbol{\sigma}') \int d\mathbf{U} P(\mathbf{U}) e^{\beta J \boldsymbol{\sigma} \cdot \mathbf{U} \boldsymbol{\sigma}'} \quad \int_{S_{d-1}} d\boldsymbol{\sigma} \Psi(\boldsymbol{\sigma}) = 0$$

commuting operators:

$$KL\Psi = c^{-1} I_{0,d}(\beta J) \Psi$$

$$(K\Psi)(\boldsymbol{\sigma}) = \int_{S_{d-1}} \frac{d\boldsymbol{\tau}}{|S_{d-1}|} e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\tau}} \Psi(\boldsymbol{\tau}) \quad (L\Psi)(\boldsymbol{\sigma}) = \int d\mathbf{U} P(\mathbf{U}) \Psi(\mathbf{U}^\dagger \boldsymbol{\sigma})$$

Lowest order ϵ^2

$$\Psi(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \frac{c}{I_{0,d}^2(\beta J)} \int_{S_{d-1}} \frac{d\boldsymbol{\tau}^1 d\boldsymbol{\tau}^2}{|S_{d-1}|^2} \Psi(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2) \int d\mathbf{U} P(\mathbf{U}) e^{\beta J (\boldsymbol{\sigma}^1 \cdot \mathbf{U} \boldsymbol{\tau}^1 + \boldsymbol{\sigma}^2 \cdot \mathbf{U} \boldsymbol{\tau}^2)}$$

$$\int_{S_{d-1}} d\boldsymbol{\sigma}^1 \Psi(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \int_{S_{d-1}} d\boldsymbol{\sigma}^2 \Psi(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = 0$$

commuting operators:

$$KL\Psi = c^{-1} I_{0,d}^2(\beta J) \Psi$$

$$(K\Psi)(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \int_{S_{d-1}} \frac{d\boldsymbol{\tau}^1 d\boldsymbol{\tau}^2}{|S_{d-1}|^2} e^{\beta J (\boldsymbol{\sigma}^1 \cdot \boldsymbol{\tau}^1 + \boldsymbol{\sigma}^2 \cdot \boldsymbol{\tau}^2)} \Psi(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2)$$

$$(L\Psi)(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \int d\mathbf{U} P(\mathbf{U}) \Psi(\mathbf{U}^\dagger \boldsymbol{\sigma}^1, \mathbf{U}^\dagger \boldsymbol{\sigma}^2)$$

Continuous phase transitions for $d = 3$: classical Heisenberg spins

*Euler angle representation
of random rotations in \mathbb{R}^3 :*

$$P(\mathbf{U}) = \epsilon \delta[\mathbf{U} - \mathbb{1}] + \frac{1 - \epsilon}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^\pi d\beta \int_0^{2\pi} d\gamma \sin(\beta) \delta[\mathbf{U} - R_z(\alpha) R_y(\beta) R_z(\gamma)]$$

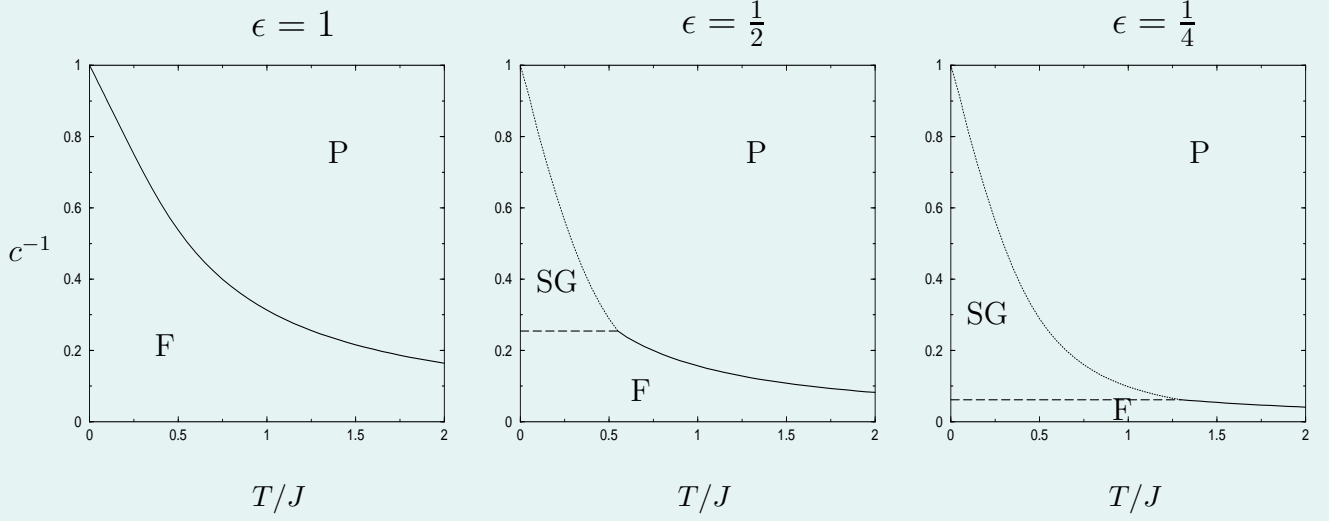
$$R_z(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_y(\beta) = \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \quad R_z(\gamma) = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P \rightarrow F : \quad \int_0^1 dy I_0(\beta J \sqrt{1-x^2} \sqrt{1-y^2}) \sinh(\beta J x y) \rho(y) = \frac{\sinh(\beta J)}{\epsilon \beta J c} \rho(x)$$

$$P \rightarrow SG : \quad \int_{-1}^1 \frac{dy ds dt}{4\pi} I_0(\beta J \sqrt{1-s^2} \sqrt{1-x^2}) e^{\beta J [sx+t]} \frac{\theta[(1-s^2)(1-t^2) - (y-st)^2]}{\sqrt{(1-s^2)(1-t^2) - (y-st)^2}} \psi(y) \\ = \frac{\sinh^2(\beta J)}{c(\beta J)^2} \psi(x)$$

$$\text{subject to constraint} \quad \int_{-1}^1 dy \psi(y) = 0$$

Phase diagrams for $d = 3$



$$P(\mathbf{U}) = \epsilon \delta[\mathbf{U} - \mathbb{1}] + \frac{1-\epsilon}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^\pi d\beta \int_0^{2\pi} d\gamma \sin(\beta) \delta[\mathbf{U} - R_z(\alpha) R_y(\beta) R_z(\gamma)]$$

Theory versus simulations

population dynamics with truncated parametrizations $w(\boldsymbol{\mu})$ of functional $W[\{P\}]$

$$m = \sqrt{m_x^2 + m_y^2 + m_z^2} \quad q = \frac{1}{3}(q_x + q_y + q_z)$$

$$m_x = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \cos(\phi_i) \sin(\theta_i) \rangle}$$

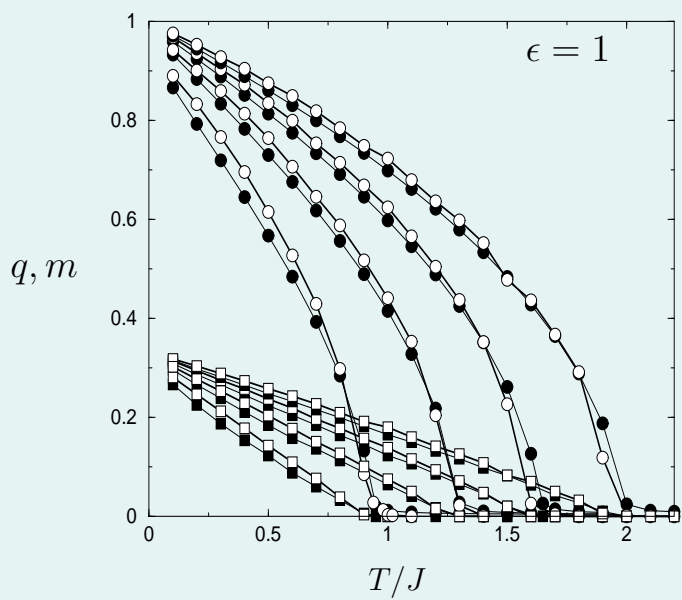
$$q_x = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \cos(\phi_i) \sin(\theta_i) \rangle^2}$$

$$m_y = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sin(\phi_i) \sin(\theta_i) \rangle}$$

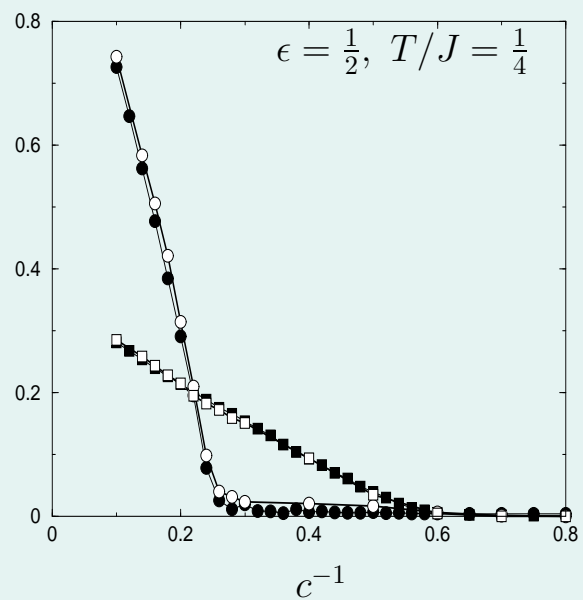
$$q_y = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sin(\phi_i) \sin(\theta_i) \rangle^2}$$

$$m_z = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \cos(\theta_i) \rangle}$$

$$q_z = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \cos(\theta_i) \rangle^2}$$



$c = 3, 4, 5, 6$ (left to right)



confirmation of Parisi-Toulouse

SUMMARY

- *finitely connected vector spin models with random matrix interactions are solvable with presently available methods*
- *RS order parameter is a functional $W[\{P\}]$*
- *continuous transition lines within RS can be calculated exactly using Guzai (functional moment) expansions of the functional $W[\{P\}]$*
- *Parisi-Toulouse hypothesis ($F \rightarrow SG$ transition) appears correct*
- *RSB effects are only modest, limited to very low temperatures*
- *obvious possible extensions:*
 - *RSB*
 - *non-Poissonian graphs*
- *less obvious extensions: dynamics*