

Theory of agent-based market models with controlled levels of greed and anxiety

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Abstract

We use generating functional analysis to study minority-game-type market models with generalized strategy valuation updates that control the psychology of agents' actions. The agents' choice between trend-following and contrarian trading, and their vigor in each, depends on the overall state of the market. Even in 'fake history' models, the theory now involves an effective overall bid process (coupled to the effective agent process) which can exhibit profound remanence effects and new phase transitions. For some models the bid process can be solved directly, others require Maxwell-construction-type approximations.

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1. Introduction

Minority games (MG) [1, 2] are simple models proposed about a decade ago to understand the origin of the cooperative phenomena and non-trivial fluctuations observed in financial markets, on the basis of so-called inductive decision making by imperfect interacting agents [3]. With the recent exposure of the inadequacy of classical economic modelling that assumes full rationality, intelligence and honesty of market players and evolution to an efficient stable market, the case for the alternative MG approach to market models is probably beyond discussion. The further strength of MG-type models is that they can be solved analytically. In particular, the generating functional analysis (GFA) method has proven to be an effective tool in this field [4–6]. We refer to the recent textbooks [7, 8] for historical backgrounds, the connection between MGs and markets, details on mathematical methods and full references.

Standard MG models are now understood quite well. We regard it as our next task to generalize the mathematical technology that has been developed in the study of standard MGs, so that this technology can also be applied to market models that are more realistic economically. To do so we generalize the MG agents' strategy valuation rules, allowing in these rules for a non-trivial (and possibly non-monotonic) dependence on the overall market

bid A . The overall market bid is, as usual, interpreted as a measure of the supply/demand mismatch in the market. For instance, agents in our generalized models will be allowed to switch from a contrarian to a trend-following strategy and adjust their level of caution on the basis of how anomalous they perceive the market to be. It is, however, not at all clear what form the above dependence should take. On the one hand, one could argue that when markets are booming agents are trend-followers, as in the majority game [10], leading to a steady rise of equity prices. The boom would continue until the magnitude $|A|$ of the overall market bid becomes too large. At that point agents lose their confidence in the sustainability of the boom, leading to turmoil and contrarian behaviour. This was the view of [11, 12]. Alternatively, one could equally argue that for small $|A|$ the market would be regarded as normal, and agents would seek advantage by speculative contrarian trading. In this latter picture, large values of $|A|$ (regarded as anomalous by the agents) would prompt panic-driven trend-following.

In this paper we first develop the theory for MG-type models with arbitrary valuation update functions $F[A]$. In doing so we limit ourselves to so-called fake market histories, advocated first in [13]. We then focus on specific choices for $F[A]$ that represent three distinct agent behaviour scenarios. The first scenario describes a market of strict contrarians, with controlled levels of caution versus greed. The second is one where non-volatile markets (i.e. those with modest values of $|A|$) are interpreted by agents as risk free and prompt trend-following, with agents becoming contrarians under volatile conditions. The third scenario is one where in non-volatile markets agents are contrarians, who switch to panic-driven trend-following under volatile conditions. Although the latter two cases are mathematically similar, they lead to qualitatively different solutions. We will find that certain regions of the phase diagram are characterized by prominent remanence effects with multiple locally stable states, prompting us to rely on approximations of the Maxwell-construction type. In all non-remanent cases our predictions are confirmed satisfactorily by numerical simulations. When there is remanence, there remain deviations and gaps in our understanding which will require further investigation.

2. Definitions

Let there to be N agents in the game, labelled by $i \in \{1, \dots, N\}$, where N is odd. The game proceeds at discrete steps $t = 0, 1, 2, \dots$. At every step t each agent takes a binary decision $b_i(t) \in \{-1, 1\}$, the ‘bid’, representing e.g. buying (-1) versus selling ($+1$). A rescaled aggregate overall bid at step t is defined as $A(t) = N^{-1/2} \sum_{i=1}^N b_i(t) + A_e(t)$, where $A_e(t)$ is an external contribution, representing perturbations, the impact of large market operators, or the action of regulators. At each step, the agents are provided with external information, which in fake history MGs is a random number $\mu(t) \in \{1, \dots, p\}$. Each agent i has two strategies \mathbf{R}_{ia} , with $a = 1, 2$, with which to map each μ to a trading action:

$$\mathbf{R}_{ia} : \{1, \dots, p\} \rightarrow \{-1, 1\}.$$

A strategy functions as a lookup table with p entries, all of which are drawn independently at random from $\{-1, 1\}$ (with equal probabilities) before the start of the game. If agent i uses strategy a at step t , then he/she will act (deterministically) according to $b_i(t) = R_{ia}^{\mu(t)}$.

In the standard MG an agent makes a profit if he/she finds himself/herself in the minority, i.e. if $A(t)b_i(t) < 0$. Each agent monitors the performance of his/her two strategies (irrespective of whether they were used) by measuring the so-called valuations $p_{ia}(t)$, defined dynamically via the update rule $p_{ia}(t+1) = p_{ia}(t) - \tilde{\eta}A(t)R_{ia}^{\mu(t)}$. Here $\tilde{\eta} > 0$ denotes a ‘learning rate’. At each step t of the game, each agent i will select his/her best strategy $a_i(t)$ at that stage of the process, defined as $a_i(t) = \operatorname{argmax}_{a \in \{1,2\}} \{p_{ia}(t)\}$. Agents thus always

behave as ‘contrarians’. Here we generalize this rule in the spirit of [11], to allow whether agents behave as contrarians or trend-followers to depend on the instantaneous overall state $A(t)$ of the market. We define $p_{ia}(t+1) = p_{ia}(t) - \tilde{\eta} R_{ia}^{\mu(t)} F[A(t)]$, with an anti-symmetric function $F[-A] = -F[A]$ (in order to retain $A = 0$ as the ‘natural’ value for the overall bid). In the ‘batch’ version of the above dynamics one replaces in the evolution equations for the valuations the randomly drawn information $\mu(t)$ by an average over $\{1, \dots, p\}$. The result is

$$p_{ia}(t+1) = p_{ia}(t) - \frac{\tilde{\eta}}{\sqrt{N}} \sum_{\mu=1}^p F[A_{\mu}(t)] R_{ia}^{\mu} \quad (1)$$

$$A_{\mu}(t) = A_e(t) + \frac{1}{\sqrt{N}} \sum_{i=1}^N R_{ia_i(t)}^{\mu}. \quad (2)$$

We switch in the usual manner to new variables $q_i(t) = \frac{1}{2}[p_{i1}(t) - p_{i2}(t)]$, $\xi_i^{\mu} = \frac{1}{2}(R_{i1}^{\mu} - R_{i2}^{\mu})$, $\omega_i^{\mu} = \frac{1}{2}(R_{i1}^{\mu} + R_{i2}^{\mu})$ and $\Omega_{\mu} = N^{-1/2} \sum_i \omega_i^{\mu}$. We choose $\tilde{\eta} = 2$, and add perturbations $\theta_i(t)$ in order to define response functions. We introduce decision noise by replacing $\text{sgn}[q_i(t)] \rightarrow \sigma[q_i(t), z_i(t)]$, in which $\{z_j(t)\}$ are independent and zero-average Gaussian random numbers (the standard examples are additive and multiplicative noise definitions, corresponding to $\sigma[q, z] = \text{sgn}[q + Tz]$ and $\sigma[q, z] = \text{sgn}[q] \text{sgn}[1 + Tz]$). This leaves us with

$$q_i(t+1) = q_i(t) + \theta_i(t) - \frac{2}{\sqrt{N}} \sum_{\mu=1}^p \xi_i^{\mu} F[A_{\mu}(t)] \quad (3)$$

$$A_{\mu}(t) = A_e(t) + \Omega_{\mu} + \frac{1}{\sqrt{N}} \sum_{j=1}^N \xi_j^{\mu} \sigma[q_j(t), z_j(t)]. \quad (4)$$

One obtains the standard batch MG for $F[A] = A$, and a batch *majority* game for $F[A] = -A$. We will develop the theory initially for arbitrary $F[A]$, but focus ultimately on the choices

$$F[A] = \text{sgn}(A)|A|^{\gamma}, \quad F[A] = \tau A[1 - A^2/A_0^2] \quad (5)$$

with $\tau = \pm 1$ and $A_0 > 0$. In the second formula, for $\tau = -1$ the agents behave as a trend-followers (majority game play) when $|A| < A_0$, but switch to contrarians (minority game play) for $|A| > A_0$; this is the model of [11, 12]. For $\tau = 1$ the situation is reversed.

We will write averaging over the stochastic process (3) as $\langle \dots \rangle$. The global bid fluctuations in the system are characterized by the volatility matrix $\Xi_{tt'}$:

$$\Xi_{tt'} = \frac{1}{p} \sum_{\mu=1}^p \left\langle \left[A_{\mu}(t) - \frac{1}{p} \sum_{v=1}^p \langle A_v(t) \rangle \right] \left[A_{\mu}(t') - \frac{1}{p} \sum_{v=1}^p \langle A_v(t') \rangle \right] \right\rangle. \quad (6)$$

From this follows the conventional market volatility σ via $\sigma^2 = \lim_{\tau \rightarrow \infty} \tau^{-1} \sum_{t=1}^{\tau} \Xi_{tt}$.

3. Generating functional analysis

The appropriate moment generating functional for a stochastic process of the type (3), given that now we will be interested also in overall bid statistics, is

$$Z[\psi, \phi] = \left\langle e^{i \sum_{t \geq 0} \sum_i \psi_i(t) \sigma[q_i(t), z_i(t)] + i \sqrt{2} \sum_{t \geq 0} \sum_{\mu} \phi_{\mu}(t) A_{\mu}(t)} \right\rangle \quad (7)$$

with $A_\mu(t)$ as defined in (4). For (7) to be well defined, we specify an upper time limit t_{\max} . Taking suitable derivatives of (7) with respect to the variables $\{\psi_i(t), \phi_\mu(t)\}$ generates all moments of the random variables $\{\sigma[q_i(t), z_i(t)]\}$ and $\{A_\mu(t)\}$, at arbitrary times, e.g.

$$\langle \sigma[q_i(t), z_i(t)] \rangle = -i \lim_{\psi, \phi \rightarrow \mathbf{0}} \frac{\partial Z[\psi, \phi]}{\partial \psi_i(t)} \quad (8)$$

$$\langle \sigma[q_i(t), z_i(t)] \sigma[q_j(t'), z_j(t')] \rangle = - \lim_{\psi, \phi \rightarrow \mathbf{0}} \frac{\partial^2 Z[\psi, \phi]}{\partial \psi_i(t) \partial \psi_j(t')} \quad (9)$$

$$\langle A^\mu[\mathbf{q}(t), \mathbf{z}(t)] \rangle = - \frac{i}{\sqrt{2}} \lim_{\psi, \phi \rightarrow \mathbf{0}} \frac{\partial Z[\psi, \phi]}{\partial \phi_\mu(t)} \quad (10)$$

$$\langle A^\mu[\mathbf{q}(t), \mathbf{z}(t)] A^\nu[\mathbf{q}(t'), \mathbf{z}(t')] \rangle = - \frac{1}{2} \lim_{\psi, \phi \rightarrow \mathbf{0}} \frac{\partial^2 Z[\psi, \phi]}{\partial \phi_\mu(t) \partial \phi_\nu(t')} \quad (11)$$

Averaging (7) over the disorder (the strategies) allows us to obtain from (8)–(11) expressions for disorder-averaged observables, such as correlation and response functions:

$$C_{tt'} = \frac{1}{N} \sum_i \overline{\langle \sigma[q_i(t), z_i(t)] \sigma[q_i(t'), z_i(t')] \rangle} = - \lim_{\psi \rightarrow \mathbf{0}} \frac{1}{N} \sum_i \frac{\partial^2 \overline{Z[\psi]}}{\partial \psi_i(t) \partial \psi_i(t')} \quad (12)$$

$$G_{tt'} = \frac{1}{N} \sum_i \frac{\partial}{\partial \theta_i(t')} \overline{\langle \sigma[q_i(t), z_i(t)] \rangle} = - \lim_{\psi \rightarrow \mathbf{0}} \frac{i}{N} \sum_i \frac{\partial^2 \overline{Z[\psi]}}{\partial \psi_i(t) \partial \theta_i(t')} \quad (13)$$

3.1. Evaluation of the disorder average

We write (7) in the usual way as an integral over all possible values of $q_i(t)$ and $A_\mu(t)$ at all times, and insert δ -functions to select the solution of (3), followed by averaging over the noise $\mathbf{z} = \{z_i(t)\}$. With the short-hand $s_i(t) = \sigma[q_i(t), z_i(t)]$ we obtain

$$\begin{aligned} Z[\psi, \phi] = & \int \left[\prod_{\mu t} \frac{dA_\mu(t) d\hat{A}_\mu(t)}{2\pi} e^{i\hat{A}_\mu(t)[A_\mu(t) - A_c(t)] + i\sqrt{2}\phi_\mu(t)A_\mu(t)} \right] \\ & \times \left\langle \int p_0(\mathbf{q}(0)) \left[\prod_{it} \frac{dq_i(t) d\hat{q}_i(t)}{2\pi} e^{i\hat{q}_i(t)[q_i(t+1) - q_i(t) - \theta_i(t)] + i\psi_i(t)s_i(t)} \right] \right. \\ & \left. \times \prod_{i\mu} e^{\frac{i}{\sqrt{N}} \sum_t \{2\xi_i^\mu \hat{q}_i(t) F[A_\mu(t)] - \hat{A}_\mu(t) [\omega_i^\mu + \xi_i^\mu s_i(t)]\}} \right]_{\mathbf{z}} \end{aligned} \quad (14)$$

The disorder variables appear only in the last line, so

$$\begin{aligned} \overline{Z[\psi, \phi]} = & \int \left[\prod_{\mu t} \frac{dA_\mu(t) d\hat{A}_\mu(t)}{2\pi} e^{i\hat{A}_\mu(t)[A_\mu(t) - A_c(t)] + i\sqrt{2}\phi_\mu(t)A_\mu(t)} \right] \\ & \times \left\langle \int p_0(\mathbf{q}(0)) \left[\prod_{it} \frac{dq_i(t) d\hat{q}_i(t)}{2\pi} e^{i\hat{q}_i(t)[q_i(t+1) - q_i(t) - \theta_i(t)] + i\psi_i(t)s_i(t)} \right] \Xi \right\rangle_{\mathbf{z}} \end{aligned} \quad (15)$$

with

$$\begin{aligned} \Xi = & \prod_{i\mu} e^{\frac{i}{\sqrt{N}} \sum_t \{(R_1 - R_2)\hat{q}_i(t) F[A_\mu(t)] - \frac{1}{2}\hat{A}_\mu(t)[R_1 + R_2 + (R_1 - R_2)s_i(t)]\}} \\ = & \prod_{\mu} e^{-\frac{1}{4} \sum_{tt'} \hat{A}_\mu(t) \hat{A}_\mu(t') - \sum_{tt'} L_{tt'} F[A_\mu(t)] F[A_\mu(t')] - \frac{1}{4} \sum_{tt'} C_{tt'} \hat{A}_\mu(t) \hat{A}_\mu(t')} \\ & \times e^{\sum_{tt'} K_{tt'} \hat{A}_\mu(t) F[A_\mu(t')] + \mathcal{O}(N^{-1})} \end{aligned} \quad (16)$$

where we introduced the temporary abbreviations

$$C_{tt'} = \frac{1}{N} \sum_i s_i(t) s_i(t') K_{tt'} = \frac{1}{N} \sum_i s_i(t) \hat{q}_i(t') L_{tt'} = \frac{1}{N} \sum_i \hat{q}_i(t) \hat{q}_i(t').$$

We isolate these as usual by inserting appropriate δ -functions (in integral representation, which generates conjugate integration variables), and define the short-hands $\mathcal{DC} = \prod_{tt'} [dC_{tt'}/\sqrt{2\pi}]$ and $\mathcal{DA} = \prod_t [dA(t)/\sqrt{2\pi}]$ (with similar definitions for the other kernels and functions). Substitution into $\overline{Z[\psi, \phi]}$, followed by re-arranging terms, then leads to

$$\begin{aligned} \overline{Z[\psi, \phi]} &= \int [\mathcal{DC} \mathcal{D}\hat{C}] [\mathcal{DK} \mathcal{D}\hat{K}] [\mathcal{DL} \mathcal{D}\hat{L}] e^{iN \sum_{tt'} [\hat{C}_{tt'} C_{tt'} + \hat{K}_{tt'} K_{tt'} + \hat{L}_{tt'} L_{tt'}] + \mathcal{O}(\log(N))} \\ &\quad \times \left\langle \int \mathcal{D}\mathbf{q} \mathcal{D}\hat{\mathbf{q}} p_0(\mathbf{q}(0)) e^{i \sum_{ii} \hat{q}_i(t) [q_i(t+1) - q_i(t) - \theta_i(t)] + i \sum_{ii} \psi_i(t) s_i(t)} \right. \\ &\quad \times e^{-i \sum_i \sum_{tt'} [\hat{C}_{tt'} s_i(t) s_i(t') + \hat{K}_{tt'} s_i(t) \hat{q}_i(t') + \hat{L}_{tt'} \hat{q}_i(t) \hat{q}_i(t')]} \Bigg\rangle_{\mathbf{z}} \\ &\quad \times \prod_{\mu} \int \mathcal{D}\mathbf{A} \mathcal{D}\hat{\mathbf{A}} e^{i \sum_t [\hat{A}(t) [A(t) - A_e(t)] + i\sqrt{2}\phi_{\mu}(t) A(t)] - \frac{1}{4} \sum_{tt'} \hat{A}(t) [1 + C_{tt'}] \hat{A}(t')} \\ &\quad \times e^{-\sum_{tt'} L_{tt'} F[A(t)] F[A(t')] + \sum_{tt'} K_{tt'} \hat{A}(t) F[A(t')]} \end{aligned} \quad (17)$$

Upon assuming simple initial conditions of the form $p_0(\mathbf{q}) = \prod_i p_0(q_i)$ we then arrive at

$$\overline{Z[\psi, \phi]} = \int [\mathcal{DC} \mathcal{D}\hat{C}] [\mathcal{DK} \mathcal{D}\hat{K}] [\mathcal{DL} \mathcal{D}\hat{L}] e^{N[\Psi + \Phi + \Omega] + \mathcal{O}(\log(N))} \quad (18)$$

with

$$\Psi = i \sum_{tt'} [\hat{C}_{tt'} C_{tt'} + \hat{K}_{tt'} K_{tt'} + \hat{L}_{tt'} L_{tt'}] \quad (19)$$

$$\begin{aligned} \Phi &= \frac{1}{N} \sum_{\mu} \log \int \mathcal{D}\mathbf{A} \mathcal{D}\hat{\mathbf{A}} e^{i \sum_t [\hat{A}(t) [A(t) - A_e(t)] + i\sqrt{2}\phi_{\mu}(t) A(t)]} \\ &\quad \times e^{-\frac{1}{4} \sum_{tt'} \hat{A}(t) [1 + C_{tt'}] \hat{A}(t') - \sum_{tt'} L_{tt'} F[A(t)] F[A(t')] + \sum_{tt'} K_{tt'} \hat{A}(t) F[A(t')]} \end{aligned} \quad (20)$$

$$\begin{aligned} \Omega &= \frac{1}{N} \sum_i \log \left\langle \int \mathcal{D}q \mathcal{D}\hat{q} p_0(q(0)) e^{i \sum_t [\hat{q}(t) [q(t+1) - q(t) - \theta_i(t)] + \psi_i(t) \sigma[q(t), z(t)]] - i \sum_{tt'} \hat{q}(t) \hat{L}_{tt'} \hat{q}(t')} \right. \\ &\quad \times e^{-i \sum_{tt'} [\hat{C}_{tt'} \sigma[q(t), z(t)] \sigma[q(t'), z(t')] + \hat{K}_{tt'} \sigma[q(t), z(t)] \hat{q}(t')]} \Bigg\rangle_{\mathbf{z}} \end{aligned} \quad (21)$$

The $\mathcal{O}(\log(N))$ term in the exponent of (18) is independent of $\{\psi_i(t), \phi_{\mu}(t), \theta_i(t)\}$, and the external bid $A_e(\ell)$ and the introduced valuation update function $F[A]$ appear only in Φ .

3.2. Saddle-point equations and interpretation of order parameters

The disorder-averaged functional (18) is evaluated by steepest descent integration, leading to coupled equations from which to solve the dynamic order parameters $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$:

$$C_{tt'} = \langle \sigma[q(t), z(t)] \sigma[q(t'), z(t')] \rangle_{\star} \quad (22)$$

$$K_{tt'} = \langle \sigma[q(t), z(t)] \hat{q}(t') \rangle_{\star} \quad (23)$$

$$L_{tt'} = \langle \hat{q}(t) \hat{q}(t') \rangle_{\star} \quad (24)$$

$$\hat{C}_{tt'} = \frac{i\partial\Phi}{\partial C_{tt'}} \hat{K}_{tt'} = \frac{i\partial\Phi}{\partial K_{tt'}} \hat{L}_{tt'} = \frac{i\partial\Phi}{\partial L_{tt'}}. \quad (25)$$

The notation $\langle \dots \rangle_\star$ in the above expressions is a short-hand for

$$\langle g[\{q, \hat{q}, z\}] \rangle_\star = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \frac{\int \mathcal{D}q \mathcal{D}\hat{q} \langle M_i[\{q, \hat{q}, z\}] g[\{q, \hat{q}, z\}] \rangle_z}{\int \mathcal{D}q \mathcal{D}\hat{q} \langle M_i[\{q, \hat{q}, z\}] \rangle_z} \quad (26)$$

$$\begin{aligned} M_i[\{q, \hat{q}, z\}] &= p_0(q(0)) e^{i \sum_t \hat{q}(t)[q(t+1) - q(t) - \theta_i(t)] + i \sum_t \psi_i(t) \sigma[q(t), z(t)]} \\ &\times e^{-i \sum_{t'} [\hat{C}_{tt'} \sigma[q(t), z(t)] \sigma[q(t'), z(t')] + \hat{K}_{tt'} \sigma[q(t), z(t)] \hat{q}(t')]} \\ &\times e^{-i \sum_{t'} \hat{L}_{tt'} \hat{q}(t) \hat{q}(t')}. \end{aligned} \quad (27)$$

Since the bid contribution $A_e(t)$ appears only in Φ , one can take over standard results from the theory of batch MGs with regard to interpretation and simplification of order parameters (see e.g. [8]). Using the normalization identity $\lim_{\psi, \phi \rightarrow \mathbf{0}} \overline{Z}[\psi, \phi] = 1$ we find

$$C_{tt'} = \lim_{\psi, \phi \rightarrow \mathbf{0}} \langle \sigma[q(t), z(t)] \sigma[q(t'), z(t')] \rangle_\star \quad (28)$$

$$G_{tt'} = -i \lim_{\psi, \phi \rightarrow \mathbf{0}} \langle \sigma[q(t), z(t)] \hat{q}(t') \rangle_\star \quad (29)$$

$$0 = - \lim_{\psi, \phi \rightarrow \mathbf{0}} \langle \hat{q}(t) \hat{q}(t') \rangle_\star. \quad (30)$$

These expressions (28)–(30) are to be evaluated *at* the physical saddle-point of $\Psi + \Omega + \Phi$. We conclude from (29) and (30), in combination with (23) and (24), that for all (t, t')

$$K_{tt'} = iG_{tt'}, \quad L_{tt'} = 0. \quad (31)$$

We now send the fields $\{\psi_i, \phi_\mu\}$ to zero, and choose $\{\theta_i\}$ to be independent of i . The measure $M_i[\{q, \hat{q}, z\}]$ then loses its dependence on i , and equations (22)–(25) simplify to

$$C_{tt'} = \langle \sigma[q(t), z(t)] \sigma[q(t'), z(t')] \rangle_\star \quad (32)$$

$$G_{tt'} = -i \langle \sigma[q(t), z(t)] \hat{q}(t') \rangle_\star \quad (33)$$

$$\hat{C}_{tt'} = \lim_{L \rightarrow 0} \frac{i\partial\Phi}{\partial C_{tt'}}, \quad \hat{K}_{tt'} = \lim_{L \rightarrow 0} \frac{i\partial\Phi}{\partial K_{tt'}}, \quad \hat{L}_{tt'} = \lim_{L \rightarrow 0} \frac{i\partial\Phi}{\partial L_{tt'}} \quad (34)$$

with

$$\langle g[\{q, \hat{q}, z\}] \rangle_\star = \frac{\int \mathcal{D}q \mathcal{D}\hat{q} \langle M[\{q, \hat{q}, z\}] g[\{q, \hat{q}, z\}] \rangle_z}{\int \mathcal{D}q \mathcal{D}\hat{q} \langle M[\{q, \hat{q}, z\}] \rangle_z} \quad (35)$$

$$\begin{aligned} M[\{q, \hat{q}, z\}] &= p_0(q(0)) e^{i \sum_t \hat{q}(t)[q(t+1) - q(t) - \theta(t)] - i \sum_{t'} \hat{L}_{tt'} \hat{q}(t) \hat{q}(t')} \\ &\times e^{-i \sum_{t'} [\hat{C}_{tt'} \sigma[q(t), z(t)] \sigma[q(t'), z(t')] + \hat{K}_{tt'} \sigma[q(t), z(t)] \hat{q}(t')]} \end{aligned} \quad (36)$$

3.3. Evaluation of Φ

We turn our attention to the function Φ (20), which, according to (34), we only need to know for small L . We define the matrices \mathbb{I} and D , with entries $\mathbb{I}_{tt'} = \delta_{tt'}$ and $D_{tt'} = 1 + C_{tt'}$, and the short-hands $\mathbf{A} = \{A(t)\}$, $\mathbf{A}_e = \{A_e(t)\}$ and $F[\mathbf{A}] = \{F[A(t)]\}$. This allows us to write

$$\begin{aligned} \Phi &= \frac{1}{N} \sum_\mu \log \int d\mathbf{A} d\xi P(\xi) \delta[\mathbf{A} - \mathbf{A}_e + GF[\mathbf{A}] - \xi] \\ &\times e^{i\sqrt{2} \sum_t \phi_\mu(t) A(t) - \sum_{t'} L_{tt'} F[A(t)] F[A(t')]} \end{aligned} \quad (37)$$

with¹

$$P(\xi) = \frac{\prod_t (1/\sqrt{2\pi})}{\sqrt{\det[\frac{1}{2}\mathbf{D}]}} e^{-\xi \cdot \mathbf{D}^{-1} \xi}. \quad (38)$$

The derivation from (20) of the more intuitive form (37) for Φ is based on the identity

$$\int \frac{d\hat{\mathbf{A}}}{\prod_t (2\pi)} e^{i\hat{\mathbf{A}} \cdot \xi - \frac{1}{4} \hat{\mathbf{A}} \cdot \mathbf{D} \hat{\mathbf{A}}} = P(\xi). \quad (39)$$

At the saddle-point, G must obey causality, so we can interpret the above expression in terms of an effective stochastic process for the bid, with a zero-average Gaussian noise field $\xi(t)$:

$$A(t) = A_e(t) - \sum_{t' < t} G_{tt'} F[A(t')] + \xi(t) \quad (40)$$

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = \frac{1}{2} D_{tt'}. \quad (41)$$

The situation is clearly reminiscent of real history MGs [9], although the equations are simpler. From now on we write averages over (40) and (41) simply as $\langle \dots \rangle$. We also define

$$\Phi_0 = \alpha \log \int d\mathbf{A} d\xi P(\xi) \delta[\mathbf{A} - \mathbf{A}_e + G F[\mathbf{A}] - \xi]. \quad (42)$$

At the saddle-point we have $\Phi_0 = 0$. We expand Φ for small L and small ϕ :

$$\begin{aligned} \Phi = \Phi_0 + \frac{i\sqrt{2}}{N} \sum_{\mu t} \phi_\mu(t) \langle A(t) \rangle - \alpha \sum_{tt'} L_{tt'} \langle F[A(t)] F[A(t')] \rangle \\ - \frac{1}{N} \sum_{\mu t t'} \phi_\mu(t) \phi_\mu(t') [\langle A(t) A(t') \rangle - \langle A(t) \rangle \langle A(t') \rangle] + \mathcal{O}(\phi^3, L^2). \end{aligned} \quad (43)$$

Clearly, $\lim_{\phi \rightarrow 0, L \rightarrow 0} \Phi = \Phi_0$, and Φ_0 depends on G only, not on C or L . From our expansion of Φ we can extract all the quantities we are interested in. We only have to be careful that causality can only be assumed *after* the differentiations:

$$\hat{C}_{tt'} = \lim_{L \rightarrow 0, \phi \rightarrow 0} \frac{i\partial\Phi}{\partial C_{tt'}} = 0 \quad (44)$$

$$\hat{K}_{tt'} = \lim_{L \rightarrow 0, \phi \rightarrow 0} \frac{\partial\Phi}{\partial G_{tt'}} = \frac{\partial\Phi_0}{\partial G_{tt'}} = -\alpha \frac{\partial}{\partial A_e(t)} \langle F[A(t')] \rangle \quad (45)$$

$$\hat{L}_{tt'} = \lim_{L \rightarrow 0} \frac{i\partial\Phi}{\partial L_{tt'}} = -i\alpha \langle F[A(t)] F[A(t')] \rangle. \quad (46)$$

We define the bid response function $R_{tt'} = \partial \langle F[A(t)] \rangle / \partial A_e(t')$ and the bid covariance function $\Sigma_{tt'} = 2 \langle F[A(t)] F[A(t')] \rangle$, so $\hat{K}_{tt'} = -\alpha R_{tt'}$ and $\hat{L}_{tt'} = -\frac{1}{2} i\alpha \Sigma_{tt'}$. Causality ensures that $R_{tt'} = 0$ for $t < t'$, and $R_{tt} = 1$. We can also calculate bid statistics:

$$\overline{\langle A^\mu[\mathbf{q}(t), \mathbf{z}(t)] \rangle} = -\frac{i}{\sqrt{2}} \lim_{\psi, \phi \rightarrow 0} \frac{\partial \overline{Z[\psi, \phi]}}{\partial \phi_\mu(t)} = \langle A(t) \rangle \quad (47)$$

$$\langle A^\mu[\mathbf{q}(t), \mathbf{z}(t)] A^\nu[\mathbf{q}(t'), \mathbf{z}(t')] \rangle = -\frac{1}{2} \lim_{\psi, \phi \rightarrow 0} \frac{\partial^2 Z[\psi, \phi]}{\partial \phi_\mu(t) \partial \phi_\nu(t')} = \langle A(t) A(t') \rangle. \quad (48)$$

¹ The symbol ξ has been chosen here in order to follow conventions laid down in earlier papers, e.g. [9]. Since ξ has so far in this paper been used only in a context that involved site indices, namely ξ_t^μ , and since the latter variables stopped being used after equation (14), this cannot cause ambiguity.

3.4. The effective single-agent equation

The above results lead to a further simplification of our saddle-point equations (32)–(34):

$$C_{tt'} = \langle \sigma[q(t), z(t)] \sigma[q(t'), z(t')] \rangle_{\star} \quad (49)$$

$$G_{tt'} = -i \langle \sigma[q(t), z(t)] \hat{q}(t') \rangle_{\star} \quad (50)$$

with

$$\langle g[\{q, \hat{q}, z\}] \rangle_{\star} = \frac{\int \mathcal{D}q \mathcal{D}\hat{q} \langle M[\{q, \hat{q}, z\}] g[\{q, \hat{q}, z\}] \rangle_{\mathbf{z}}}{\int \mathcal{D}q \mathcal{D}\hat{q} \langle M[\{q, \hat{q}, z\}] \rangle_{\mathbf{z}}} \quad (51)$$

$$M[\{q, \hat{q}, z\}] = p_0(q(0)) e^{i \sum_t \hat{q}(t)[q(t+1) - q(t) - \theta(t) + \alpha \sum_{t'} R_{tt'} \sigma[q(t'), z(t')] - \frac{1}{2} \alpha \sum_{t'} \hat{q}(t) \Sigma_{tt'} \hat{q}(t')}. \quad (52)$$

We eliminate $\{\hat{q}(t)\}$ by exploiting causality: the term $\sum_{t'} R_{tt'} \sigma[q(t'), z(t')]$ in (52) involves only values of $q(t')$ with $t' \leq t$. This allows us to calculate the denominator of the fraction (51) by integrating out the variables $\{q(t)\}$ iteratively, first over $q(t_{\max})$ (which gives us $\delta[\hat{q}(t_{\max} - 1)]$), followed by integration over $q(t_{\max} - 1)$, etc. The result is simply

$$\int \mathcal{D}q \mathcal{D}\hat{q} M[\{q, \hat{q}, z\}] = \int dq(0) p_0(q(0)) = 1. \quad (53)$$

This, in turn, implies that

$$\langle \sigma[q(t), z(t)] \hat{q}(t') \rangle_{\star} = i \frac{\partial}{\partial \theta(t')} \langle \sigma[q(t), z(t)] \rangle_{\star}.$$

We do the remaining integrals over $\{q\}$, and write our equations in the simpler form

$$C_{tt'} = \langle \sigma[q(t), z(t)] \sigma[q(t'), z(t')] \rangle_{\star} \quad G_{tt'} = \frac{\partial \langle \sigma[q(t), z(t)] \rangle_{\star}}{\partial \theta(t')} \quad (54)$$

with $\langle g[\{q, z\}] \rangle_{\star} = \int [\prod_t dq(t)] \langle M[\{q, z\}] g[\{q, z\}] \rangle_{\mathbf{z}}$ and

$$M[\{q, z\}] = p_0(q(0)) \int \prod_t \left[\frac{d\eta(t)}{\sqrt{2\pi}} \right] \frac{e^{-\frac{1}{2} \sum_{t'} \eta(t) (\Sigma^{-1})_{tt'} \eta(t')}}{\sqrt{\det \Sigma}} \times \prod_{t \geq 0} \delta \left[q(t+1) - q(t) - \theta(t) + \alpha \sum_{t'} R_{tt'} \sigma[q(t'), z(t')] - \sqrt{\alpha} \eta(t) \right]. \quad (55)$$

We recognize that (55) is the measure corresponding to a single-agent process of the form

$$q(t+1) = q(t) + \theta(t) - \alpha \sum_{t' \leq t} R_{tt'} \sigma[q(t'), z(t')] + \sqrt{\alpha} \eta(t) \quad (56)$$

in which $\eta(t)$ is a zero-mean Gaussian noise, with temporal correlations $\langle \eta(t) \eta(t') \rangle = \Sigma_{tt'}$. The two kernels Σ and R are to be calculated from

$$R_{tt'} = \frac{\partial}{\partial A_e(t')} \langle F[A(t)] \rangle, \quad \Sigma_{tt'} = 2 \langle F[A(t)] F[A(t')] \rangle \quad (57)$$

where the averages refer to the process (40), (41). The correlation and response functions (12), (13), the order parameters of our problem, are to be solved from (54), in which $\langle \dots \rangle_{\star}$ now denotes averaging over (56) and the zero-average Gaussian noise $\{z(t)\}$, with $\langle z(t) z(t') \rangle = \delta_{tt'}$. These results represent a fully exact and closed theory for $N \rightarrow \infty$.

As a simple test we could go back to the standard MG. If we choose $F[A] = A$, the effective bid process (40), (41) becomes linear, and is solved easily:

$$A(t) = \sum_{t'} (\mathbb{I} + G)_{tt'}^{-1} [A_e(t') + \xi(t')] \quad (58)$$

so that we can calculate the kernels Σ and R explicitly:

$$R_{tt'} = \frac{\partial}{\partial A_e(t')} \langle A(t) \rangle = \frac{\partial}{\partial A_e(t')} \sum_s (\mathbb{I} + G)_{ts}^{-1} A_e(s) = (\mathbb{I} + G)_{tt'}^{-1} \quad (59)$$

$$\begin{aligned} \Sigma_{tt'} &= 2 \langle A(t) A(t') \rangle = 2 \sum_{ss'} (\mathbb{I} + G)_{ts}^{-1} (\mathbb{I} + G)_{t's'}^{-1} \langle [A_e(s) + \xi(s)] [A_e(s') + \xi(s')] \rangle \\ &= [(\mathbb{I} + G)^{-1} D[A_e] (\mathbb{I} + G^\dagger)^{-1}]_{tt'} \end{aligned} \quad (60)$$

with $D[A_e]_{tt'} = 2A_e(t)A_e(t') + 1 + C_{tt'}$. One confirms readily that this is the correct solution.

4. Ergodic stationary states for $A_e(t) = A_e$

We now take $A_e(t) = A_e$ for all t . In time-translation invariant stationary states without long-term memory one has $G_{tt'} = G(t - t')$, $C_{tt'} = C(t - t')$, $\Sigma_{tt'} = \Sigma(t - t')$ and $R_{tt'} = R(t - t')$; all three operators $\{C, G, \Sigma\}$ and their powers commute. We try to calculate the four persistent order parameters, $\chi = \sum_{t>0} G(t)$, $\chi_R = \sum_{t \geq 0} R(t)$, $c = \lim_{t \rightarrow \infty} C(t)$ and $S_0^2 = \lim_{t \rightarrow \infty} \Sigma(t)$, from the closed equations (54) and (57). We assume a stationary state without anomalous response, i.e. both χ and χ_R are finite numbers. From now on we will use the the following notation for time averages: $\bar{x} = \lim_{\tau \rightarrow \infty} \tau^{-1} \sum_{t=1}^{\tau} x(t)$. Much of the analysis is standard, and we will where appropriate skip those details that are familiar.

4.1. Equations for persistent order parameters—the effective agent process

We define $\tilde{q} = \lim_{t \rightarrow \infty} q(t)/t$, assuming that this limit exists, and send $t \rightarrow \infty$ in the integrated version of (56). The result is

$$\tilde{q} = \sqrt{\alpha} \bar{\eta} + \bar{\theta} - \alpha \chi_R \bar{\sigma}. \quad (61)$$

Here $\bar{\sigma} = \lim_{\tau \rightarrow \infty} \tau^{-1} \sum_{t \leq \tau} \int Dz \sigma[\tilde{q}t, z]$ and $Dz = (2\pi)^{-1/2} e^{-z^2/2} dz$. Given the properties of $\sigma[q]$, in cases where $\tilde{q} \neq 0$ we must have $\bar{\sigma} = \text{sgn}[\tilde{q}] \cdot \sigma[\infty]$; for $\tilde{q} = 0$ we know that $|\bar{\sigma}| \leq \sigma[\infty]$. We inspect the possible solutions ‘fickle’ ($\tilde{q} = 0$) versus ‘frozen’ ($\tilde{q} \neq 0$), noting that in both cases we must have $\text{sgn}[\chi_R \bar{\sigma}] = \text{sgn}[\bar{\eta} \sqrt{\alpha} + \bar{\theta}]$. For the fickle solution the situation is clear:

$$\text{‘fickle’}: \quad \frac{\bar{\sigma}}{\sigma[\infty]} = \frac{\sqrt{\alpha} \bar{\eta} + \bar{\theta}}{\alpha \chi_R \sigma[\infty]} \quad \text{exists if} \quad |\bar{\eta} \sqrt{\alpha} + \bar{\theta}| \leq \sigma[\infty] |\chi_R| \alpha. \quad (62)$$

For the frozen solution we must distinguish between the cases $\chi_R > 0$ and $\chi_R < 0$, representing negative versus positive feedback in the effective agent equation.

- $\chi_R > 0$. Here due to the absence of positive feedback in the system, we can write directly

$$|\bar{\eta} \sqrt{\alpha} + \bar{\theta}| > \sigma[\infty] \chi_R \alpha: \quad \text{‘frozen’ solution,} \quad \bar{\sigma} = \sigma[\infty] \text{sgn}[\bar{\eta} \sqrt{\alpha} + \bar{\theta}] \quad (63)$$

$$|\bar{\eta} \sqrt{\alpha} + \bar{\theta}| \leq \sigma[\infty] \chi_R \alpha: \quad \text{‘fickle’ solution,} \quad \bar{\sigma} = [\sqrt{\alpha} \bar{\eta} + \bar{\theta}] / \alpha \chi_R. \quad (64)$$

- $\chi_R < 0$. As was found earlier in other MG versions with positive feedback, the effective agent equation now allows for remanence effects, leading to the potential for multiple solutions:

$$|\bar{\eta} \sqrt{\alpha} + \bar{\theta}| > \sigma[\infty] |\chi_R| \alpha: \quad \text{‘frozen’ solution,} \quad \bar{\sigma} = \sigma[\infty] \text{sgn}[\bar{\eta} \sqrt{\alpha} + \bar{\theta}] \quad (65)$$

$$|\bar{\eta} \sqrt{\alpha} + \bar{\theta}| \leq \sigma[\infty] |\chi_R| \alpha: \quad \begin{cases} \text{‘frozen’ solutions,} & \bar{\sigma} = \pm \sigma[\infty] \\ \text{‘fickle’ solution,} & \bar{\sigma} = [\sqrt{\alpha} \bar{\eta} + \bar{\theta}] / \alpha \chi_R. \end{cases} \quad (66)$$

For $|\bar{\eta}\sqrt{\alpha} + \bar{\theta}| \leq \sigma[\infty]|\chi_R|\alpha$ there are two stable solutions $\bar{\sigma} = \pm\sigma[\infty]$ separated by the unstable ‘fickle’ one. The two branches $\bar{\sigma} = -\sigma[\infty] \operatorname{sgn}[\bar{\eta}\sqrt{\alpha} + \bar{\theta}]$ in this region must be remanent ones, and the Maxwell construction tells us to choose in the stationary state

$$\chi_R < 0: \quad \text{‘frozen’ solution, } \bar{\sigma} = \sigma[\infty] \operatorname{sgn}[\bar{\eta}\sqrt{\alpha} + \bar{\theta}]. \quad (67)$$

The variance of $\bar{\eta}$ is simply equal to S_0^2 , since

$$\langle \bar{\eta}^2 \rangle = \lim_{\tau \rightarrow \infty} \frac{1}{\tau^2} \sum_{t'=0}^{\tau} \Sigma_{t't'} = \lim_{t \rightarrow \infty} \Sigma(t) = S_0^2. \quad (68)$$

We can now calculate equations for the persistent order parameters, sending $\bar{\theta} \rightarrow 0$ as soon as possible. As before we have to distinguish between $\chi_R > 0$ and $\chi_R < 0$.

- $\chi_R > 0$. The persistent correlations follow from $c = \langle \bar{\sigma}^2 \rangle_*$:

$$\begin{aligned} c &= \int d\bar{\eta} P(\bar{\eta}) \left\{ \theta[\chi_R\sigma[\infty]\sqrt{\alpha} - |\bar{\eta}|] \frac{\bar{\eta}^2}{\alpha\chi_R^2} + \theta[|\bar{\eta}| - \chi_R\sigma[\infty]\sqrt{\alpha}] \sigma^2[\infty] \right\} \\ &= \frac{2S_0^2}{\alpha\chi_R^2} \left\{ \frac{1}{2} \operatorname{Erf} \left[\frac{\chi_R\sigma[\infty]\sqrt{\alpha}}{\sqrt{2S_0^2}} \right] - \frac{\chi_R\sigma[\infty]\sqrt{\alpha}}{\sqrt{2\pi S_0^2}} e^{-\frac{\alpha\chi_R^2\sigma^2[\infty]}{2S_0^2}} \right\} \\ &\quad + 2\sigma^2[\infty] \left\{ \frac{1}{2} - \frac{1}{2} \operatorname{Erf} \left[\frac{\chi_R\sigma[\infty]\sqrt{\alpha}}{\sqrt{2S_0^2}} \right] \right\}. \end{aligned} \quad (69)$$

The frozen fraction ϕ and the susceptibility χ are calculated similarly:

$$\phi = \int d\bar{\eta} P(\bar{\eta}) \theta[|\bar{\eta}| - \chi_R\sigma[\infty]\sqrt{\alpha}] = 1 - \operatorname{Erf} \left[\frac{\chi_R\sigma[\infty]\sqrt{\alpha}}{S_0\sqrt{2}} \right] \quad (70)$$

$$\chi = \int d\bar{\eta} P(\bar{\eta}) \frac{\partial}{\partial \bar{\theta}} \bar{\sigma} = \frac{1}{\alpha\chi_R} \operatorname{Erf} \left[\frac{\chi_R\sigma[\infty]\sqrt{\alpha}}{S_0\sqrt{2}} \right]. \quad (71)$$

In terms of the usual short-hand $v = \chi_R\sigma[\infty]\sqrt{\alpha}/S_0\sqrt{2}$, we then arrive at

$$c = \sigma^2[\infty] \left\{ 1 + \frac{1 - 2v^2}{2v^2} \operatorname{Erf}[v] - \frac{1}{v\sqrt{\pi}} e^{-v^2} \right\} \quad (72)$$

$$\phi = 1 - \operatorname{Erf}[v] \quad (73)$$

$$\chi = \operatorname{Erf}[v]/\alpha\chi_R. \quad (74)$$

- $\chi_R < 0$. This situation is simpler: one has $c = \langle \bar{\sigma}^2 \rangle_* = \sigma^2(\infty)$, and $\phi = 1$. The susceptibility χ for $\bar{\theta} = 0$ becomes

$$\chi = \lim_{\bar{\theta} \rightarrow 0} \int d\bar{\eta} P(\bar{\eta}) \frac{\partial}{\partial \bar{\theta}} \bar{\sigma} = \frac{2\sigma[\infty]}{\sqrt{2\pi\alpha S_0^2}}. \quad (75)$$

If $\chi_R < 0$ the system is fully frozen, and nothing further happens. To close our persistent order parameter equations for c and ϕ if $\chi_R > 0$, we need the ratio χ_R/S_0 ; to get also χ we need χ_R and S_0 . We now take $A_e(t) = A_e$, and define the asymptotic time averages $\bar{A} = \lim_{\tau \rightarrow \infty} \tau^{-1} \sum_{t \leq \tau} A(t)$ and $\bar{F} = \lim_{\tau \rightarrow \infty} \tau^{-1} \sum_{t \leq \tau} F[A(t)]$. This allows us to write

$$\chi_R = \partial \langle \bar{F} \rangle / \partial A_e, \quad S_0 = 2 \langle \bar{F}^2 \rangle. \quad (76)$$

4.2. Analysis of the effective overall bid process

Closing our stationary-state equations requires extracting the values of χ_R and S_0 from the effective process (40) and (41) for the overall bids. We separate in the bid noise $\xi(t)$ and the bids $A(t)$ the persistent from the non-persistent terms:

$$A(t) = \bar{A} + \tilde{A}(t), \quad \xi(t) = z\sqrt{\frac{1}{2}(1+c)} + \tilde{\xi}(t). \quad (77)$$

Here z is a zero-average unit-variance frozen Gaussian variable, and $\tilde{\xi}(t)$ is also a zero-average Gaussian variable, uncorrelated with z and with covariances $\langle \tilde{\xi}(t)\tilde{\xi}(t') \rangle = \frac{1}{2}\tilde{C}(t-t')$. Here $\tilde{C}(t) = C(t) - c$. Our bid process (40) now becomes

$$\bar{A} = A_e - \chi\bar{F} + z\sqrt{\frac{1}{2}(1+c)} \quad (78)$$

$$\tilde{A}(t) = \tilde{\xi}(t) - \sum_{s>0} G(s)\{F[\bar{A} + \tilde{A}(t-s)] - \bar{F}\}. \quad (79)$$

The two quantities \bar{A} and \bar{F} are both parametrized by z , so we write $\bar{A}(z)$ and $\bar{F}(z)$. Also the non-persistent bid parts $\tilde{A}(t)$ depend on z since \bar{A} occurs in (79), so we write $\tilde{A}(t, z)$. Our static objects S_0 and χ_R are known once we have $\bar{F}(z)$. In (78) we have already one relation for the two objects, so we need one more equation connecting $\bar{A}(z)$ to $\bar{F}(z)$ to obtain closed formulae. To get this second relation we work out $\bar{F}(z)$:

$$\bar{F}(z) = \int d\tilde{A} W(\tilde{A}|z) F[\bar{A}(z) + \tilde{A}], \quad W(a|z) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \sum_{t \leq \tau} \delta[a - \tilde{A}(t, z)]. \quad (80)$$

So far our analysis is direct and fully exact. What is left is to find the statistics $W(a|z)$ of the non-frozen bid contributions, which requires ansätze.

To calculate $W(a|z)$ we assume the response function G to decay much more slowly than the times over which $\tilde{A}(t, z)$ are correlated, so that to the time summation in (79) we can apply the central limit theorem. This tells us that also $\tilde{A}(t, z)$ must be zero-average Gaussian. Let us define the covariance matrix $\Xi_{tt'}(z) = \langle \tilde{A}(t, z)\tilde{A}(t', z) \rangle$; it must be time-translation invariant, so we write $\Xi(t, z) = \Xi_{s+t, s}(z)$. In the appendix we show that

$$\Xi(t, z) = \frac{1}{2}\tilde{C}(t) + \mathcal{O}(\tau_C/\tau_G). \quad (81)$$

If, as in earlier MG analyses [8], we can rely on $\lim_{N \rightarrow \infty} \tau_C/\tau_G = 0$ in the ergodic regime (this will be our present ansatz), then we have simply $\Xi(t, z) = \frac{1}{2}\tilde{C}(t)$, and in particular $\Xi(0, z) = \frac{1}{2}(1-c)$, which closes our equations:

$$\bar{A}(z) = A_e - \chi\bar{F}(z) + z\sqrt{\frac{1}{2}(1+c)}, \quad \bar{F}(z) = \int Dx F[\bar{A}(z) + x\sqrt{\frac{1}{2}(1-c)}] \quad (82)$$

with the short-hand $Dx = (2\pi)^{-1/2} e^{-x^2/2} dx$. We eliminate $\bar{A}(z)$ and get $\bar{F}(z) = f(z, A_e)$, where $f(z, A_e)$ is the solution of the fixed-point equation:

$$f = \int Dx F \left[A_e - \chi f + z\sqrt{\frac{1}{2}(1+c)} + x\sqrt{\frac{1}{2}(1-c)} \right]. \quad (83)$$

This shows that $\bar{F}(z) = \Phi(A_e + z\sqrt{\frac{1}{2}(1+c)})$, where $\Phi(u)$ is to be solved from

$$\Phi(u) = \int Dx F \left[u - \chi\Phi(u) + x\sqrt{\frac{1}{2}(1-c)} \right]. \quad (84)$$

Clearly $\Phi(u)$ is anti-symmetric, since $F[A]$ is anti-symmetric. Finally, we will solve the nonlinear functional equation (84). To do this we define a new function $\Delta(z)$ via the identity $\Phi(u) = [u - \Delta^{-1}(u)]/\chi$, insertion into (84) of which gives after some rewriting

$$\Delta(z) = z + \chi \int Dx F \left[z + x \sqrt{\frac{1}{2}(1-c)} \right]. \quad (85)$$

Equation (85) is explicit, but shows that in the case of positive feedback (as for the majority game, corresponding to $F[A] = -A$) there is again the possibility of multiple solutions, which would here take the form of non-invertibility of the function $\Delta(z)$. Non-invertibility is signalled by finding $\Delta'(z) = 0$ for finite z , i.e. by

$$1 + \frac{\chi\sqrt{2}}{\sqrt{1-c}} \int Dx x F \left[z + x \sqrt{\frac{1}{2}(1-c)} \right] = 0. \quad (86)$$

4.3. Equations for χ_R and S_0 —closure of the stationary-state theory

Given the solution of (84), we obtain closure of our stationary-state equations:

$$\chi_R = \frac{\sqrt{2}}{\sqrt{1+c}} \int Dz z \Phi \left(A_e + z \sqrt{\frac{1}{2}(1+c)} \right) \quad (87)$$

$$S_0^2 = 2 \int Dz \Phi^2 \left(A_e + z \sqrt{\frac{1}{2}(1+c)} \right). \quad (88)$$

When expressing these equations in terms of the function $\Delta^{-1}(z)$ (which is parametrized by c and χ), we find that both are expressed in terms of the following Gaussian integrals:

$$I_0(c, \chi) = \int Dz \Delta^{-1} \left(A_e + z \sqrt{\frac{1}{2}(1+c)} \right) \quad (89)$$

$$I_1(c, \chi) = \int Dz z \Delta^{-1} \left(A_e + z \sqrt{\frac{1}{2}(1+c)} \right) \quad (90)$$

$$I_2(c, \chi) = \int Dz \left[\Delta^{-1} \left(A_e + z \sqrt{\frac{1}{2}(1+c)} \right) \right]^2. \quad (91)$$

We note that the validity of the last step depends crucially on non-invertibility issues being absent or resolved. Substitution of (85) into (87) and (88), followed by re-arrangements, gives

$$\chi_R = \frac{1}{\chi} \left[1 - \frac{\sqrt{2}}{\sqrt{1+c}} I_1(c, \chi) \right] \quad (92)$$

$$S_0^2 = \frac{2}{\chi^2} \left[A_e^2 - 2A_e I_0(c, \chi) + \frac{1}{2}(1+c) + I_2(c, \chi) - \sqrt{2(1+c)} I_1(c, \chi) \right]. \quad (93)$$

For $A_e = 0$ one has $I_0(c, \chi) = 0$ and the above integrals simplify. Equations (92) and (93) are to be solved in combination with (72) and (74) for c and χ . If this results in $\chi_R > 0$, the problem is solved and the observable ϕ follows from (73). As soon as $\chi_R \leq 0$, we enter the fully frozen state $c = \phi = 1$ induced by positive feedback in the valuation dynamics. From (82) we can also extract an expression for a static overall bid susceptibility $\chi_A(z) = \partial \bar{A}(z) / \partial A_e$.

It will be helpful to develop intuition for the physical meaning of $\Delta(z)$ and its inverse. According to (78) and the derivation leading from there on to (85) one has

$$\bar{A}(z) = A_e + z\sqrt{\frac{1}{2}(1+c)} - \chi\bar{F}(z) \tag{94}$$

$$\bar{F}(z) = \frac{1}{\chi}\left(A_e + z\sqrt{\frac{1}{2}(1+c)}\right) - \frac{1}{\chi}\Delta^{-1}\left(A_e + z\sqrt{\frac{1}{2}(1+c)}\right) \tag{95}$$

which can be combined to give

$$\bar{A}(z) = \Delta^{-1}\left(A_e + z\sqrt{\frac{1}{2}(1+c)}\right). \tag{96}$$

Careful retracing of our derivations shows that the Gaussian variable z originates from the integer variable $\mu \in \{1, \dots, p\}$ in the averaging over strategies in the generating functional, and hence represents the frozen variability in agents' strategies in terms of their dependence on the $p = \alpha N$ fake histories. We also note that for $F[A] = 0$, i.e. in the absence of valuation updates, one basically studies the initial state of the model. It then follows from the above relations that the function $\Delta^{-1}(z)$ tells us how the 'prior' time-averaged market bids $\bar{A}(z)_{\text{prior}} = A_e + z\sqrt{\frac{1}{2}(1+c)}$, as found for individual fake histories μ before agent adaptation has taken place, are modified as a result of the agent's valuation updates (upon equilibration) into the 'posterior' time-averaged market bids $\bar{A}(z)_{\text{posterior}} = \Delta^{-1}\left(A_e + z\sqrt{\frac{1}{2}(1+c)}\right)$. So $\Delta^{-1}(z)$ represents the collective impact of agent interaction on the overall bid. Non-invertibility of $\Delta(z)$ implies a potential dependence on the route taken by the system from the initial to the final state (i.e. overall market bid remanence).

4.4. Simple model examples

At this stage it is appropriate to inspect specific choices for $F[A]$, to serve as tests. We set $A_e = 0$ for simplicity; the external bids were needed to calculate the overall bid susceptibility χ_R , but are no longer essential.

- $F[A] = A$, the standard minority game. Here $\Delta(z) = z(1 + \chi)$, so $\Delta^{-1}(z) = z/(1 + \chi)$ and the Gaussian integrals $I_\ell(c, \chi)$ become

$$I_0(c, \chi) = 0, \quad I_1(c, \chi) = \frac{\sqrt{1+c}}{\sqrt{2}(1+\chi)}, \quad I_2(c, \chi) = \frac{1+c}{2(1+\chi)^2}. \tag{97}$$

This then reproduces the correct relations

$$\chi_R = 1/(1 + \chi), \quad S_0 = \sqrt{1+c}/(1 + \chi). \tag{98}$$

The onset of non-invertibility is according to (86) marked by $1 + \chi = 0$ (which never happens in the ergodic phase of the standard MG, where $\chi \geq 0$). Since always $\chi_R > 0$ we never enter the fully frozen (remanent) state obtained via the Maxwell construction.

- $F[A] = -A$, the standard majority game. Here $\Delta(z) = z(1 - \chi)$, so $\Delta^{-1}(z) = z/(1 - \chi)$ and the Gaussian integrals $I_\ell(c, \chi)$ become

$$I_0(c, \chi) = 0, \quad I_1(c, \chi) = \frac{\sqrt{1+c}}{\sqrt{2}(1-\chi)}, \quad I_2(c, \chi) = \frac{1+c}{2(1-\chi)^2}. \tag{99}$$

The result for χ_R and S_0 is

$$\chi_R = 1/(\chi - 1), \quad S_0 = \sqrt{1+c}/|\chi - 1|. \tag{100}$$

If $\chi_R > 0$ (no remanence) one extracts from (74) and (100) that $\chi = \text{Erf}(v)/[\text{Erf}(v) - \alpha]$, so for $\alpha < 1$ we can be sure that $\chi < 0$ and run into the contradiction $\chi_R < 0$. Apparently,

the general scenario is that where $\chi_R < 0$ and the system is in the fully frozen remanent state. Hence $\chi < 1$, and its value is given by formula (75), which reduces to

$$\chi = \left[1 + \frac{\sqrt{2\pi\alpha(1+c)}}{2\sigma[\infty]} \right]^{-1} \in (0, 1). \tag{101}$$

The condition for leaving the frozen remanent state, namely $\chi = 1$, is seen to coincide with the condition for having non-invertibility for the overall bid process, but this condition will clearly never be met; the system is *always* in the fully frozen remanent state.

5. Applications—greedy versus cautious contrarians

We now apply our theory to specific choices for the function $F[A]$, all corresponding to models that so far could only be studied via numerical simulations. For simplicity we choose $A_e = 0$.

5.1. Preparation

Due to $A_e = 0$, equations (92) and (93) that close our equations for persistent order parameters simplify to

$$\chi\chi_R = 1 - \sqrt{2}I_1(c, \chi)/\sqrt{1+c} \tag{102}$$

$$|\chi|S_0 = \sqrt{1+c+2I_2(c, \chi) - 2\sqrt{2(1+c)}I_1(c, \chi)} \tag{103}$$

with

$$I_1(c, \chi) = \int Dz z \Delta^{-1} \left(z \sqrt{\frac{1}{2}(1+c)} \right), \quad I_2(c, \chi) = \int Dz \left[\Delta^{-1} \left(z \sqrt{\frac{1}{2}(1+c)} \right) \right]^2 \tag{104}$$

and with $\Delta(z)$ as given in (85). It will prove efficient to define the function

$$c(v) = \sigma^2[\infty] \left\{ 1 + \frac{1-2v^2}{2v^2} \text{Erf}(v) - \frac{1}{v\sqrt{\pi}} e^{-v^2} \right\}. \tag{105}$$

Combining our relations so far then allows us to conclude that ergodic solutions with $\chi_R > 0$ follow from solving the following two coupled equations for the basic unknowns (v, χ) , after which the order parameters ϕ and c follow via $\phi = 1 - \text{Erf}(v)$ and $c = c(v)$ (so always $v \geq 0$):

$$\frac{\sigma[\infty]\sqrt{\text{Erf}(v)}}{v\sqrt{2[1+c(v)]}} = \text{sgn}(\chi) \left\{ \frac{1+c(v)+2I_2(c(v), \chi) - 2\sqrt{2(1+c(v))}I_1(c(v), \chi)}{1+c(v) - \sqrt{2[1+c(v)]}I_1(c(v), \chi)} \right\}^{\frac{1}{2}} \tag{106}$$

$$\alpha = \frac{\text{Erf}(v)}{1 - \sqrt{2}I_1(c(v), \chi)/\sqrt{1+c(v)}}. \tag{107}$$

Only $\chi > 0$ is possible, so we are allowed to introduce a further function $U(v) \geq 0$,

$$U(v) = \frac{\sigma^2[\infty] \text{Erf}(v)}{2[1+c(v)]v^2} \tag{108}$$

and compactify our equations for (v, χ) , describing solutions with $\chi > 0$ and $\chi_R > 0$, to

$$U(v) = \frac{1 + \frac{I_2(c(v), \chi)}{[1+c(v)]/2} - 2 \frac{I_1(c(v), \chi)}{\sqrt{[1+c(v)]/2}}}{1 - \frac{I_1(c(v), \chi)}{\sqrt{[1+c(v)]/2}}} \alpha = \frac{\text{Erf}(v)}{1 - \frac{I_1(c(v), \chi)}{\sqrt{[1+c(v)]/2}}}. \tag{109}$$

One proves from the definitions of $I_{1,2}(c, \chi)$ that always $I_2(c, \chi) \geq 2|I_1(c, \chi)|\sqrt{[1+c(v)]/2}$, so the numerator of the first of our compact equations is always nonnegative. To have a solution with $\chi_R > 0$ we must demand that the denominator is also positive.

In contrast, in states with $\chi_R < 0$ one has $c = \sigma^2(\infty)$, $\phi = 1$ i.e. the system is always fully frozen. Here the only order parameter left to be calculated is χ , which follows from

$$\frac{2\sigma[\infty]}{\sqrt{2\pi\alpha[1+\sigma^2(\infty)]}} = \text{sgn}(\chi) \sqrt{1 + \frac{I_2(\sigma^2(\infty), \chi)}{[1+\sigma^2(\infty)]/2} - 2 \frac{I_1(\sigma^2(\infty), \chi)}{\sqrt{[1+\sigma^2(\infty)]/2}}}. \quad (110)$$

Again only $\chi > 0$ is possible (which we may rely upon in the rest of this paper); hence

$$\frac{2\sigma^2[\infty]}{\pi\alpha[1+\sigma^2(\infty)]} = 1 + \frac{I_2(\sigma^2(\infty), \chi)}{[1+\sigma^2(\infty)]/2} - 2 \frac{I_1(\sigma^2(\infty), \chi)}{\sqrt{[1+\sigma^2(\infty)]/2}}. \quad (111)$$

To confirm that indeed $\chi_R < 0$ we must verify the outcome of

$$\chi_R = \chi^{-1} \left[1 - \frac{\sqrt{2}}{\sqrt{1+\sigma^2(\infty)}} I_1(\sigma^2(\infty), \chi) \right]. \quad (112)$$

5.2. Invertible overall bid impact functions

Here we focus on those models, which include the original MG, in which $F[A]$ is monotonically increasing. The ansatz $\chi > 0$ now guarantees that $\Delta(z)$ is invertible. These models have a remarkable universality property: all those which have an ergodic phase exhibit an ergodicity-breaking transition, marked by $\chi \rightarrow \infty$, exactly at the value α_c of the standard minority game, irrespective of the precise form of the function $F[A]$. To demonstrate this we first rewrite either (84) or (85):

$$\frac{u - \Delta^{-1}(u)}{\chi} = \int Dx F \left[\Delta^{-1}(u) + x \sqrt{\frac{1}{2}(1-c)} \right]. \quad (113)$$

From this we conclude that, for all $z \in \mathbb{R}$,

$$\lim_{\chi \rightarrow \infty} \int Dx F \left[\Delta^{-1}(z) + x \sqrt{\frac{1}{2}(1-c)} \right] = 0. \quad (114)$$

Since $F[A]$ is monotonic, the obvious solution $\Delta^{-1}(z) = 0$ (guaranteed by the anti-symmetry of $F[A]$) must be unique, and hence we know that generally $\lim_{\chi \rightarrow \infty} \Delta^{-1}(z) = 0$. It now follows that $\lim_{\chi \rightarrow \infty} I_1(c, \chi) = \lim_{\chi \rightarrow \infty} I_2(c, \chi) = 0$, and therefore via (102) and (103) we find

$$\lim_{\chi \rightarrow \infty} \chi \chi_R = 1, \quad \lim_{\chi \rightarrow \infty} |\chi| S_0 = \sqrt{1+c}, \quad \lim_{\chi \rightarrow \infty} \frac{S_0}{\chi_R} = \text{sgn}(\chi_R) \sqrt{1+c}. \quad (115)$$

The ansatz $\chi > 0$ guarantees $\text{sgn}(\chi_R) = 1$ (with $\chi_R \downarrow 0$ as $\chi \rightarrow \infty$), and our persistent order parameter equations (72)–(74) close at the transition $\chi \rightarrow \infty$ exactly in the same way as they would for the conventional MG, i.e.

$$c = \sigma^2[\infty] \left\{ 1 + \frac{1-2v^2}{2v^2} \text{Erf}[v] - \frac{1}{v\sqrt{\pi}} e^{-v^2} \right\} \quad (116)$$

$$\alpha = \text{Erf}[v], \quad v = \sigma[\infty] \sqrt{\alpha} / \sqrt{2(1+c)}. \quad (117)$$

So all systems with monotonically increasing anti-symmetric $F[A]$ will, provided they have an ergodic regime, *always* exhibit a $\chi \rightarrow \infty$ transition at the conventional MG value $\alpha_c \approx 0.3374$, irrespective of the actual form of $F[A]$.

Away from the transition one should expect a dependence of the values of the persistent order parameters on the choice made for $F[A]$. However, for ‘sensible’ choices of $F[A]$ one generally finds this dependence to be weak. To illustrate this we now focus on a particular class of models, with monotonic valuation update functions of the form

$$F[A] = \text{sgn}[A]|A|^\gamma, \quad \gamma \geq 0. \quad (118)$$

These models are relatively straightforward generalizations of the original MG (which corresponds to $\gamma = 1$); nevertheless, for $\gamma \neq 1$ the standard solution route (i.e. generating functional analysis of the strategy selection process, without including the overall bid dynamics explicitly in the formalism) would not have worked. For $\gamma > 1$ the agents with (118) place more importance on making money by exploiting large fluctuations ($|A| > 1$), so can be described as greedy high-risk contrarians, whereas for $\gamma < 1$ they prefer to exploit small market fluctuations, and operate as cautious low-risk contrarians.

We know from the above argument that their $\chi \rightarrow \infty$ ergodicity breaking transition point α_c will be identical to that of the standard MG. Let us now calculate the persistent order parameters in the ergodic region $\alpha > \alpha_c$. On physical grounds one does not expect a negative susceptibility and $F[A]$ increases monotonically; hence $\Delta(z)$ will be invertible, and we can transform the integrals (104) via the substitution $z = \Delta(x)\sqrt{2}/\sqrt{1+c}$, and find

$$I_1(c, \chi) = \int \frac{dx}{\sqrt{2\pi}} e^{-\Delta^2(x)/(1+c)} \quad (119)$$

$$I_2(c, \chi) = \frac{\sqrt{2}}{\sqrt{1+c}} \int \frac{dx}{\sqrt{2\pi}} e^{-\Delta^2(x)/(1+c)} x^2 \Delta'(x). \quad (120)$$

This removes the need for inversion of $\Delta(z)$. For the models (118) one then finds

$$\Delta(z) = z + \chi \int Dx \text{sgn} \left[z + x \sqrt{\frac{1}{2}(1-c)} \right] \left| z + x \sqrt{\frac{1}{2}(1-c)} \right|^\gamma. \quad (121)$$

For non-integer γ the integral has to be calculated numerically. For integer γ one finds

$$\gamma = 0: \quad \Delta(z) = z + \chi \text{Erf} \left[\frac{z}{\sqrt{1-c}} \right] \quad (122)$$

$$\gamma = 1: \quad \Delta(z) = z + \chi z \quad (123)$$

$$\gamma = 2: \quad \Delta(z) = z + \left[z^2 + \frac{1}{2}(1-c) \right] \chi \text{Erf} \left[\frac{z}{\sqrt{1-c}} \right] + \frac{\chi z \sqrt{1-c}}{\sqrt{\pi}} e^{-z^2/(1-c)} \quad (124)$$

$$\gamma = 3: \quad \Delta(z) = z + \chi \left[z^3 + \frac{3}{2}(1-c)z \right]. \quad (125)$$

Since on physical grounds one does not expect a negative susceptibility, invertibility is expected to hold and the present family of models should behave qualitatively as the ordinary MG. One also expects that as the susceptibility goes to zero for large alpha the behavior of the system should be almost independent of the value of γ . For $\gamma \leq 3$ these predictions are borne out by numerical simulations (all carried out with $N = 4097$, and measured during 1000 steps after a 2000 steps equilibration period), as shown in figures 1–4. For large γ (excessive agent greed), however, in spite of the agents still operating as contrarians, the ergodic phase appears to be destroyed by their increasing focus on big-gain big-risk decisions, and there is no longer an efficient market regime with low volatility.

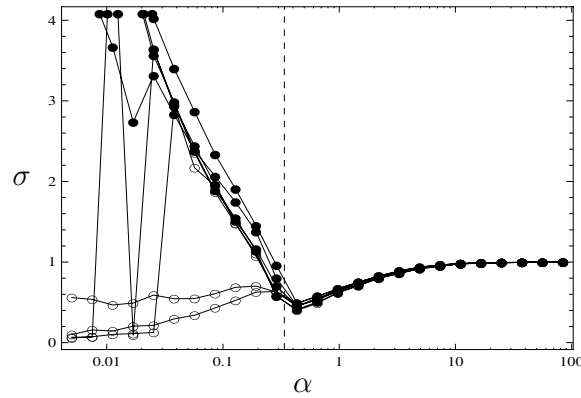


Figure 1. Simulation results for the volatility σ of MG-type models with $F[A] = \text{sgn}[A]|A|^\gamma$, for $\gamma = 0, 1, 2, 3$. Vertical dashed line: predicted location of the $\chi \rightarrow \infty$ transition. Empty/full markers correspond to biased/unbiased initial conditions. The location of the phase transition seems indeed independent of γ . Even for $\alpha > \alpha_c$ the volatility appears only weakly dependent upon the greed exponent γ . However, for values of $\gamma \geq 4$ (excessive agent greed) the ergodic region is destroyed and the efficient phase of the market vanishes (see figure 4).

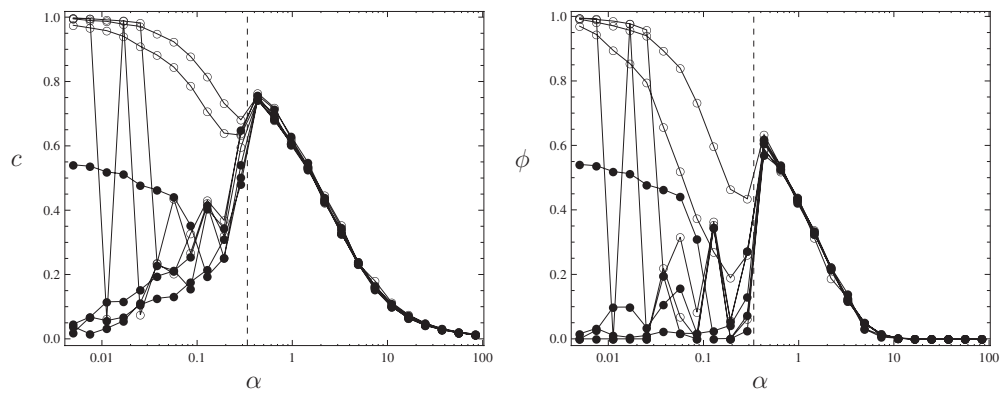


Figure 2. Simulation results for the fraction of frozen agents ϕ and correlations c for MG-type models with $F[A] = \text{sgn}[A]|A|^\gamma$, for $\gamma = 0, 1, 2, 3$. Vertical dashed line: predicted location of the $\chi \rightarrow \infty$ transition. Empty/full markers correspond to biased/unbiased initial conditions. As observed for the volatility, also ϕ and c appear to be only weakly dependent on γ in the ergodic region $\alpha < \alpha_c$.

6. Applications—dynamic switching between contrarian trading and trend-following

Next we inspect a class of models in which agents switch from trend-following to contrarian behaviour, dependent on the absolute value $|A|$ of the overall market bid. The rationale is to create more realistic agent behaviour, based on interpreting A as a measure of the price returns in the market. One example was proposed in [11], corresponding to $F[A] = \epsilon A^3 - A$, with $\epsilon > 0$. This model, in which agents are trend-followers for $|A| < 1/\sqrt{\epsilon}$ but contrarians for $|A| > 1/\sqrt{\epsilon}$, cannot be solved analytically using the standard generating functional analysis

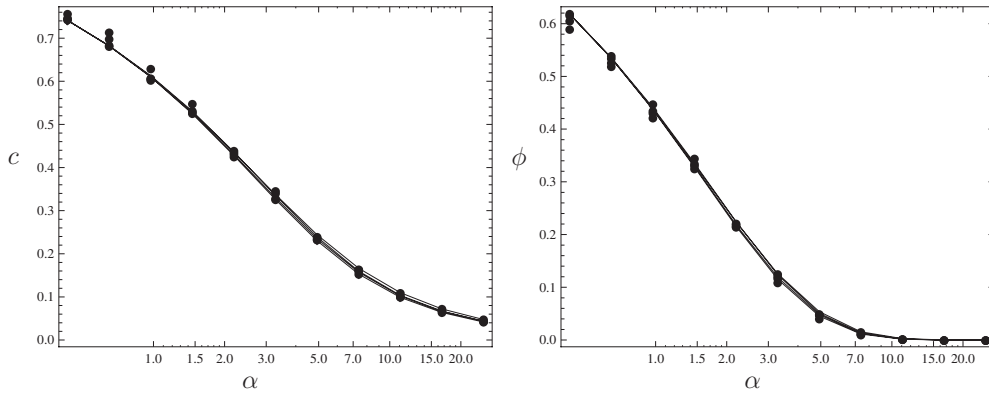


Figure 3. Theory versus simulation results for MG-type models with $F[A] = \text{sgn}[A]|A|^\gamma$, for $\gamma = 0, 1, 2, 3$. Observed versus predicted values of the persistent correlations c and the fraction of frozen agents ϕ , for the ergodic regime (i.e. for $\alpha > \alpha_c$). Lines: theoretical prediction; markers: simulation results. There is clearly excellent agreement between the theoretical predictions and the computer simulations.

route. Here we generalize their model slightly, allowing also for the reverse tendency, where agents become trend-followers for *large* instead of small $|A|$, and analyse the case

$$F[A] = \tau A(1 - A^2/A_0^2), \quad \text{with } \tau = \pm 1, \quad A_0 > 0. \quad (126)$$

The model of [11] corresponds to $\tau = -1$ and $A_0 = 1/\sqrt{\epsilon}$. For $\tau = 1$, in contrast, agents behave as contrarians for modest deviations of the returns from their average value, but switch to herding when they perceive the market to be anomalous, i.e. for $|A| > A_0$. For $A_0 \rightarrow \infty$, the model (126) reduces either to the standard MG (for $\tau = 1$) or to a majority-type game (for $\tau = -1$); for $A_0 \rightarrow 0$ one anticipates the opposite.

6.1. Non-invertible overall bid impact functions

For (126) one finds

$$\Delta(z) = z[1 + \tilde{\chi} - \frac{3}{2}(1 - c)\tilde{\chi}/A_0^2] - \tilde{\chi}z^3/A_0^2 \quad (127)$$

where $\tilde{\chi} = \tau\chi$. Whether $\Delta(z)$ is invertible will depend on A_0 . If $A_0^2(1 + \tilde{\chi}^{-1}) > \frac{3}{2}(1 - c)$, there will always be a region of non-invertibility, with three solutions z of the equation $\Delta(z) = \Delta$. The latter are the roots of a cubic equation and can therefore be calculated analytically,

$$z^3 - z \left[A_0^2(1 + \tilde{\chi})/\tilde{\chi} - \frac{3}{2}(1 - c) \right] + \frac{A_0^2\Delta}{\tilde{\chi}} = 0. \quad (128)$$

Following [14] we write the cubic equation in the form $z^3 + 3qz - 2r = 0$ by defining

$$q = \frac{1}{2}(1 - c) - A_0^2(1 + \tilde{\chi})/3\tilde{\chi}, \quad r = -A_0^2\Delta/2\tilde{\chi}. \quad (129)$$

We can then classify the solution(s) of the equation $\Delta(z) = \Delta$ as follows:

$$q^3 + r^2 > 0: \quad \text{one soln,} \quad z = [r + \sqrt{q^3 + r^2}]^{1/3} + [r - \sqrt{q^3 + r^2}]^{1/3} \quad (130)$$

$$q^3 + r^2 < 0: \quad \text{three solns,} \quad z_1 = 2|q|^{1/3} \cos \left(\frac{1}{3} \arccos \left(\frac{r}{\sqrt{|q|^3}} \right) \right) \quad (131)$$

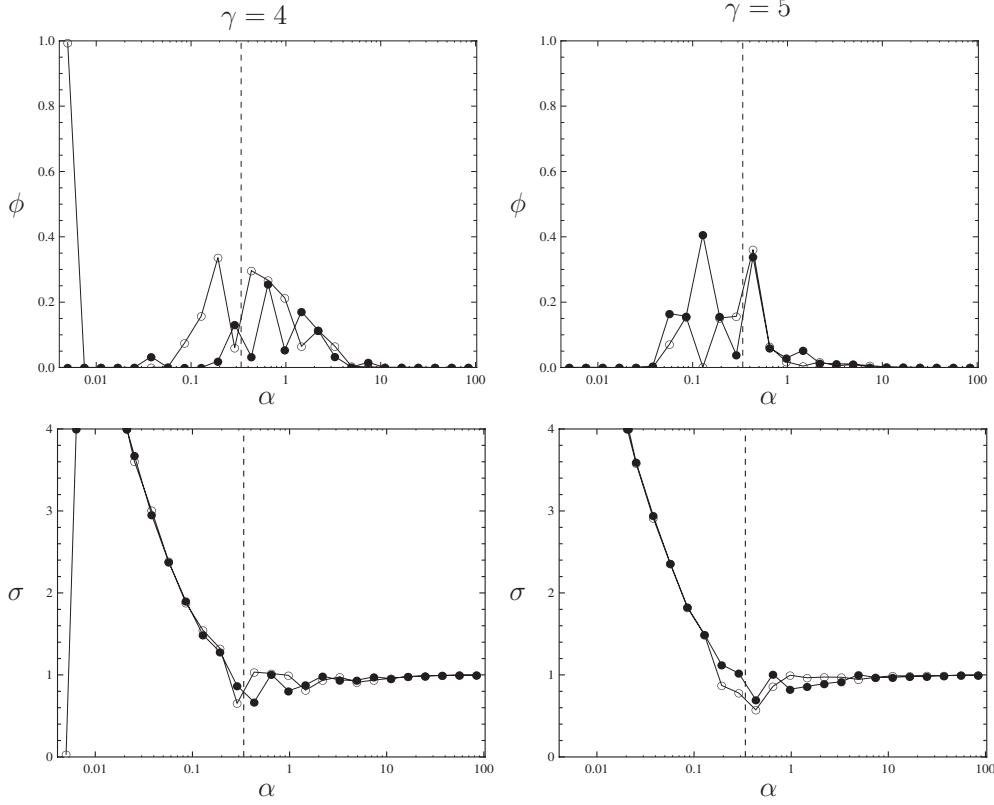


Figure 4. Fraction of frozen agents ϕ and volatility σ , measured in simulations of MG-type systems with $F[A] = \text{sgn}[A]|A|^\gamma$, for $\gamma = 4, 5$. Dashed line: predicted location of the $\chi \rightarrow \infty$ transition. Empty/full markers correspond to biased/unbiased initial conditions. For $\gamma \geq 4$ we no longer appear to have the usual ergodic phase, and more extensive simulations with different durations and system sizes are required to determine the nature of the macroscopic state(s).

$$z_2 = -|q|^{\frac{1}{2}} \cos\left(\frac{1}{3} \arccos\left(\frac{r}{\sqrt{|q|^3}}\right)\right) + \sqrt{3}|q|^{\frac{1}{2}} \sin\left(\frac{1}{3} \arccos\left(\frac{r}{\sqrt{|q|^3}}\right)\right) \quad (132)$$

$$z_3 = -|q|^{\frac{1}{2}} \cos\left(\frac{1}{3} \arccos\left(\frac{r}{\sqrt{|q|^3}}\right)\right) - \sqrt{3}|q|^{\frac{1}{2}} \sin\left(\frac{1}{3} \arccos\left(\frac{r}{\sqrt{|q|^3}}\right)\right). \quad (133)$$

Our solution must respect the symmetry $\Delta^{-1}(-z) = -\Delta^{-1}(z)$, which translates into searching for a root with $z(-r) = -z(r)$. Inspection reveals that for $q^3 + r^2 > 0$ the solution (130) has the desired symmetry. For $q^3 + r^2 < 0$ we find that $z_2(-r) = -z_2(r)$, and that $z_3(-r) = -z_1(r)$. Hence $z_{1,3}(r)$ represent two ‘extremal’ solution branches (related to each other by inversion symmetry), and $z_2(r)$ represents a middle branch. The region of multiple solutions is defined by $|r| < |q|^{3/2}$, where we have infinitely many options for assigning a value to $\Delta^{-1}(z)$. The conventional one is the Maxwell construction, based on assuming the multiplicity of solutions to be caused by remanence. Here the middle branch $z_2(r)$ is taken to be dynamically unstable, and one takes for $0 < r < |q|^{3/2}$ the continuation $z_1(r)$ of the $r > |q|^{3/2}$ solution, and for $-|q|^{3/2} < r < 0$ the continuation $z_3(r)$ of the $r < -|q|^{3/2}$ solution, with a discontinuity at

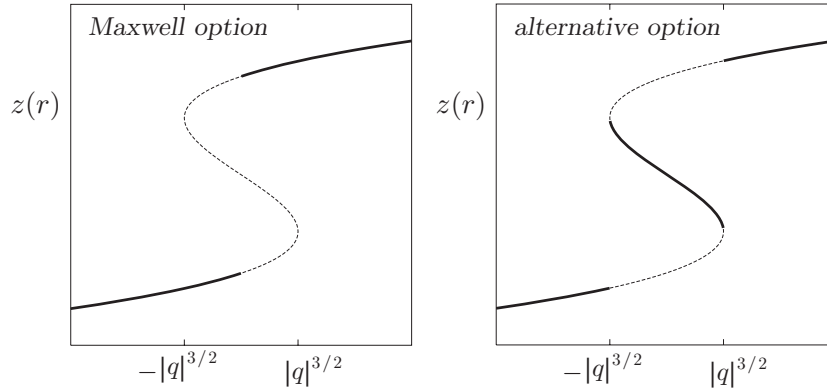


Figure 5. The options for constructing an anti-symmetric solution $z(r)$ of the cubic equation $z^3 + 3qz - 2r = 0$ for $q < 0$, shown as a function of r . Dashed: all solutions. Left: the Maxwell construction, which assumes the middle solution for $|r| < |q|^{3/2}$ to be unstable, and the others to reflect remanence. Right: the consequence of assuming the middle solution to be stable. Only the one on the right turns out to have the correct $A_0 \rightarrow \infty$ limit (see the main text).

$r = 0$. Our second option implies assuming the middle branch to be stable, i.e. choosing $z_2(r)$ for $|r| < |q|^{3/2}$. The two options are illustrated in figure 5. Assessing which of these choices (if any) is correct would in principle require a stability analysis of the asymptotic solution of the overall bid dynamics.

Upon translating the above arguments into the language of $\Delta^{-1}(z)$, the resulting picture is the following, where still $q = \frac{1}{2}(1 - c) - \frac{1}{3}A_0^2(1 + \tilde{\chi})/\tilde{\chi}$ but now with $r(z) = -\frac{1}{2}A_0^2z/\tilde{\chi}$:

$q > 0$ or $q < 0, |r(z)| > |q|^{3/2}$:

$$\Delta^{-1}(z) = [r(z) + \sqrt{q^3 + r^2(z)}]^{1/3} + [r(z) - \sqrt{q^3 + r^2(z)}]^{1/3}. \quad (134)$$

Here $\Delta(z)$ is fully invertible. In the alternative scenario we have the Maxwell option:

$$q < 0, \quad |r(z)| < |q|^{3/2}: \quad \Delta^{-1}(z) = 2 \operatorname{sgn}[r(z)]|q|^{1/2} \cos\left(\frac{1}{3} \arccos\left(\frac{|r(z)|}{\sqrt{|q|^3}}\right)\right). \quad (135)$$

But we also have the non-Maxwell solution, which can be rewritten as

$$q < 0, \quad |r(z)| < |q|^{3/2}: \quad \Delta^{-1}(z) = -2|q|^{1/2} \sin\left(\frac{1}{3} \arcsin\left(\frac{r(z)}{\sqrt{|q|^3}}\right)\right). \quad (136)$$

Only for $A_0 \rightarrow \infty$ can we decide between our candidate solutions for $q < 0$ and $|r(z)| < |q|^{3/2}$ without analysing dynamic stability in the underlying bid process. There (126) reduces to the standard MG for $\tau = 1$ and to the standard majority game for $\tau = -1$, both of which we analysed in section 4.4. The correct solution must reproduce $\lim_{A_0 \rightarrow \infty} \Delta^{-1}(z) = z/(1 + \tilde{\chi})$. For $A_0 \rightarrow \infty$, $\tilde{\chi} \notin (-1, 0)$, and finite z one has $q < 0$ and $|r(z)| < |q|^{3/2}$ (in fact $\lim_{A_0 \rightarrow \infty} r(z)/|q|^{3/2} = 0$), so we do indeed probe the region of ambiguity. Our test reveals, using $\sqrt{3} \sin\left(\frac{1}{3} \arccos(x)\right) - \cos\left(\frac{1}{3} \arccos(x)\right) = -\frac{2}{3}x + \mathcal{O}(x^3)$, that for $A_0 \rightarrow \infty$:

$$\text{Maxwell soln:} \quad \Delta^{-1}(z) = A_0|1 + \tilde{\chi}^{-1}|^{1/2} \operatorname{sgn}(z) + \mathcal{O}(A_0^0), \quad \text{incorrect} \quad (137)$$

$$\text{alternative soln:} \quad \Delta^{-1}(z) = z/(1 + \tilde{\chi}) + \mathcal{O}(A_0^{-1}), \quad \text{correct.} \quad (138)$$

For $\tilde{\chi} \in (-1, 0)$ we have $q > 0$, so there is no ambiguity: $\Delta^{-1}(z)$ is given by (134), which for $A_0 \rightarrow \infty$ reproduces correctly $\Delta^{-1}(z) = z/(1 + \tilde{\chi}) + \mathcal{O}(A_0^{-1})$. Hence, for $A_0 \rightarrow \infty$ the only acceptable solution is (134) and (136)². We resolve the remaining ambiguity as follows: since the pure majority and minority games always exhibit $\lim_{z \rightarrow 0} \frac{d}{dz} \Delta^{-1}(z) > 0$, we take this property to hold generally. In view of the physical meaning of $\Delta(z)$ as relating posterior overall market bids to prior overall market bids, via (96), this means we are assuming that for weak random bids the effect of agent interaction is to push these bids back to zero (contrarian action) or to amplify them (trend-following), but there is no change of sign. For $\tau = -1$ we now have either $\frac{3}{2}(1 - c)/A_0^2 > 1 - \chi^{-1}$ and the unambiguous solution (134), or we have $\frac{3}{2}(1 - c)/A_0^2 < 1 - \chi^{-1}$ and the Maxwell option (135). For $\tau = 1$ we must always demand $\frac{3}{2}(1 - c)/A_0^2 < 1 + \chi^{-1}$ and the non-Maxwell option (136).

6.2. Conventional $\chi \rightarrow \infty$ transitions

We can now calculate transition lines. In the present model we can have a traditional $\chi \rightarrow \infty$ transition, a transition where χ_R changes sign (switching from a minority to a majority-type game), and a transition marking the emergence of jumps in $\Delta^{-1}(z)$. We start with the calculation of the $\chi \rightarrow \infty$ transitions.

For $\tau = 1$, $\chi \rightarrow \infty$ and finite z one has $r(z) = \mathcal{O}(\chi^{-1})$ and $q = \frac{1}{2}(1 - c) - \frac{1}{3}A_0^2 + \mathcal{O}(\chi^{-1})$, from which it follows that if $\lim_{\chi \rightarrow \infty} \frac{3}{2}(1 - c) \neq A_0^2$ then

$$\Delta^{-1}(z) = -\frac{z}{\chi} \left[\frac{3}{2}(1 - c)/A_0^2 - 1 \right]^{-1} + \mathcal{O}(\chi^{-2}). \quad (139)$$

From this one extracts $\lim_{\chi \rightarrow \infty} I_1(c, \chi) = \lim_{\chi \rightarrow \infty} I_2(c, \chi) = 0$, and via the same arguments that applied to models with invertible bid impact functions one is led to

$$\lim_{\chi \rightarrow \infty} \chi \chi_R = 1, \quad \lim_{\chi \rightarrow \infty} \chi S_0 = \sqrt{1 + c}, \quad \lim_{\chi \rightarrow \infty} \frac{S_0}{\chi_R} = \text{sgn}(\chi_R) \sqrt{1 + c}. \quad (140)$$

If $\chi_R > 0$, our equations (72)–(74) close for $\chi \rightarrow \infty$ exactly as they would for the conventional MG, with the standard $\chi \rightarrow \infty$ transition at the value $\alpha_c \approx 0.3374$. If $\chi_R < 0$, on the other hand, no $\chi \rightarrow \infty$ transition is possible. Taking into account the requirement $\lim_{z \rightarrow 0} \frac{d}{dz} \Delta^{-1}(z) > 0$ implies that for $\tau = 1$ the $\chi \rightarrow \infty$ transition exists only if $A_0^2 > \frac{3}{2}(1 - c)$. Here $q < 0$ and $r(z) = \mathcal{O}(\chi^{-1})$, so we are in the remanent region.

For $\tau = -1$, $\chi \rightarrow \infty$ and finite z one still has $r(z) = \mathcal{O}(\chi^{-1})$ and $q = \frac{1}{2}(1 - c) - \frac{1}{3}A_0^2 + \mathcal{O}(\chi^{-1})$. So if $\lim_{\chi \rightarrow \infty} \frac{3}{2}(1 - c) \neq A_0^2$ then for $q < 0$ we will have $|r(z)| \ll |q|^{3/2}$. Hence

$$A_0^2 < \frac{3}{2}(1 - c): \quad \Delta^{-1}(z) = \frac{z}{\chi} \left[\frac{3}{2}(1 - c)/A_0^2 - 1 \right]^{-1} + \mathcal{O}(\chi^{-2}) \quad (141)$$

$$A_0^2 > \frac{3}{2}(1 - c): \quad \Delta^{-1}(z) = \text{sgn}(z) \sqrt{A_0^2 - \frac{3}{2}(1 - c)} + \mathcal{O}(\chi^{-1}). \quad (142)$$

If $A_0^2 < \frac{3}{2}(1 - c)$ we return as expected to the conventional MG transition line at $\alpha_c \approx 0.3374$; however, in contrast to $\tau = 1$ this line is now in the non-remanent region. If $A_0^2 > \frac{3}{2}(1 - c)$,

² It is surprising that in this problem the Maxwell construction is not always the correct way to handle the multiplicity of solutions, given its track record in physical many-particle systems. However, minority games do not obey detailed balance, so intuition developed on the basis of bifurcation analyses obtained via free energy minimization in equilibrium systems may be misleading.

on the other hand, the $\chi \rightarrow \infty$ line will be in the remanent region. With some foresight we now write $A_0^2 = \frac{3}{2}(1 - c) + \frac{1}{2}(1 + c)\Xi^2$ with $\Xi \geq 0$, and obtain

$$I_1(c, \chi) = \Xi \sqrt{\frac{1+c}{\pi}} + \mathcal{O}(\chi^{-1}), \quad I_2(c, \chi) = \frac{1}{2}(1+c)\Xi^2 + \mathcal{O}(\chi^{-1}) \quad (143)$$

which leads to

$$U(v) - 1 = \frac{\Xi^2 - \Xi\sqrt{2/\pi}}{1 - \Xi\sqrt{2/\pi}}, \quad \alpha = \text{Erf}(v)/[1 - \Xi\sqrt{2/\pi}]. \quad (144)$$

Upon solving the first equation for Ξ ,

$$\Xi_{\pm}(v) = \frac{1}{\sqrt{2\pi}} \{2 - U(v) \pm \sqrt{[2 - U(v)]^2 + 2\pi[U(v) - 1]}\} \quad (145)$$

we then arrive at a convenient parametrization of the transition line in the (α, A_0) plane, with $v \geq 0$ as a parameter. The line turns out to have two branches (indicated by \pm):

$$A_0^{\pm}(v) = \sqrt{\frac{3}{2}[1 - c(v)] + \frac{1}{2}[1 + c(v)]\Xi_{\pm}^2(v)} \quad (146)$$

$$\alpha_{\pm}(v) = \text{Erf}(v)/[1 - \Xi_{\pm}(v)\sqrt{2/\pi}]. \quad (147)$$

We must demand $0 \leq \Xi_{\pm}(v) \leq \sqrt{\pi/2}$, to guarantee $\chi_R > 0$, and $[2 - U(v)]^2 + 2\pi[U(v) - 1] \geq 0$, to ensure $\Xi_{\pm}(v) \in \mathbb{R}$. We have now found the $\chi \rightarrow \infty$ transition line for both $A_0^2 < \frac{3}{2}(1 - c)$ (in the non-remanent region, where it is independent of A_0), and for $\frac{3}{2}(1 - c) < A_0^2 < \frac{3}{2}(1 - c) + \frac{1}{4}\pi(1 + c)$ (in the remanent region, where it depends on A_0). At the moment where $A_0^2 = \frac{3}{2}(1 - c) + \frac{1}{4}\pi(1 + c)$, the bid susceptibility χ_R goes through zero, marking a switch to majority game behavior; we find below that this occurs for the + branch at $v = 0$.

To aid and test numerical evaluation it will be helpful to assess the limits $v \rightarrow 0$ and $v \rightarrow \infty$ of the above parametrized branches. For small v one finds, using $\text{Erf}(v) = (2v/\sqrt{\pi})[1 - \frac{1}{3}v^2 + \mathcal{O}(v^4)]$ and the fact that $\Xi(v)$ cannot be negative,

$$c(v) = \sigma^2[\infty] \left[1 - \frac{4v}{3\sqrt{\pi}}\right] + \mathcal{O}(v^3), \quad U(v) = \frac{\sigma^2[\infty]}{v[1 + \sigma^2[\infty]]\sqrt{\pi}} + \mathcal{O}(1) \quad (148)$$

$$\Xi_+(v) = \sqrt{\frac{\pi}{2}} - \frac{v[1 + \sigma^2[\infty]]\pi(\pi - 2)}{2\sigma^2[\infty]\sqrt{2}} + \mathcal{O}(v^2). \quad (149)$$

Hence

$$\lim_{v \rightarrow 0} A_0^+(v) = \sqrt{\frac{3}{2}[1 - \sigma^2[\infty]] + \frac{\pi}{4}[1 + \sigma^2[\infty]]} \quad (150)$$

$$\lim_{v \rightarrow 0} \alpha_+(v) = \frac{4\sigma^2[\infty]}{[1 + \sigma^2[\infty]]\pi(\pi - 2)}. \quad (151)$$

For $\sigma[\infty] = 1$ (no decision noise) this gives $\lim_{v \rightarrow 0} \alpha_+(v) = 2/\pi(\pi - 2) \approx 0.5577$ and $\lim_{v \rightarrow 0} A_0^+(v) = \sqrt{\pi/2} \approx 1.2533$. For $v \rightarrow \infty$, in contrast, we observe due to $\lim_{v \rightarrow \infty} U(v) = 0$ that the $\Xi_{\pm}(v)$ are no longer real-valued; both branches terminate and meet at the value v_{\max} such that $[U(v_{\max}) - 2]^2 = 2\pi[1 - U(v_{\max})]$, i.e. where

$$U(v_{\max}) = \sqrt{\pi - 2}[\sqrt{\pi} - \sqrt{\pi - 2}]. \quad (152)$$

6.3. Onset of remanence-induced discontinuities in the $\chi_R > 0$ region

It is clear that discontinuities emerge in our equations as soon as $q \leq 0$. Even if it is not yet clear which precise observables will be affected by these discontinuities, it implies that there exists an alternative transition marked by the condition $q = 0$, i.e. by

$$c = 1 - \frac{2}{3}A_0^2(1 + \tilde{\chi}^{-1}). \tag{153}$$

When (153) holds, we find upon expanding for small q that $\lim_{q \rightarrow 0} \Delta^{-1}(z) = -(A_0^2 z / \tilde{\chi})^{1/3}$, and so along the line (153) the integrals (104) reduce to gamma functions:

$$I_1(c, \chi) = -\left(\frac{A_0^2 \sqrt{2(1+c)}}{2\tilde{\chi}}\right)^{1/3} \int Dz |z|^{4/3} = -\left(\frac{A_0^2 \sqrt{1+c}}{2\tilde{\chi}\sqrt{2}}\right)^{1/3} \frac{1}{3\sqrt{\pi}} \Gamma\left(\frac{1}{6}\right) \tag{154}$$

$$I_2(c, \chi) = \left(\frac{A_0^2 \sqrt{2(1+c)}}{2\tilde{\chi}}\right)^{2/3} \int Dz |z|^{2/3} = \left(\frac{A_0^2 \sqrt{1+c}}{\tilde{\chi}}\right)^{2/3} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5}{6}\right). \tag{155}$$

This leads to

$$U(v) - 1 = \frac{6\Gamma(\frac{5}{6})\Xi^2 + \Gamma(\frac{1}{6})\Xi}{3\sqrt{\pi} + \Gamma(\frac{1}{6})\Xi} \alpha = \text{Erf}(v) / \left[1 + \Gamma\left(\frac{1}{6}\right) \Xi / 3\sqrt{\pi}\right] \tag{156}$$

in which now $\Xi = [A_0^2 / \tilde{\chi} [1 + c(v)]]^{1/3}$. Solving the first equation for Ξ gives

$$\Xi_{\pm}(v) = \frac{\Gamma(\frac{1}{6})}{12\Gamma(\frac{5}{6})} \left\{ U(v) - 2 \pm \sqrt{[2 - U(v)]^2 - 72[1 - U(v)]\sqrt{\pi}\Gamma\left(\frac{5}{6}\right) / \Gamma^2\left(\frac{1}{6}\right)} \right\}. \tag{157}$$

Again, after combination with the $q = 0$ condition, we thereby arrive at a parametrization of the transition line in the (α, A_0) plane, with $v \geq 0$ as a parameter, with two \pm branches:

$$A_0^{\pm}(v) = \sqrt{\frac{3}{2}[1 - c(v)] - [1 + c(v)]\Xi_{\pm}^3(v)} \tag{158}$$

$$\alpha_{\pm}(v) = \text{Erf}(v) / \left[1 + \frac{\Gamma(\frac{1}{6})}{3\sqrt{\pi}} \Xi_{\pm}(v)\right]. \tag{159}$$

The corresponding value of χ then follows from

$$\chi = \tau \left[\frac{3[1 - c(v)]}{2[1 + c(v)]\Xi_{\pm}^3(v)} - 1 \right]. \tag{160}$$

To have $\Xi_{\pm}(v) \in \mathbb{R}$ we must demand $[U(v) + 2(\xi - 1)]^2 \geq 4\xi(\xi - 1)$, where $\xi = 18\sqrt{\pi}\Gamma(\frac{5}{6})/\Gamma^2(\frac{1}{6}) \approx 1.1623$. Since $U(v) \geq 0$ this implies $U(v) \geq 2\sqrt{\xi - 1}(\sqrt{\xi} - \sqrt{\xi - 1}) \approx 0.5441$. Secondly, to have $A_0 \in \mathbb{R}$ and $\alpha > 0$ we must demand, respectively, $\Xi_{\pm}(v) \leq [\frac{3}{2}[1 - c(v)]/[1 + c(v)]]^{1/3}$ and $\Xi_{\pm}(v) \geq -3\sqrt{\pi}/\Gamma(\frac{1}{6})$. The third condition is $\chi \geq 0$, which translates into: $\Xi_{\pm}(v) \geq 0$ for $\tau = 1$, and $\Xi_{\pm}(v) \leq 0$ for $\tau = -1$. The second and third conditions can be combined into

$$\tau = 1: \quad 0 \leq \Xi_{\pm}(v) \leq \left[\frac{3}{2}[1 - c(v)]/[1 + c(v)]\right]^{1/3} \tag{161}$$

$$\tau = -1: \quad -3\sqrt{\pi}/\Gamma\left(\frac{1}{6}\right) \leq \Xi_{\pm}(v) \leq 0. \tag{162}$$

For small v we may use expansion (148), which tells us that $U(v)$ diverges at $v = 0$, to find

$$\Xi_{+}(v) = \frac{U(v)\Gamma(\frac{1}{6})}{6\Gamma(\frac{5}{6})} + \mathcal{O}(1) = \frac{\sigma^2[\infty]\Gamma(\frac{1}{6})}{6v[1 + \sigma^2[\infty]]\sqrt{\pi}\Gamma(\frac{5}{6})} + \mathcal{O}(1) \tag{163}$$

$$\Xi_-(v) = -\frac{3\sqrt{\pi}}{\Gamma(\frac{1}{6})} + \frac{3\pi v[1 + \sigma^2[\infty]]}{\sigma^2[\infty]\Gamma(\frac{1}{6})} \left[\frac{18\sqrt{\pi}\Gamma(\frac{5}{6})}{\Gamma^2(\frac{1}{6})} - 1 \right] + \mathcal{O}(v^2). \quad (164)$$

Hence

$$\lim_{v \rightarrow 0} \alpha_-(v) = \frac{2\sigma^2[\infty]\Gamma^2(\frac{1}{6})}{\pi[1 + \sigma^2[\infty]](18\sqrt{\pi}\Gamma(\frac{5}{6}) - \Gamma^2(\frac{1}{6}))} \quad (165)$$

$$\lim_{v \rightarrow 0} A_0^-(v) = \sqrt{\frac{3}{2}[1 - \sigma^2[\infty]] + 27[1 + \sigma^2[\infty]]\pi^{3/2}/\Gamma^3\left(\frac{1}{6}\right)}. \quad (166)$$

For $\sigma[\infty] = 0$ (i.e. no decision noise) this result gives $\lim_{v \rightarrow 0} \alpha_-(v) = \Gamma^2(\frac{1}{6})/\pi[18\sqrt{\pi}\Gamma(\frac{5}{6}) - \Gamma^2(\frac{1}{6})] \approx 1.9611$ and $\lim_{v \rightarrow 0} A_0^-(v) = 3\sqrt{6}\pi^{3/4}/\Gamma^{3/2}(\frac{1}{6}) \approx 1.3204$. For the ‘+’ branch the limit $v \rightarrow 0$ does not exist; instead when v is decreased the line terminates at the value v_{\min} such that $A_0^+(v_{\min}) = 0$. Finally, the two branches terminate and meet at the value v_{\max} such that $[U(v_{\max}) - 2]^2 = 2\lambda[1 - U(v_{\max})]$, in which $\lambda = 36\sqrt{\pi}\Gamma(\frac{5}{6})/\Gamma^2(\frac{1}{6})$, i.e. where

$$U(v_{\max}) = \sqrt{\lambda - 2}[\sqrt{\lambda} - \sqrt{\lambda - 2}] \quad (167)$$

(note: via the relation $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$ one could simplify our expressions further).

6.4. The $\chi_R = 0$ transition lines

We saw earlier that χ_R can change sign at the $\chi = \infty$ line. Here we inspect the possibility of having a $\chi_R = 0$ transition for finite χ and finite α . The effective agent equation (61) would now give $\tilde{q} = \bar{\eta}\sqrt{\alpha} + \theta$, leading to $c = \sigma^2[\infty]$ and $\phi = 1$. The susceptibility would become

$$\chi = \frac{1}{\sqrt{\alpha}} \left\langle \frac{\partial}{\partial \bar{\eta}} \bar{\sigma} \right\rangle_{\star} = \frac{2\sigma[\infty]}{\sqrt{2\pi\alpha S_0^2}}. \quad (168)$$

The χ_R transition line is hence to be solved from the following coupled equations for (χ, α, A_0) :

$$I_1(\sigma^2(\infty), \chi) = \sqrt{\frac{1}{2}[1 + \sigma^2(\infty)]}, \quad \alpha = \frac{\sigma^2[\infty]/\pi}{I_2(\sigma^2(\infty), \chi) - \frac{1}{2}[1 + \sigma^2(\infty)]}. \quad (169)$$

One can use $\chi > 0$ as a parameter, solve the first equation for $A_0(\chi)$, and then calculate $\alpha(\chi)$ via the second. For $\sigma(\infty) = 1$ (no decision noise) the two equations simplify further to

$$1 = \int Dz z \Delta^{-1}(z) 1 + \frac{1}{\alpha\pi} = \int Dz [\Delta^{-1}(z)]^2. \quad (170)$$

We will explore this latter case, where $\Delta(z) = z(1 + \tau\chi) - z^3/A_0^2$ and $r(z) = \frac{1}{2}z(A_0^2 + 3q)$, in more detail. This requires working out the function $\Delta^{-1}(z)$ for the two model instances.

If $\tau = 1$ we have $q = -\frac{1}{3}A_0^2(1 + \chi^{-1}) \leq -\frac{1}{3}A_0^2$; here we must in the remanent regime always select the non-Maxwell option (136). We replace χ as a parameter by $\lambda = 2|q|^{3/2}/(3|q| - A_0^2) \geq 0$. This implies that $r(z)/|q|^{3/2} = -z/\lambda$, and hence

$$\Delta^{-1}(z) = \text{sgn}(z)\sqrt{|q|}\Omega(|z|/\lambda) \quad (171)$$

$$\Omega(x) = 2\theta[1 - x] \sin\left(\frac{1}{3}\arcsin(x)\right) - \theta[x - 1] \left\{ [x + \sqrt{x^2 - 1}]^{\frac{1}{3}} + [x - \sqrt{x^2 - 1}]^{\frac{1}{3}} \right\}. \quad (172)$$

The first equation of (170) then gives $\sqrt{|q|} = \left[\int D_z |z| \Omega(|z|/\lambda) \right]^{-1}$, and we obtain the following explicit parametrization of the $\chi_R = 0$ line:

$$A_0(\lambda) = \frac{1}{\int D_z |z| \Omega(|z|/\lambda)} \sqrt{3 - \frac{2}{\lambda \int D_z |z| \Omega(|z|/\lambda)}} \tag{173}$$

$$\alpha(\lambda) = \frac{1}{\pi} \left[\frac{\int D_z \Omega^2(|z|/\lambda)}{\left[\int D_z |z| \Omega(|z|/\lambda) \right]^2} - 1 \right]^{-1}. \tag{174}$$

Let us determine the extremal points. For $\lambda \rightarrow \infty$ we use $\Omega(x) = \frac{2}{3}x + \frac{8}{81}x^3 + \mathcal{O}(x^5)$ and get

$$\lim_{\lambda \rightarrow \infty} A_0(\lambda) = \sqrt{3}, \quad \lim_{\lambda \rightarrow \infty} \alpha^{-1}(\lambda) = 0. \tag{175}$$

For small λ we note that the $\chi_R = 0$ line terminates at the value λ_c where A_0 ceases to be real-valued. This value λ_c is the solution of

$$\frac{3}{2} \lambda \int D_z |z| \Omega(|z|/\lambda) = 1 \tag{176}$$

and corresponds to the following point in the (α, A_0) plane:

$$A_0 = 0, \quad \alpha = \frac{1}{\pi} \left[\frac{9}{4} \lambda_c^2 \int D_z \Omega^2(|z|/\lambda_c) - 1 \right]^{-1}. \tag{177}$$

If $\tau = -1$ we have $q = -\frac{1}{3}A_0^2(1 - \chi^{-1})$ and $r(z) = \frac{1}{2}A_0^2 z / \chi$. Here $3q + A_0^2 = A_0^2 / \chi \geq 0$ so $q \geq -\frac{1}{3}A_0^2$, and in the remanent region we select the Maxwell option (135). If now we define as our line parameter $\lambda = 2|q|^{3/2} / (3q + A_0^2) \geq 0$ we will have $r(z)/|q|^{3/2} = z/\lambda$, and hence

$$\Delta^{-1}(z) = \text{sgn}(z) \sqrt{|q|} \Omega(|z|/\lambda) \tag{178}$$

$$q > 0: \quad \Omega(x) = \left[x + \sqrt{1+x^2} \right]^{\frac{1}{3}} + \left[x - \sqrt{1+x^2} \right]^{\frac{1}{3}} \tag{179}$$

$$q < 0: \quad \Omega(x) = \theta[x-1] \left\{ \left[x + \sqrt{x^2-1} \right]^{\frac{1}{3}} + \left[x - \sqrt{x^2-1} \right]^{\frac{1}{3}} \right\} + 2\theta[1-x] \cos\left(\frac{1}{3} \arccos(x)\right). \tag{180}$$

Again we find the first equation of (170) giving $\sqrt{|q|} = \left[\int D_z |z| \Omega(|z|/\lambda) \right]^{-1}$, but now our parametrization becomes

$$A_0(\lambda) = \frac{1}{\int D_z |z| \Omega(|z|/\lambda)} \sqrt{\frac{2}{\lambda \int D_z |z| \Omega(|z|/\lambda)} - 3 \text{sgn}(q)} \tag{181}$$

$$\alpha(\lambda) = \frac{1}{\pi} \left[\frac{\int D_z \Omega^2(|z|/\lambda)}{\left[\int D_z |z| \Omega(|z|/\lambda) \right]^2} - 1 \right]^{-1}. \tag{182}$$

In the limit $\lambda \rightarrow \infty$ we use the expansions $\Omega(x) = \frac{2}{3}x - \frac{8}{81}x^3 + \mathcal{O}(x^5)$ for $q > 0$, and $\Omega(x) = \sqrt{3} + \frac{1}{3}x - \frac{\sqrt{3}}{18}x^2 + \mathcal{O}(x^3)$ for $q < 0$. We then find

$$q > 0: \quad \lim_{\lambda \rightarrow 0} A_0(\lambda) = \sqrt{3} \approx 1.7321, \quad \lim_{\lambda \rightarrow \infty} \alpha^{-1}(\lambda) = 0 \tag{183}$$

$$q < 0: \quad \lim_{\lambda \rightarrow \infty} A_0(\lambda) = \sqrt{\frac{\pi}{2}} \approx 1.2533, \quad \lim_{\lambda \rightarrow \infty} \alpha(\lambda) = \frac{2}{\pi(\pi-2)} \approx 0.5576 \tag{184}$$

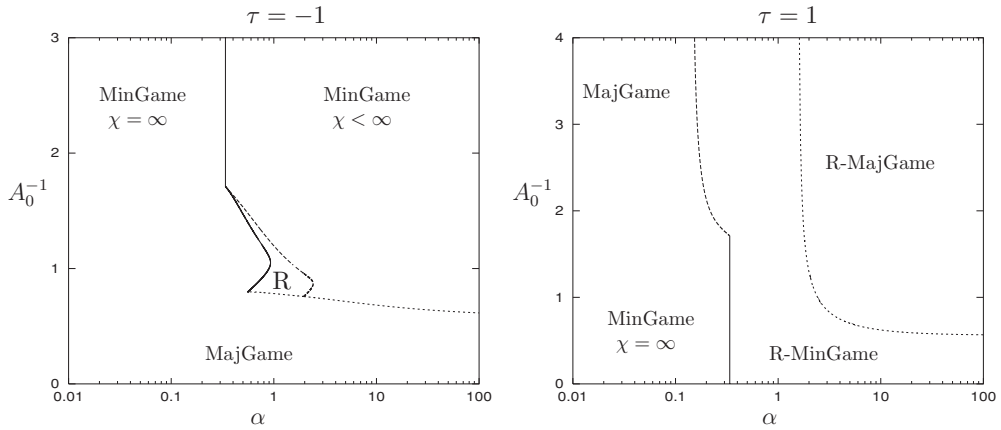


Figure 6. Phase diagrams for the models with $F[A] = \tau A [1 - A^2/A_0^2]$, $\tau = \pm 1$. Solid lines: $\chi = \infty$ transition; dashed lines: remanence onset transition; dotted lines: $\chi_R = 0$ (minority-to-majority) transition. The MinGame phase is characterized by $\chi_R > 0$ (contrarian trading), the MajGame phase by $\chi_R < 0$ (trend-following). R indicates remanence. In the left picture ($\tau = -1$, giving trend-following for $|A| < A_0$ and contrarian trading for $|A| > A_0$), the phase diagram is reasonably complete and reliable. For small A_0 we have only the MinGame phase, with the standard $\chi \rightarrow \infty$ transition at $\alpha_c \approx 0.3374$. As A_0 is increased a small remanent region is formed, until we switch to a MajGame state for large A_0 . In the right picture ($\tau = 1$, giving contrarian trading for $|A| < A_0$ and trend-following for $|A| > A_0$), we find the MinGame phase, with the standard $\chi \rightarrow \infty$ transition at $\alpha_c \approx 0.3374$, for large A_0 (as expected). However, the overall picture is now more uncertain since most of the diagram is remanent. Since we can never be sure of picking the right solution in the remanent phase, the location of the $\chi_R = 0$ line cannot be taken as certain, and indeed intuition suggests that it should have connected to the point where the $\chi = \infty$ and the remanence onset lines meet. In both diagrams we cannot draw the line separating the non-ergodic MinGame phase from the MajGame phase for small α ; as this line involves non-ergodic phases, we have no access to it with a theory that assumes ergodicity.

The latter point coincides with the termination point of the $\chi \rightarrow \infty$ transition line for $\tau = -1$. The $\chi_R = 0$ line for $\tau = -1$ can only have a finite λ termination point with $A_0 = 0$ if $q > 0$, i.e. in the remanent region. In practice we find no such points for $\tau = -1$, irrespective of q .

6.5. Phase diagrams

We can now summarize the picture obtained by analysing time-translation invariant states in terms of a phase diagram. The control parameters are $\alpha \geq 0$, $A_0 \geq 0$ and $\tau = \pm 1$. We have so far identified three transition types: $\chi \rightarrow \infty$, $\chi_R \rightarrow 0$ and $q \rightarrow 0$. The result is shown in figure 6, and discussed in detail in the corresponding figure caption. Most of the technical subtleties in the present models are generated by remanence, so the phase diagram for $\tau = -1$ (giving the model of [11]) is the most complete and reliable, since the remanent region is relatively contained. The situation is different for $\tau = 1$, where most of the phase diagram is remanent, and consequently we cannot be certain of the location of the $\chi_R = 0$ line. The reason for this fundamental difference would appear to be that in the latter case we are always led to the non-Maxwell option for the relevant saddle-point, which (see figure 5) involves two discontinuities as opposed to one. A proper resolution of the remaining uncertainties here would require going beyond the asymptotic limit of the effective agent and the effective overall

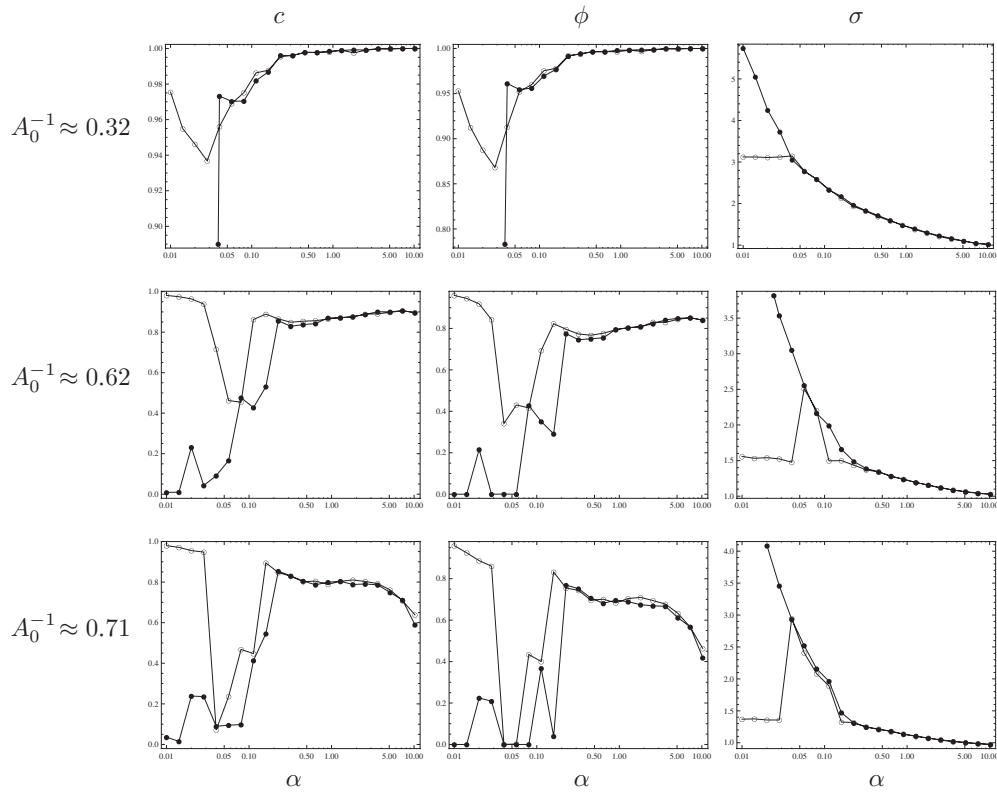


Figure 7. Simulation results for the persistent correlations c , the fraction of frozen agents ϕ , and the volatility σ , for $\tau = -1$ (i.e. $F[A] = -A + A^3/A_0^2$). The chosen values of A_0^{-1} should, according to the phase diagram, probe the MajGame phase of the model, without any $\chi = \infty$ transition. Full markers: *tabula rasa* initial conditions; empty markers: biased initial conditions.

bid processes, in order to determine dynamic stability and dependence on initial conditions of the values of the persistent order parameters.

6.6. Simulations

We have tested our predictions for the values of persistent order parameters and the locations of phase transition lines, for the models (126) and for both values of τ . All simulations were carried out without decision noise, for systems of size $N = 4097$ and for both unbiased ($q_i(0) = \pm 10^{-4}$) and biased ($q_i(0) = \pm 1$) random initializations. Observables were measured during 1000 batch steps, following an equilibration stage of 2000 batch steps.

We begin with $\tau = -1$ (the model of [11]). In figure 7 we present results for the volatility σ , the persistent correlations c , and the fraction of frozen agents ϕ , for values of A_0^{-1} corresponding to the MajGame regime, without remanence. The simulations indeed reproduce a majority-game-type state for large A_0 (top row), but as A_0 is reduced (middle and bottom rows) the system exhibits behaviour that is less clear-cut than what is suggested by the phase diagram. We are still in the non-remanent region, so this cannot be explained by remanence effects. Close to the remanent region R of the $\tau = -1$ phase diagram, the simulations are in rough agreement with the predictions, see figure 8. The vertical dashed line marks the $\chi = \infty$

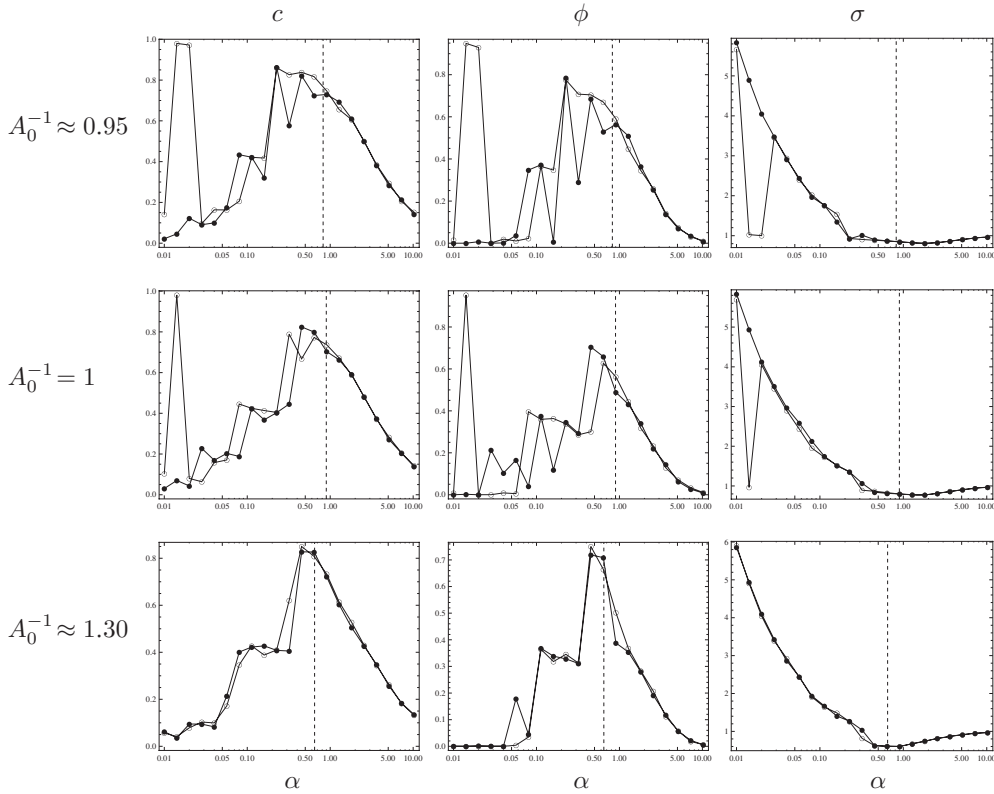


Figure 8. Simulation results for the persistent correlations c , the fraction of frozen agents ϕ , and the volatility σ , for $\tau = -1$ (i.e. $F[A] = -A + A^3/A_0^2$). The chosen values of A_0^{-1} should, according to the phase diagram, probe the region close where a A_0 -dependent $\chi = \infty$ transition occurs (indicated with a vertical dashed line), with a MinGame phase for large α . Full markers: *tabula rasa* initial conditions; empty markers: biased initial conditions.

transition, and seems to agree with simulations within the finite size limitations. In figure 9 we probe the region where we expect MG-type behaviour. This is borne out in the ergodic region, for $\alpha > \alpha_c$ (and the transition is found in the right place, in agreement with the phase diagram). However, for $\alpha < \alpha_c$ the behaviour is found to be far from typical; in contrast to the standard MG, the differences between the biased and unbiased initial conditions are small, and more likely to result of instabilities than long-term memory. In fact one notes similarities in behavior with figure 4 for $F[A] = \text{sgn}(A)|A|^\gamma$ with $\gamma = 4, 5$; this suggest as a possible explanation that if A_0^{-1} gets larger, the first term of the present $F[A] = A^3/A_0^2 - A$ plays a role similar to $\text{sgn}(A)|A|^\gamma$ with $\gamma \geq 4$.

In figure 10 we present simulation results for the most difficult case $\tau = 1$, i.e. $F[A] = A - A^3/A_0^2$. These show that the anticipated transition from a MinGame phase at small values of A_0^{-1} to an MajGame phase for larger A_0^{-1} indeed occurs, but for some value $0.25 \leq A_0^{-1} \leq 0.5$, which differs from the prediction in the phase diagram. Furthermore, one notes instabilities of multiplicities of states in the remanent regime, to the right of the $\chi = \infty$ transition. These deviations for $\tau = 1$ are of course not unexpected, since the $\tau = 1$ phase diagram is plagued by remanence. They could be caused by many things,

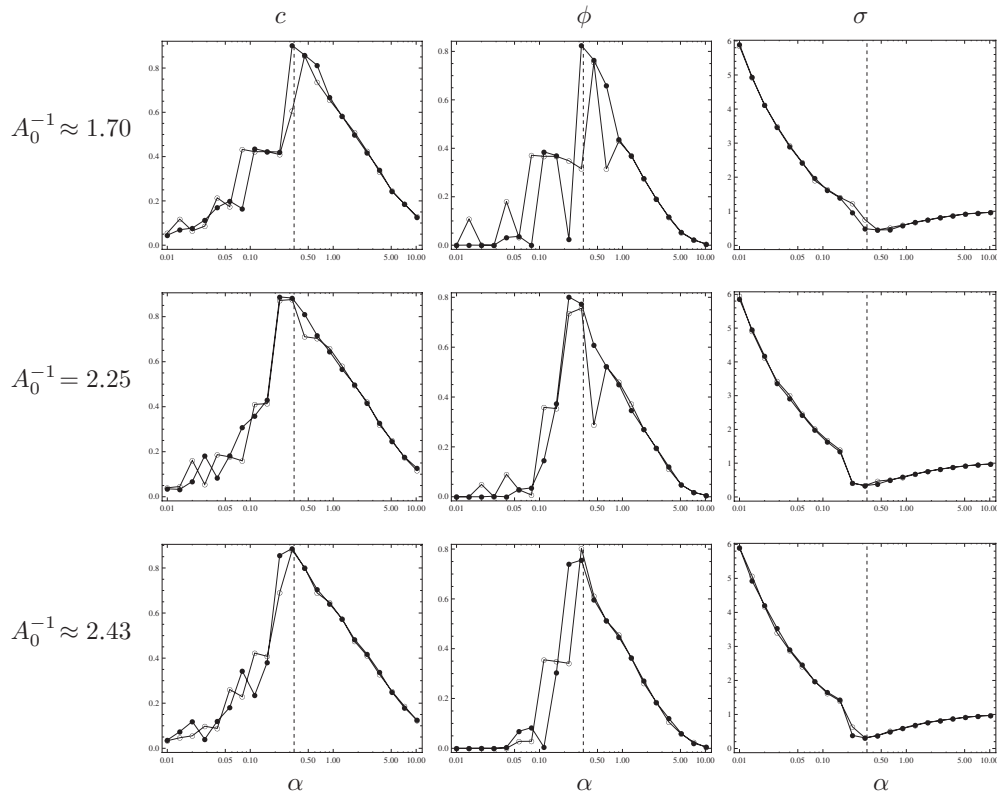


Figure 9. Simulation results for the persistent correlations c , the fraction of frozen agents ϕ , and the volatility σ , for $\tau = -1$ (i.e. $F[A] = -A + A^3/A_0^2$). The chosen values of A_0^{-1} should, according to the phase diagram, probe the region close where a $\chi = \infty$ transition should occur at the conventional MG value $\alpha_c \approx 0.3374$ (indicated with a vertical dashed line), with an ergodic MinGame phases for large α and a non-ergodic MinGame phase for small α . Full markers: *tabula rasa* initial conditions; empty markers: biased initial conditions. We note that the behavior in the non-ergodic regime is not typical of the standard MG.

possibly in combination: incorrect selection of mathematical solutions in the remanent phase, insufficiently equilibrated simulations made worse by remanence, alternative transitions (e.g. onset of weak long-term memory), etc.

7. Discussion

Minority-game-type models have an amazing ability to describe new phenomena within an accessible mathematical framework, and to throw up new mathematical surprises and puzzles. In this paper we have studied generalized agent-based market models of the MG-type (in their so-called ‘fake history’ batch version), in which the agents’ strategy valuation update rule is allowed to depend on the overall market bid, via an impact function $F[A]$ (which would be $F[A] = \pm A$ for the standard minority and majority games, respectively). The function $F[A]$ allows us to model the effect of agents’ interpretation of the state of the market (the magnitude of the fluctuations), assuming that whether price fluctuations are perceived to be large or small should somehow influence how agents trade in financial markets. Our motive is to incorporate

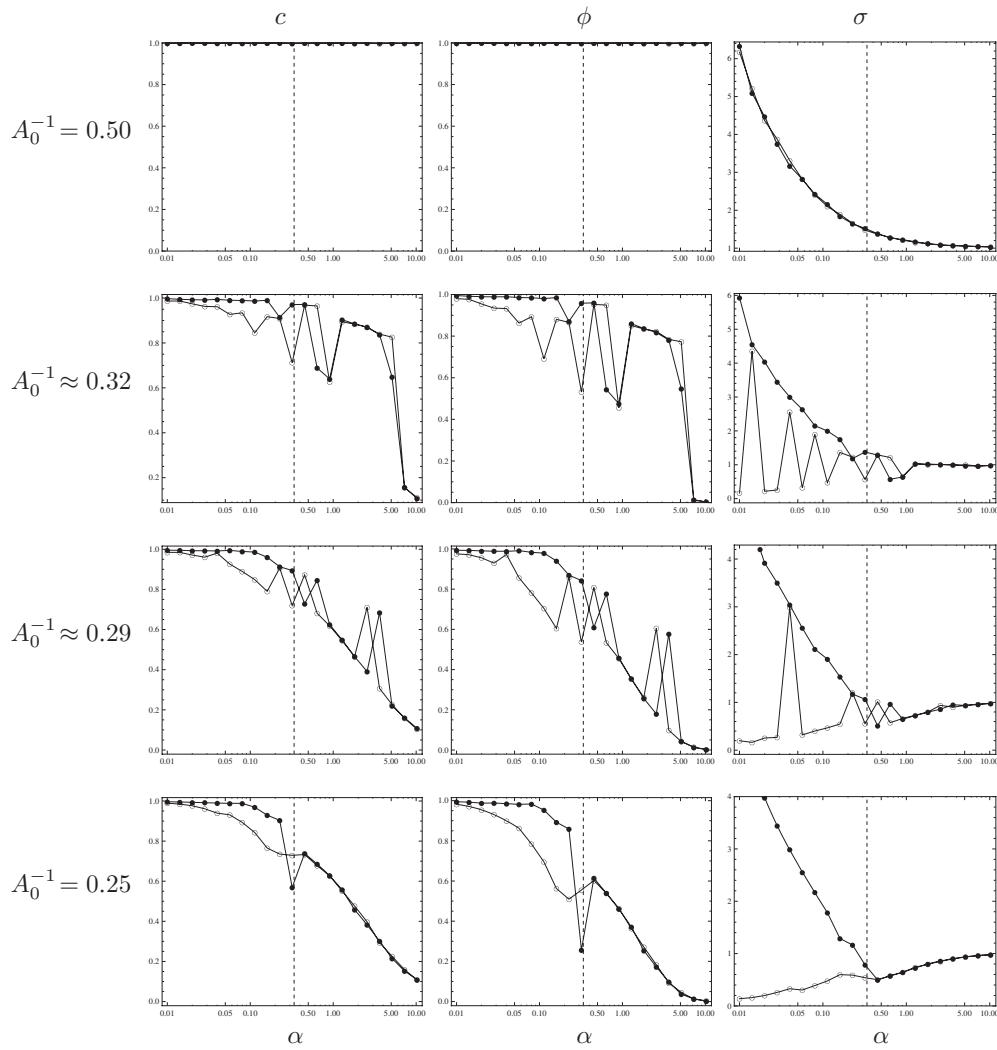


Figure 10. Simulation results for the persistent correlations c , the fraction of frozen agents ϕ , and the volatility σ , for the troublesome case $\tau = 1$ (i.e. $F[A] = A - A^3/A_0^2$). According to the phase diagram, one should for small A_0^{-1} find a conventional $\chi = \infty$ phase separating an ergodic from a non-ergodic MinGame phase. For larger A_0^{-1} we should at some point enter a MajGame phase. Full markers: *tabula rasa* initial conditions; empty markers: biased initial conditions. Although the observed behaviour agrees with the theory for small A_0^{-1} , and indeed a MajGame phase appears as A_0^{-1} is increased, the point where this happens does not agree with the phase diagram. This underlines the problems in the identification of transition lines in the remanent region of the phase diagram, which are indeed much more profound for $\tau = 1$.

into solvable market models such behavioral elements. In this study we focus on two model classes: one in which agents are always contrarians, but where we can control their greed and willingness to take risks, namely $F[A] = \text{sgn}(A)|A|^\gamma$, and one in which agents can switch between trend-following and contrarian trading, namely $F[A] = \pm A[1 - A^2/A_0^2]$, somewhat reminiscent of how they would adapt to booming and ‘bear’ markets.

At a mathematical level, the generating functional analysis of the present class of models requires studying the overall bid evolution explicitly, similar to how one would study models with real market history. It turns out that a key issue in working towards closed self consistent equations for persistent order parameters is whether the overall bid process is remanent. In non-remanent cases, such as $F[A] = \text{sgn}(A)|A|^\gamma$, solution is direct and relatively easy, and the agreement between theory and simulations is excellent (for this particular model: unless γ becomes too large, where we lose the ergodic phase altogether). In remanent cases, such as $F[A] = \tau A[1 - A^2/A_0^2]$ with $\tau = \pm 1$, we need to rely on ansätze and Maxwell-type approximations to select a solution from the possible stationary states, especially for $\tau = 1$, and the agreement between theory and experiment is consequently limited.

Once more, what appears at first sight to be simple modifications of the standard MG lead to highly non-trivial and unexpected behaviour (even in batch models with fake histories). As soon as agents are allowed to adapt their trading style to the magnitude of the fluctuations, in the spirit of [11], one introduces a non-trivial effective overall bid process, with new instabilities and new transitions. We are now approaching the point where theories based only on persistent order parameter equations, the ones that benefit most from the simplifications induced by having ‘frozen agents’, are no longer giving us the information we need. In models dominated by remanence, which are the type one needs when including more realistic agent behaviour, we can no longer avoid solving the full dynamics more explicitly.

Appendix. Scaling of overall market bid covariances

We defined the overall bid covariance matrix as $\Xi_{tt'}(z) = \langle \tilde{A}(t, z)\tilde{A}(t', z) \rangle$. It must be time-translation invariant, so we write $\Xi(t, z) = \Xi_{s+t, s}(z)$, i.e. $\Xi(t, z) = \langle \tilde{A}(s, z)\tilde{A}(t+s, z) \rangle$. From (79) we extract:

$$\begin{aligned} \Xi(t, z) = & \frac{1}{2}\tilde{C}(t) - \sum_s G(s)\{\langle F[\bar{A}(z) + \tilde{A}(t-s, z)]\tilde{A}(0, z) \rangle + \langle F[\bar{A}(z) + \tilde{A}(-s, z)]\tilde{A}(t, z) \rangle\} \\ & - \sum_{ss'} G(s)G(s')\{\langle F[\bar{A}(z) + \tilde{A}(t-s, z)]F[\bar{A}(z) + \tilde{A}(-s', z)] - \bar{F}^2(z) \rangle\}. \end{aligned} \tag{A.1}$$

Since the non-persistent bids are Gaussian this equation gives an explicit relation from which to solve their covariances; along the lines of [16] we apply the general relation

$$\begin{aligned} \langle G[\tilde{A}(t, z), \tilde{A}(t', z)] \rangle = & \int Dx Dy G \left[\frac{x}{\sqrt{2}}\sqrt{S_1(t-t')} \right. \\ & \left. + \frac{y}{\sqrt{2}}\sqrt{S_2(t-t')}, \frac{x}{\sqrt{2}}\sqrt{S_1(t-t')} - \frac{y}{\sqrt{2}}\sqrt{S_2(t-t')} \right] \end{aligned} \tag{A.2}$$

where $S_1(t-t') = \Xi(0, z) + \Xi(t-t', z)$, $S_2(t-t') = \Xi(0, z) - \Xi(t-t', z)$. In particular we need the two quantities

$$\langle F[\bar{A}(z) + \tilde{A}(u, z)]\tilde{A}(v, z) \rangle = \Xi(u-v, z)\bar{F}'(z) \tag{A.3}$$

and

$$\begin{aligned} \langle F[\bar{A}(z) + \tilde{A}(u, z)]F[\bar{A}(z) + \tilde{A}(v, z)] \rangle \\ = \int Dx Dy F \left[\bar{A}(z) + \frac{x}{\sqrt{2}}\sqrt{S_1(u-v)} + \frac{y}{\sqrt{2}}\sqrt{S_2(u-v)} \right] \\ \times F \left[\bar{A}(z) + \frac{x}{\sqrt{2}}\sqrt{S_1(u-v)} - \frac{y}{\sqrt{2}}\sqrt{S_2(u-v)} \right] \end{aligned} \tag{A.4}$$

For $u = v$ this gives simply $\langle F^2[\bar{A}(z) + \tilde{A}(u, z)] \rangle = \bar{F}^2(z)$. The first average above is of order Ξ . The second average can for $u \neq v$ be expanded in powers of $\Xi(u - v, z)$ (which should decay to zero quickly), using

$$\begin{aligned} I(\xi) &= \int Dx Dy F \left[\bar{A} + \frac{x}{\sqrt{2}} \sqrt{\Xi(0) + \xi} + \frac{y}{\sqrt{2}} \sqrt{\Xi(0) - \xi} \right] \\ &\quad \times F \left[\bar{A} + \frac{x}{\sqrt{2}} \sqrt{\Xi(0) + \xi} - \frac{y}{\sqrt{2}} \sqrt{\Xi(0) - \xi} \right] \\ &= \left\{ \int Dx F[\bar{A} + x\sqrt{\Xi(0)}] \right\}^2 + \xi \left\{ \int Dx F'[\bar{A} + x\sqrt{\Xi(0)}] \right\}^2 + \mathcal{O}(\xi^2). \end{aligned} \quad (\text{A.5})$$

Thus, we conclude that

$$\begin{aligned} \langle F[\bar{A}(z) + \tilde{A}(u, z)] F[\bar{A}(z) + \tilde{A}(v, z)] \rangle - \bar{F}^2(z) &= \delta_{uv} [\bar{F}^2(z) - \bar{F}^2(z)] \\ &\quad + (1 - \delta_{uv}) \Xi(u - v, z) [\bar{F}'(z)]^2 + \mathcal{O}(\Xi(u - v, z)) \end{aligned} \quad (\text{A.6})$$

which leads to

$$\begin{aligned} \Xi(t, z) &= \frac{1}{2} \tilde{C}(t) - \bar{F}'(z) \sum_{s>0} G(s) \{ \Xi(-s - t, z) + \Xi(-s + t, z) \} \\ &\quad - [\bar{F}^2(z) - \bar{F}^2(z)] \sum_{s>0} G(s) G(s - t) \\ &\quad - [\bar{F}'(z)]^2 \sum_{s>0} \sum_{v>t-s, v \neq 0} G(s) G(s - t + v) \Xi(v, z) [1 + \mathcal{O}(\Xi)]. \end{aligned} \quad (\text{A.7})$$

At this point we inspect how various terms scale. We know from earlier MG work (especially from approximate calculations of the volatility, see e.g. [8]) that the relaxation of the response function G is very slow. Upon making an ansatz of the form $G(t) = \chi(e^\mu - 1)e^{-\mu t}$ one can show that in the stationary state one must put $\mu \rightarrow 0$ (the relaxation time seems to diverge with N in an as yet undetermined way). For such an exponential ansatz one would find $\sum_{s>0} G(s)G(s - t) = \chi^2 e^{-\mu|t|} (e^\mu - 1)/(e^\mu + 1) = \mathcal{O}(\mu\chi^2)$. Thus if $\mu \rightarrow 0$ the second term in (A.7) vanishes in the ergodic regime. In the final term we use our earlier ansatz that the correlations $\Xi(s)$ decay on short times; formula (A.7) suggests that this relaxation time will be that of $\tilde{C}(t)$. If this relaxation time is τ_C , we may estimate the scaling of the last term in (A.7) as $\mathcal{O}(\mu\tau_C\chi^2)$. So, upon writing the relaxation time of the response function as τ_G , we may replace $\mu \rightarrow 1/\tau_G$ and write (A.7) as

$$\Xi(t, z) = \frac{1}{2} \tilde{C}(t) - \bar{F}'(z) \mathcal{O}(\tau_C/\tau_G) - [\bar{F}^2(z) - \bar{F}^2(z)] \mathcal{O}(1/\tau_G) - [\bar{F}'(z)]^2 \mathcal{O}(\tau_C/\tau_G). \quad (\text{A.8})$$

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