Generating functional analysis of the dynamics of the batch minority game with random external information

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(Received 4 December 2000; published 24 April 2001)

We study the dynamics of the batch minority game, with random external information, using generating functional techniques introduced by De Dominicis. The relevant control parameter in this model is the ratio $\alpha=p/N$ of the number p of possible values for the external information over the number N of trading agents. In the limit $N\rightarrow\infty$ we calculate the location α_c of the phase transition (signaling the onset of anomalous response), and solve the statics for $\alpha>\alpha_c$ exactly. The temporal correlations in global market fluctuations turn out not to decay to zero for infinitely widely separated times. For $\alpha<\alpha_c$ the stationary state is shown to be nonunique. For $\alpha\rightarrow 0$ we analyze our equations in leading order in α, and find asymptotic solutions with diverging volatility $\sigma=O(\alpha^{-1/2})$ (as regularly observed in simulations), but also asymptotic solutions with vanishing volatility $\sigma=O(\alpha^{1/2})$. The former, however, are shown to emerge only if the agents' initial strategy valuations are below a specific critical value.

DOI: 10.1103/PhysRevE.63.056121 PACS number(s): 02.50.Le, 87.23.Ge, 05.70.Ln, 64.60.Ht

I. INTRODUCTION

The minority game has been the subject of much (and at times heated) debate in the physics literature recently. It was originally introduced in [1], as a variation of the El Farol-Bar problem [2], to serve as a simple model for a situation where adaptive agents are competing for limited resources. It has since attracted much attention, especially as a model for financial markets (see, e.g., [3]). The players in the minority game are trading agents who, at every stage of the game, have to make a decision whether to buy or to sell, on the basis of both publicly available information (i.e., past market dynamics, weather forecasts, political developments, or stock prices) and their personal strategies. Those agents who find themselves having made the minority decision make a profit, while those agents who opted for the majority choice lose money. After each round all agents revalue their strategies. There are many variations on the precise implementation of this game, yet most share the same main features of the emerging market fluctuations. The important control parameter in the model is the ratio $\alpha=p/N$ of the number p of possible values for the external information over the number N of trading agents. If this ratio α is very large, the agents exhibit essentially random behavior. This is reflected in the fluctuations of the total bid, which is the sum of all buyers minus the sum of all sellers. If less external information is available (or used) to base decisions upon, i.e., for reduced α, the mismatch between buyers and sellers is found to decrease, and the market behaves more efficiently. This behavior is now understood quite well on the basis of the replica calculations in [4–6] and the crowd-anticrowd theory of [7]. The situation is much less clear, however, when α becomes very small. One possibility is that the market becomes extremely efficient, and the number of buyers almost equals the number of sellers. Another possibility is that the mismatch between buyers and sellers diverges if the amount of shared (i.e., external) information becomes small, and the market becomes extremely inefficient (see, e.g., [8,9]).

In this paper we solve the dynamics for the original many-agent model, using the exact generating functional (or path integral) techniques introduced in [10]. After defining the rules of the game we derive in the limit $N\rightarrow\infty$ an equivalent description in terms of an effective stochastic non-Markovian single-agent process, for which we calculate the first time steps. For sufficiently large values of α, we can solve the statics exactly under the assumption of absence of anomalous response. We calculate the point α_c where this assumption breaks down, resulting in a phase transition; our value for α_c is identical to that found in [4]. The present dynamical approach allows us to study the behavior of the market below α_c. In this region there exist persistent non-static solutions that cannot be studied by the methods of [4]. Below α_c the market is nonergodic and the initial conditions of the agents determine the final stationary state of the market [4,5,13]. For $\alpha\rightarrow 0$ we calculate the market volatility to leading order in α for the case where the agents are initialized with only weak strategy preferences, leading to a diverging volatility with exactly the scaling exponent $\sigma=O(\alpha^{-1/2})$ predicted in [9] on the basis of heuristic arguments. We find a critical value for the initial strategy valuations above which this solution no longer exists and is replaced by an alternative solution with a vanishing volatility of the form $\sigma=O(\alpha^{1/2})$. Our dynamical approach allows in addition for the calculation of the two-time correlations in the global market fluctuations, by definition inaccessible with equilibrium methods (replica or otherwise), which are found to have a persistent component. Numerical simulations confirm our theoretical results convincingly.

II. MODEL DEFINITIONS

There are N agents playing the game. We will only consider the case where N is very large, and ultimately take the limit $N\rightarrow\infty$. The agents are labeled with roman indices i, j, k, etc. At iteration round l all agents are given the same (as yet unspecified) piece of external information $I_{\mu(l)}$, chosen randomly from a total number $p=\alpha N$ of possible values, i.e., $\mu(l)\in\{1,\ldots,\alpha N\}$. In the original model [1] the history of the actual market is used as the information given to the agents; however, in [11] it was shown that random information gives (almost) the same volatility. Each agent i has S strategies $\mathbf{R}_{ia}=(R_{ia}^{1},\ldots,R_{ia}^{\alpha N})\in\{-1,1\}^{\alpha N}$ at her disposal with which to determine how to convert the external information into a trading decision, with $a\in\{1,\ldots,S\}$. Each component R_{ia}^{μ} is selected randomly and independently from $\{-1,1\}$ before the start of the game, with uniform probabilities, and remains fixed throughout the game. The strategies thus introduce quenched disorder into the model. Each strategy of every agent is given an initial valuation or payoff $p_{ia}(0)$. The choice made for these initial values will turn out to be crucial for the emerging behavior of the market. Given a choice $\mu(l)$ made for the external information presented at the start of round l, every agent i selects the strategy labeled by $\tilde{a}_i(l)$ that for trader i has the highest payoff value at that point in time, i.e., $\tilde{a}_i(l)=\arg\max p_{ia}(l)$, and subsequently makes a binary bid $b_i(l)=R_{i\tilde{a}_{i(l)}}^{\mu(l)}$. The (rescaled) total bid at stage l is defined as $A(l)=N^{-1/2}\Sigma_i b_i(l)$. Next all agents update the payoff values of each strategy a on the basis of what would have happened if they had played that particular strategy:

$$p_{ia}(l+1)=p_{ia}(l)-R_{ia}^{\mu(l)}A(l).$$

The minus sign in this expression has the effect that strategies that would have produced a minority decision are rewarded.

This setup so far allows for an arbitrary number of strategies S. The qualitative behavior of the market fluctuations, however, is found to be very much the same for all nonextensive numbers of strategies larger than 1 [12,9]. We therefore present results here only for the $S=2$ model, where the equations can be simplified considerably upon introducing for each agent the instantaneous difference between the two strategy valuations $q_i(l)=[p_{i1}(l)-p_{i2}(l)]/2$ as well as their common part $\boldsymbol{\omega}_i=(\mathbf{R}_{i1}+\mathbf{R}_{i2})/2$ and the difference between the strategies $\boldsymbol{\xi}_i=(\mathbf{R}_{i1}-\mathbf{R}_{i2})/2$. The strategy actually selected in round l can now be written explicitly as a function of $s_i(l)=\mathrm{sgn}[q_i(l)]$, viz., $\mathbf{R}_{i\tilde{a}_i(l)}=\boldsymbol{\omega}_i+s_i(l)\boldsymbol{\xi}_i$, and the evolution of the difference will now be given by

$$q_i(l+1)=q_i(l)-\xi_i^{\mu(l)}\left[\Omega^{\mu(l)}+N^{-1/2}\sum_j \xi_j^{\mu(l)}s_j(l)\right], \quad (1)$$

with $\boldsymbol{\Omega}=N^{-1/2}\Sigma_j\boldsymbol{\omega}_j\in\Re^{\alpha N}$. It has been observed in numerical simulations (see, e.g., [13]) that the magnitude of the market fluctuations remains almost unchanged if a large number of bids are performed before a reevaluation of the strategies is carried out. This is the motivation for us to study a modified (and simpler) version of the dynamics of the game, where, rather than allowing the strategy payoff valuations to be changed at each round, only the accumulated effect of a large number of market decisions is used to change an agent's strategy payoff valuations. This amounts to performing an average in the above dynamic equations over the choices to be made for the external information. If we also change the time unit accordingly from l (which measured individual rounds of the game) to a new unit t which is proportional to the number of payoff validation updates, we arrive at

$$q_i(t+1)=q_i(t)-h_i-\sum_j J_{ij}s_j(t), \quad (2)$$

where $J_{ij}=\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j/N\tau^2$ and $h_i=\boldsymbol{\xi}_i\cdot\boldsymbol{\Omega}/\sqrt{N}\tau^2$, and with $\tau^2=\langle(\Omega^{\mu})^2\rangle=\langle(\xi_i^{\mu})^2\rangle=\langle(\omega_i^{\mu})^2\rangle$; here $\tau^2=\frac{1}{2}$. The above particular choice of time scaling has been made only because it gives the simplest equations later. To make a connection with the original game, one must interpret the evolution of the $q_i(t)$ as described by Eq. (2) as the accumulated effect of order N iterations in the original model. Equation (2) defines the version of the minority game analyzed in this paper. Note that Eq. (2) cannot be converted into a continuous time equation, upon replacing $[q_i(t+1)-q_i(t)]/\sqrt{N}$ by dq_i/dt. A number of agents change their preferred strategy at every iteration of Eq. (2). The size of their q's will be of the order of (half) the step size. In the continuous time limit, in contrast, this step size is lost; yet any discretization used to integrate the continuous time differential equation obtained will effectively reintroduce an (arbitrary) scale for the q's. We regard Eq. (2) as the equivalent of what in the neural network literature would be called the ''batch'' version of the conventional ''on-line'' minority game. For a more detailed discussion concerning the validity of a continuous time differential equation for the thermal minority game we refer to [14,4,15]. Finally, the magnitude of the market fluctuations, or *volatility*, is given by $\sigma^2=\langle A^2\rangle-\langle A\rangle^2$. From the starting point $A(l)=N^{-1/2}\Sigma_i[\omega_i^{\mu(l)}+s_i(l)\xi_i^{\mu(l)}]$ and on the time scales of the process (2), one easily derives

$$\langle A\rangle=\frac{1}{\alpha N\sqrt{N}}\sum_i s_i\sum_\mu \xi_i^{\mu}+O\left(\frac{1}{\sqrt{N}}\right), \quad (3)$$

$$\langle A^2\rangle=\frac{1}{2}+\frac{1}{\alpha N}\left[\sum_i h_i s_i+\frac{1}{2}\sum_{ij} s_i J_{ij}s_j\right]+O\left(\frac{1}{\sqrt{N}}\right). \quad (4)$$

Purely random trading corresponds to $\langle A\rangle=0$ and $\sigma^2=1$. We will also define a more general object, the volatility matrix $\Xi_{tt'}$,

$$\Xi_{tt'}=\langle[A_t-\langle A_t\rangle][A_{t'}-\langle A_{t'}\rangle]\rangle, \quad (5)$$

which measures the temporal correlations of the market fluctuations. Note that $\sigma_t^2=\Xi_{tt}$. In the case where the average bid $\langle A\rangle$ is zero (which will turn out to happen in the present model), the volatility measures the efficiency of the market. Zero volatility implies that supply and demand are always at the same level, and that the market is extremely efficient. A large volatility implies large mismatches between supply and demand, and is the signature of an inefficient market.

III. THE GENERATING FUNCTIONAL

There are two compelling reasons for studying the dynamics of the minority game (MG). First, dynamical techniques do not rely on the presence of a Lyapunov function, so that the MG can be studied for small α. Secondly, it is clear from simulations [13] (see also the figures below) that, at least on the relevant time scales, the stationary state of the minority game can depend quite strongly on the initial conditions. One canonical tool to deal with the dynamics of the present problem is generating functional analysis as introduced by De Dominicis [10], originally developed in the disordered systems community (to study spin glasses, in particular). This formalism allows one to carry out the disorder average (which here is an average over all strategies) and take the $N\to\infty$ limit exactly. The final result of the analysis is a set of closed equations, which can be interpreted as describing the dynamics of an effective ''single agent'' [10,16]. Due to the disorder in the process, this single agent will acquire an effective ''memory,'' i.e., she will evolve according to a nontrivial non-Markovian stochastic process.

First we rewrite Eq. (2) as a Chapman-Kolmogorov equation describing the temporal evolution of an ensemble of markets:

$$p_{t+1}(\mathbf{q})=\int d\mathbf{q}'\ W(\mathbf{q}|\mathbf{q}')p_t(\mathbf{q}'),$$

where, in the absence of noise, the transition probability density is simply

$$W(\mathbf{q}|\mathbf{q}')=\prod_i \delta\left(q_i-q_i'+h_i+\sum_j J_{ij}s_j'\right)$$
$$=\int \frac{d\hat{\mathbf{q}}}{(2\pi)^N}\exp\left[\sum_i i\hat{q}_i\left\langle q_i-q_i'+h_i+\sum_j J_{ij}s_j'\right\rangle\right]$$

with the shorthand $s_j'=\mathrm{sgn}[q_j']$. The moment generating functional for a stochastic process of the present type is defined as

$$Z[\boldsymbol{\psi}]=\left\langle \exp\left[i\sum_t \sum_i \psi_i(t)q_i(t)\right]\right\rangle$$
$$=\int \prod_t [d\mathbf{q}(t)W(\mathbf{q}(t+1)|\mathbf{q}(t))]p_0(\mathbf{q}(0))$$
$$\times\exp\left[i\sum_t\sum_i \psi_i(t)q_i(t)\right].$$

By taking suitable derivatives of the generating functional with respect to the conjugate variables $\boldsymbol{\psi}$, one can generate all moments of $\mathbf{q}$ at arbitrary times. Upon introducing the two short hand notations

$$w_t^\mu=\frac{1}{\tau\sqrt{N}}\sum_i \hat{q}_i(t)\xi_i^\mu,\quad x_t^\mu=\frac{1}{\tau\sqrt{N}}\sum_i s_i(t)\xi_i^\mu,$$

as well as $D\mathbf{q}=\Pi_{it}[dq_i(t)/\sqrt{2\pi}]$, $D\mathbf{w}=\Pi_{\mu t}[dw_t^\mu/\sqrt{2\pi}]$, and $D\mathbf{x}=\Pi_{\mu t}[dx_t^\mu/\sqrt{2\pi}]$ (with similar definitions for $D\hat{\mathbf{q}}$, $D\hat{\mathbf{w}}$, and $D\hat{\mathbf{x}}$, respectively), the generating functional takes the following form:

$$Z[\boldsymbol{\psi}]=\int D\mathbf{w}\,D\hat{\mathbf{w}}\,D\mathbf{x}\,D\hat{\mathbf{x}}\exp\left\{i\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu\right.$$
$$\left.+w_t^\mu(\Omega^\mu/\tau+x_t^\mu)]\right\}\int D\mathbf{q}\,D\hat{\mathbf{q}}\ p_0(\mathbf{q}(0))$$
$$\times\exp\left\{\frac{-i}{\tau\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]\right\}$$
$$\times\exp\left(i\sum_{ti}\{\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]\right.$$
$$\left.+\psi_i(t)q_i(t)\}\right),\tag{6}$$

where we have introduced auxiliary driving forces $\theta_i(t)$ to generate averages involving $\hat{q}_i(t)$ (these can be removed later).

IV. DISORDER AVERAGING

At this stage we can carry out the disorder averages, to be denoted as $\overline{\cdots}$, which involve the variables $\xi_i^\mu=\tau^2(R_{i1}^\mu-R_{i2}^\mu)$ and $\Omega^\mu=N^{-1/2}\tau^2\Sigma_j(R_{j1}^\mu+R_{j2}^\mu)$ only. For times that do not scale with N one obtains

$$\overline{\exp\left(\frac{i}{\tau}\sum_{t\mu}w_t^\mu\Omega^\mu-\frac{i}{\tau\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]\right)}$$
$$=\prod_{i\mu}\overline{\exp\left(\frac{i\tau}{\sqrt{N}}\sum_t\{w_t^\mu(R_1+R_2)-(R_1-R_2)[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]\}\right)}$$
$$=\exp\left(-\frac{1}{2}\sum_{\mu tt'}[w_t^\mu w_{t'}^\mu+\hat{w}_t^\mu L_{tt'}\hat{w}_{t'}^\mu+2\hat{x}_t^\mu K_{tt'}\hat{w}_{t'}^\mu+\hat{x}_t^\mu C_{t,t'}\hat{x}_t^\mu]+O(N^0)\right),$$

where we have introduced $C_{tt'}=N^{-1}\Sigma_i s_i(t)s_i(t')$, $K_{tt'}=N^{-1}\Sigma_i s_i(t)\hat{q}_i(t')$, and $L_{tt'}=N^{-1}\Sigma_i\hat{q}_i(t)\hat{q}_i(t')$. We isolate these functions via the insertion of appropriate δ functions (in integral representation), and define the corresponding shorthand notation $DC=\Pi_{tt'}[dC_{tt'}/\sqrt{2\pi}]$, $DK=\Pi_{tt'}[dK_{tt'}/\sqrt{2\pi}]$, and $DL=\Pi_{tt'}[dL_{tt'}/\sqrt{2\pi}]$ (with similar definitions for $D\hat{C}$, $D\hat{K}$, and $D\hat{L}$, respectively). Upon assuming simple initial conditions of the form $p_0(\mathbf{q})=\Pi_i p_0(q_i)$, the i-dependent terms in the disorder-averaged generating functional (6) are now found to factorize fully over the N traders, and we arrive at an expression of the following form:

$$\overline{Z[\boldsymbol{\psi}]}=\int [DC\,D\hat{C}][DK\,D\hat{K}][DL\,D\hat{L}]e^{N[\Psi+\Phi+\Omega]+O(N^0)}. \tag{7}$$

The subdominant $O(N^0)$ term in the exponent is independent of the generating fields $\{\psi_i(t)\}$ and $\{\theta_i(t)\}$. There are three distinct leading contributions to the exponent in Eq. (7). The first is a ''bookkeeping'' term, linking the two-time order parameters to their conjugates:

$$\Psi=i\sum_{tt'}[\hat{C}_{tt'}C_{tt'}+\hat{K}_{tt'}K_{tt'}+\hat{L}_{tt'}L_{tt'}].$$

The second reflects the statistical properties of the players' arsenal of strategies:

$$\begin{aligned}\Phi=\alpha\ln\Bigg[\int Dw\,D\hat{w}\,Dx\,D\hat{x}\,\exp\Bigg(i\sum_t[\hat{w}_t w_t+\hat{x}_t x_t+w_t x_t]\Bigg)\\ \times\exp\Bigg(-\frac{1}{2}\sum_{tt'}[w_t w_{t'}+\hat{w}_t L_{tt'}\hat{w}_{t'}+2\hat{x}_t K_{tt'}\hat{w}_{t'}\\ +\hat{x}_t C_{tt'}\hat{x}_{t'}]\Bigg)\Bigg].\end{aligned} \tag{8}$$

The third term, which contains the generating fields, will describe the (now stochastic) evolution of the strategy valuations $q(t)$ of a single effective agent:

$$\begin{aligned}\Omega=\frac{1}{N}\sum_i\ln\Bigg[\int Dq\,D\hat{q}\,p_0(q(0))\\ \times\exp\Bigg(i\sum_t\hat{q}(t)[q(t+1)-q(t)-\theta_i(t)]\Bigg)\\ \times\exp\Bigg(i\sum_t\psi_i(t)q(t)-i\sum_{tt'}[s(t)\hat{C}_{tt'}s(t')\\ +s(t)\hat{K}_{tt'}\hat{q}(t')+\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')]\Bigg)\Bigg]\end{aligned}$$

with $s(t)=\text{sgn}[q(t)]$, $Dq=\Pi_t[dq(t)/\sqrt{2\pi}]$, $Dw=\Pi_t[dw_t/\sqrt{2\pi}]$, and $Dx=\Pi_t[dx_t/\sqrt{2\pi}]$ (and similar definitions for $D\hat{q}$, $D\hat{w}$, and $D\hat{x}$). The form of Eq. (7) is suitable for a saddle-point integration in the thermodynamic limit $N\to\infty$. With a modest amount of foresight we define $G_{tt'}=-iK_{tt'}$. Upon taking derivatives with respect to the generating fields $\{\theta_i(t),\psi_i(t)\}$, and using the built-in normalization $\overline{Z[\mathbf{0}]}=1$, we find that at the relevant saddle point

$$C_{tt'}=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle s_i(t)s_i(t')\rangle}, \tag{9}$$

$$G_{tt'}=\lim_{N\to\infty}\frac{1}{N}\sum_i\frac{\partial}{\partial\theta_i(t')}\overline{\langle s_i(t)\rangle}, \tag{10}$$

$$L_{tt'}=\lim_{N\to\infty}\frac{1}{N}\sum_i\frac{\partial^2}{\partial\theta_i(t)\partial\theta_i(t')}\overline{Z[\mathbf{0}]}=0. \tag{11}$$

The first two are recognized as representing disorder-averaged and site-averaged correlation and response functions. At this stage the generating fields are in principle no longer needed. We will put $\psi_i(t)=0$ and $\theta_i(t)=\theta(t)$, and find our expression for Ω simplifying to

$$\begin{aligned}\Omega=\ln\Bigg[\int Dq\,D\hat{q}\,p_0(q(0))\\ \times\exp\Bigg(i\sum_t\hat{q}(t)[q(t+1)-q(t)-\theta(t)]\Bigg)\\ \times\exp\Bigg(-i\sum_{tt'}[s(t)\hat{C}_{tt'}s(t')+s(t)\hat{K}_{tt'}\hat{q}(t')\\ +\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')]\Bigg)\Bigg].\end{aligned} \tag{12}$$

Extremization of the extensive exponent $\Psi+\Phi+\Omega$ of Eq. (7) with respect to $\{C,\hat{C},K,\hat{K},L,\hat{L}\}$ gives the saddle-point equations

$$C_{tt'}=\langle s(t)s(t')\rangle_\star,\quad G_{tt'}=\frac{\partial\langle s(t)\rangle_\star}{\partial\theta(t')}, \tag{13}$$

$$\hat{C}_{tt'}=\frac{i\partial\Phi}{\partial C_{tt'}},\quad\hat{K}_{tt'}=\frac{i\partial\Phi}{\partial K_{tt'}},\quad\hat{L}_{tt'}=\frac{i\partial\Phi}{\partial L_{tt'}}, \tag{14}$$

whereas $L_{tt'}=0$. The effective single-trader averages $\langle\cdots\rangle_\star$, generated by taking derivatives of Eq. (12), are defined as follows (note that $s(t)=\text{sgn}[q(t)]$):

$$\langle f[\{q\}]\rangle_\star=\frac{\int Dq\,M[\{q\}]f[\{q\}]}{\int Dq\,M[\{q\}]},$$

$$\begin{aligned}M[\{q\}]=p_0(q(0))\exp\Bigg(-i\sum_{tt'}s(t)\hat{C}_{tt'}s(t')\Bigg)\\ \times\int D\hat{q}\,\exp\Bigg(-i\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')\Bigg)\\ \times\exp\Bigg(i\sum_t\hat{q}(t)\Bigg[q(t+1)-q(t)-\theta(t)\\ -\sum_{t'}\hat{K}^T_{tt'}s(t')\Bigg]\Bigg).\end{aligned} \tag{15}$$

Upon elimination of $\{\hat{C},\hat{K},\hat{L}\}$ via Eq. (14), we have now obtained exact closed equations for the disorder-averaged correlation and response functions in the $N\to\infty$ limit: namely, Eq. (13), with the effective single-trader measure (15).

V. SIMPLIFICATION OF THE SADDLE-POINT EQUATIONS

The above procedure is quite insensitive to changing model details; alternative choices made for the statistics of traders' strategies would simply lead to a different form for the function Φ (8), whereas changing the update rules for the strategy valuations of the traders (e.g., by making these nondeterministic, as in [14,4]) would affect only the details of the term Ω (12). We now work out our equations for the present choice of model. Focusing first on Φ, we perform the x_t integrals, yielding $\Pi_t\delta[\hat{x}_t+w_t]$, and after performing the remaining $\hat{x}$ integrations we get

$$\Phi=\alpha\ln\int Dw\,D\hat{w}\,\exp\left(i\sum_t \hat{w}_t w_t\right)\times\exp\left(-\frac{1}{2}\sum_{tt'}[w_t w_{t'}+\hat{w}_t L_{tt'}\hat{w}_{t'}-2w_t K_{tt'}\hat{w}_{t'}+w_t C_{tt'}w_{t'}]\right).$$

The Gaussian integration over $\{w_t\}$ gives

$$\Phi=-\frac{1}{2}\alpha\ln\det D+\alpha\ln\int\prod_t\left[\frac{d\hat{w}_t}{\sqrt{2\pi}}\right]\times\exp\left(-\frac{1}{2}\sum_{tt'}\hat{w}_t L_{tt'}\hat{w}_{t'}\right)\times\exp\left(-\frac{1}{2}\sum_{tt'}\hat{w}_t[(\mathbb{1}-iK)^T D^{-1}(\mathbb{1}-iK)]_{tt'}\hat{w}_{t'}\right),$$

where the entries of the matrix D are given by $D_{tt'}=1+C_{tt'}$. We now take the derivative of Φ with respect to $L_{tt'}$, as dictated by Eq. (14), and subsequently put all $L_{tt'}\to 0$. This gives

$$\hat{L}=-\tfrac{1}{2}\,i\alpha(\mathbb{1}-iK)^{-1}D(\mathbb{1}-iK^T)^{-1},$$

and $\lim_{L\to 0}\Phi=-\alpha\,\mathrm{Tr}\ln(\mathbb{1}-iK)$, so that

$$\hat{K}^T=-\alpha(\mathbb{1}-iK)^{-1},\quad \hat{C}=0.$$

We now write our final result in terms of the response function (10), via the identity $K=iG$, and find our effective single-trader measure $M[\{q\}]$ of Eq. (15) reducing to

$$p_0(q(0))\int D\hat{q}\times\exp\left(-\frac{1}{2}\alpha\sum_{tt'}\hat{q}(t)[(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^T)^{-1}]_{tt'}\hat{q}(t')\right)\times\exp\left(i\sum_t\hat{q}(t)\left[q(t+1)-q(t)-\theta(t)+\alpha\sum_{t'}(\mathbb{1}+G)^{-1}_{tt'}s(t')\right]\right).\quad(16)$$

This describes a stochastic single-agent process of the form

$$q(t+1)=q(t)+\theta(t)-\alpha\sum_{t'\leq t}(\mathbb{1}+G)^{-1}_{tt'}\,\mathrm{sgn}[q(t')]+\sqrt{\alpha}\,\eta(t).\quad(17)$$

Causality ensures that $G_{tt'}=0$ for all $t'\geq t$ [so that $(\mathbb{1}+G)^{-1}_{tt'}=0$ for $t'>t$], and $\eta(t)$ is a Gaussian noise with zero mean and with temporal correlations given by $\langle\eta(t)\eta(t')\rangle=\Sigma_{tt'}$:

$$\Sigma=(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^T)^{-1}.\quad(18)$$

The correlation and response functions defined by Eqs. (9) and (10) are the dynamic order parameters of the problem, and must be solved self-consistently from the closed equations

$$C_{tt'}=\langle\mathrm{sgn}[q(t)q(t')]\rangle_\star,\quad G_{tt'}=\frac{\partial\langle\mathrm{sgn}[q(t)]\rangle_\star}{\partial\theta(t')}.\quad(19)$$

Note that $M[\{q\}]$ as given by Eq. (16) is normalized, i.e., $\int Dq\,M[\{q\}]=1$, so the associated averages reduce to $\langle f[\{q\}]_\star\rangle=\int Dq\,M[\{q\}]f[\{q\}]$. The solution of Eq. (19) can be calculated numerically with arbitrary precision, without finite size effects, using a technique described in [17].

Finally, in Appendix A we calculate the disorder-averaged rescaled average bid $\overline{\langle A_t\rangle}$ and volatility matrix $\overline{\Xi}_{tt'}=\overline{\langle A_tA_{t'}\rangle}-\overline{\langle A_t\rangle\langle A_{t'}\rangle}$, for $N\to\infty$, as defined previously in Eqs. (3) and (5). Note that objects such as $\langle A_t\rangle$ must asymptotically become self-averaging, i.e., independent of the microscopic realization of the disorder; hence $\overline{\langle A_t\rangle\langle A_{t'}\rangle}\to\overline{\langle A_t\rangle}\;\overline{\langle A_{t'}\rangle}$ for $N\to\infty$. We find the satisfactory result that the average bid is zero, and that the volatility matrix (and thus also the ordinary single-time volatility $\sigma_t^2=\Xi_{tt}$) is proportional to the covariance matrix (18) of the noise in the dynamics (17) of the effective single agent:

$$\lim_{N\to\infty}\overline{\langle A\rangle}_t=0,\quad\lim_{N\to\infty}\overline{\Xi}_{tt'}=\tfrac{1}{2}\,\Sigma_{tt'}.\quad(20)$$

Thus the noise term $\eta(t)$ in the single-agent process (17) represents the overall market fluctuations, and the covariance matrix (18) informs us of both single-time volatility and the temporal correlations of the market fluctuations.

VI. THE FIRST TIME STEPS

For the first few time steps it is possible to calculate the order parameters (correlation and response functions) and the volatility explicitly, starting from the effective single-trader measure (16). Note that $D_{tt'}=1+C_{tt'}$ and that $C_{tt}=1$ for any t. Significant simplifications can be made by using causality. For instance, we always have $(\mathbb{1}+G)^{-1}=\Sigma_{n\geq 0}(-1)^n G^n$, with causality enforcing

$$[G^n]_{tt'}=0 \quad \text{for } t'>t-n. \tag{21}$$

At $t=0$ this immediately allows us to conclude that $\Sigma_{00}=D_{00}=2$. We now obtain from Eq. (16) the joint statistics at time $t=1$:

$$p(q(1)|q(0)) = \frac{\exp(-\{q(1)-q(0)-\theta(0)+\alpha\,\text{sgn}[q(0)]\}/4\alpha)}{2\sqrt{\alpha\pi}}. \tag{22}$$

Equation (22), in turn, allows us to calculate $C_{10}=\langle \text{sgn}[q(0)q(1)]\rangle_\star$ and $G_{10}=\partial\langle\text{sgn}[q(1)]\rangle_\star/\partial\theta(0)$:

$$C_{10}=-\int dq(0)p(q(0)) \times \text{erf}\left[\frac{\sqrt{\alpha}}{2}-\frac{|q(0)|+\theta(0)\text{sgn}[q(0)]}{2\sqrt{\alpha}}\right],$$

$$G_{10}=-\frac{1}{\sqrt{\alpha\pi}}\int dq(0)p(q(0)) \times\exp\{-[\alpha\,\text{sgn}[q(0)]-q(0)-\theta(0)]^2/4\alpha\}.$$

We can now move to the next time step, again using Eq. (21), where we need the noise covariances Σ_{11} and Σ_{10}:

$$\Sigma_{10}=\sum_{tt'}[\mathbb{1}-G+O(G^2)]_{1t}D_{tt'}[\mathbb{1}-G^T+O(G^T)^2]_{t'0} =1+C_{10}-2G_{10},$$

$$\Sigma_{11}=\sum_{tt'}[\mathbb{1}-G+O(G^2)]_{1t}D_{tt'}[\mathbb{1}-G^T+O(G^T)^2]_{t'1} =2-2G_{10}[1+C_{01}]+2[G_{10}]^2.$$

Although this procedure can in principle be repeated for an arbitrary number of time steps, generating exact expressions for the various order parameters iteratively, the results become increasingly complicated when larger times are involved.

It is interesting, however, to inspect further some special limits. We first turn to the (trivial) case where α is very small, $p(q(0))=\delta[q(0)-q_0]$, and q_0 is finite. Provided $|q_0|\gg\sqrt{\alpha}$ as $\alpha\to 0$, we immediately deduce from the above results that $\lim_{\alpha\to 0}C_{10}=1$, $\lim_{\alpha\to 0}G_{10}=0$, and $\lim_{\alpha\to 0}\Sigma_{10}=\lim_{\alpha\to 0}\Sigma_{11}=2$. Hence we find in leading order in α that $q(1)=q(0)$ and $\eta(1)=\eta(0)$. One easily repeats the argument for larger times, and finds that, without perturbations, both the system variables $q(t)$ and the noise variables $\eta(t)$ will remain frozen for times $t\ll 1/\sqrt{\alpha}$, the only remaining uncertainty in the noise being the realization of $\eta(0)$:

$$q(t)=q_0+t\sqrt{\alpha}\,\eta(0)+O(\alpha t) \quad (\alpha\to 0).$$

If $\text{sgn}[q_0]\neq\text{sgn}[\eta(0)]$, the system will ‘‘defreeze’’ at the first instance where $t>|q_0/\eta(0)\sqrt{\alpha}|$. Since $\eta(0)$ is a zero average Gaussian variable, one should therefore for small α expect half of the population of traders (those with nonprofitable initial random strategy choices) to commence strategy chances at time scales $t=O(\alpha^{-1/2})$, whereas the other half will continue playing the game with their (for now profitable) initial strategy choices at least up to $t=O(\alpha^{-1})$.

It is also interesting to analyze the case where the game is initialized in a *tabula rasa* manner (which appears to have been common practice in the literature), i.e., $p(q(0))=\delta[q-q_0]$ with $q_0=0^+$, and where we have no perturbation fields, i.e., $\theta(t)=0$. Now the above results reduce to

$$C_{10}=-\text{erf}[\tfrac{1}{2}\sqrt{\alpha}], \quad G_{10}=(\alpha\pi)^{-1/2}e^{-\alpha/4},$$

$$\Sigma_{10}=1-\text{erf}[\tfrac{1}{2}\sqrt{\alpha}]-\frac{2}{\sqrt{\alpha\pi}}e^{-\alpha/4},$$

$$\Sigma_{11}=2-\frac{2}{\sqrt{\alpha\pi}}e^{-\alpha/4}(1-\text{erf}[\tfrac{1}{2}\sqrt{\alpha}])+\frac{2}{\alpha\pi}e^{-\alpha/2}.$$

The negative value of the correlation function C_{10} implies that for short times the traders will exhibit a tendency to alternate their (two) strategies. Let us now inspect the limiting behavior of the above expressions for large and small values of α. For large α one obtains

$$\lim_{\alpha\to\infty}C_{10}=-1, \quad \lim_{\alpha\to\infty}G_{10}=\lim_{\alpha\to\infty}\Sigma_{10}=0.$$

For small α, on the other hand, we find

$$C_{10}=-\frac{\sqrt{\alpha}}{\sqrt{\pi}}+O(\alpha^{3/2}), \quad G_{10}=\frac{1}{\sqrt{\alpha\pi}}+O(\sqrt{\alpha}),$$

$$\Sigma_{10}=1-\frac{2}{\sqrt{\alpha\pi}}+O(\sqrt{\alpha}), \quad \Sigma_{11}=\frac{2}{\alpha\pi}-\frac{2}{\sqrt{\alpha\pi}}+O(\alpha^0).$$

So $\eta(1)=O(\alpha^{-1/2})$, whereas $\eta(0)=O(\alpha^0)$. We also find

$$\left\langle\left[\eta(1)+\frac{\eta(0)}{\sqrt{\alpha\pi}}\right]^2\right\rangle=\Sigma_{11}+\frac{2}{\sqrt{\alpha\pi}}\Sigma_{10}+\frac{1}{\alpha\pi}\Sigma_{00}=O(\alpha^0),$$

from which it follows that $\eta(1)=-\eta(0)/\sqrt{\alpha\pi}+O(\alpha^0)$, and hence we can write the first steps of the effective single-agent equation (17) as

$$q(1)=q(0)-\alpha\,\mathrm{sgn}[q(0)]+\sqrt{\alpha}\,\eta(0)=\sqrt{\alpha}\,\eta(0)+O(\alpha),$$

$$q(2)=q(1)-\alpha\,\mathrm{sgn}[q(1)]+\alpha G_{10}\,\mathrm{sgn}[q(0)]+\sqrt{\alpha}\,\eta(1)=-\eta(0)/\sqrt{\pi}+O(\sqrt{\alpha}).$$

Thus also $C_{20}=\langle \mathrm{sgn}[q(0)q(2)]\rangle_\star=O(\sqrt{\alpha})$ and $C_{21}=\langle \mathrm{sgn}[q(1)q(2)]\rangle_\star=-1+O(\sqrt{\alpha})$. We observe that for small α the first two time steps are driven predominantly by the noise component in Eq. (17). This noise component increases in strength and starts oscillating in sign, resulting in an effective agent that is increasingly likely to alternate its strategies. Equivalently, this implies that in the initial N-agent system an increasing *fraction* of the population of agents will start alternating their strategies.

Let us finally inspect the initial behavior of Eq. (17) for the intermediate regime where $p(q(0))=\delta[q-q_0]$ with $q_0=O(\sqrt{\alpha})$, to which (as we have seen) also for $q_0=O(\alpha^0)$ about half of the traders will automatically be driven in due course. We now put $q_0=\sqrt{\alpha}\tilde{q}_0$ and find in leading order

$$C_{10}=\mathrm{erf}[\tfrac{1}{2}\,|\tilde{q}_0|]+\cdots,\quad G_{10}=\frac{1}{\sqrt{\alpha\pi}}\,e^{-\tilde{q}_0^2/4}+\cdots,$$

$$\Sigma_{10}=-\frac{2}{\sqrt{\alpha\pi}}\,e^{-\tilde{q}_0^2/4}+\cdots,\quad \Sigma_{11}=\frac{2}{\alpha\pi}\,e^{-\tilde{q}_0^2/2}+\cdots.$$

Thus we have $\langle[\eta(1)+(\alpha\pi)^{-1/2}e^{-\tilde{q}_0^2/4}\eta(0)]^2\rangle=0$, so also $\eta(1)=-(\alpha\pi)^{-1/2}e^{-\tilde{q}_0^2/4}\eta(0)$, in leading order for $\alpha\to 0$. This then, together with $q(1)=O(\sqrt{\alpha})$ [which immediately follows from Eq. (22)], leads us to

$$q(2)=-\pi^{-1/2}e^{-\tilde{q}_0^2/4}\eta(0)+O(\sqrt{\alpha}).$$

We thus find that for $q_0=O(\sqrt{\alpha})$ also the initial conditions are more or less washed out by the internal noise generated by the process, within just two iteration steps.

VII. THE STATIONARY STATE FOR $\alpha>\alpha_c$

For general α, not necessarily small, the arguments used in the second part of the previous section do not hold. In a stationary state, along with agents that will change strategy (almost) every cycle, there will generally also be agents finding themselves consistently in the minority group, which will consequently play the same strategy over and over again. For the latter ''frozen'' group (a term introduced in [18]), the differences between the valuations of the two available strategies (i.e., the values of q_i) will grow more or less linearly in time, whereas the ''fickle'' agents will have values for q_i very close to zero. In order to separate the two groups efficiently we introduce the rescaled values $\tilde{q}_i(t)=q_i(t)/t$. Frozen agents will be those for which $\lim_{t\to\infty}\tilde{q}_i(t)\neq 0$. Similarly, the effective single-agent process (17) is transformed via $\tilde{q}(t)=q(t)/t$, where now the quantity $\phi=\lim_{\epsilon\to 0}\lim_{t\to\infty}\langle\theta[|\tilde{q}(t)|-\epsilon]\rangle_\star$ will give the asymptotic fraction of frozen agents in the original N-agent system, for $N\to\infty$. The dynamical equation of the rescaled effective agent can be written as

$$\tilde{q}(t)=\frac{1}{t}\tilde{q}(1)+\frac{\sqrt{\alpha}}{t}\sum_{t'<t}\eta(t')-\frac{\alpha}{t}\sum_{t'<t}\sum_{t''}(\mathbb{1}+G)^{-1}_{t't''}\,\mathrm{sgn}[\tilde{q}(t'')].\tag{23}$$

If the game has reached a stationary state, then $G_{tt'}=G(t-t')$, $C_{tt'}=C(t-t')$, and $\Sigma_{tt'}=\Sigma(t-t')$, by definition. We will assume in this section that the stationary state is one without anomalous response, i.e., temporary perturbations will not influence the stationary state and decay sufficiently fast, such that $\lim_{\tau\to\infty}\sum_{t\le\tau}G(t)=k$ exists. This condition will be met if there is just one ergodic component; it is the dynamical equivalent of replica symmetry being stable (see, e.g., [19]) in a detailed balance model. We now define $\tilde{q}=\lim_{t\to\infty}\tilde{q}(t)$ (assuming this limit exists) and take the limit $t\to\infty$ in Eq. (23). Under the assumption of absent anomalous response, we can use the two lemmas in Appendix B to simplify the result to

$$\tilde{q}=-\frac{\alpha}{1+k}s+\sqrt{\alpha}\,\eta\tag{24}$$

with the averages $s=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\le\tau}\mathrm{sgn}[\tilde{q}_t]$ and $\eta=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\le\tau}\eta(t)$. The variance of the zero-average Gaussian random variable η follows from Eq. (18):

$$\begin{aligned}\langle\eta^2\rangle&=\lim_{\tau,\tau'\to\infty}\frac{1}{\tau\tau'}\sum_{t\le\tau}\sum_{t'\le\tau'}[(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^T)^{-1}]_{tt'}\\&=(1+k)^{-2}\left[1+\lim_{\tau,\tau'\to\infty}\frac{1}{\tau\tau'}\sum_{t\le\tau}\sum_{t'\le\tau'}C_{tt'}\right]\\&=(1+k)^{-2}[1+\langle s^2\rangle].\end{aligned}\tag{25}$$

Note that $\langle s^2\rangle=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\le\tau}C(t)=c$.

The effective agent is frozen if $\tilde{q}\neq 0$, in which case $s=\mathrm{sgn}[\tilde{q}]$. This solves Eq. (24) if and only if $|\eta|>\sqrt{\alpha}/(1+k)$. If $|\eta|<\sqrt{\alpha}/(1+k)$, on the other hand, the agent is not frozen; now $\tilde{q}=0$ and $s=(1+k)\eta/\sqrt{\alpha}$. We can now calculate $c=\langle s^2\rangle$ self-consistently, upon distinguishing between the two possibilities:

$$c=\left\langle\theta\left[|\eta|-\frac{\sqrt{\alpha}}{1+k}\right]\right\rangle+\left\langle\theta\left[\frac{\sqrt{\alpha}}{1+k}-|\eta|\right]\frac{(1+k)^2\eta^2}{\alpha}\right\rangle.$$

Working out the Gaussian integrals describing the statics of η with variance (25) then gives

$$c=1-\left(1-\frac{1+c}{\alpha}\right)\mathrm{erf}\left[\sqrt{\frac{\alpha}{2(1+c)}}\right]-2\sqrt{\frac{1+c}{2\pi\alpha}}\,e^{-\alpha/2(1+c)}.\tag{26}$$

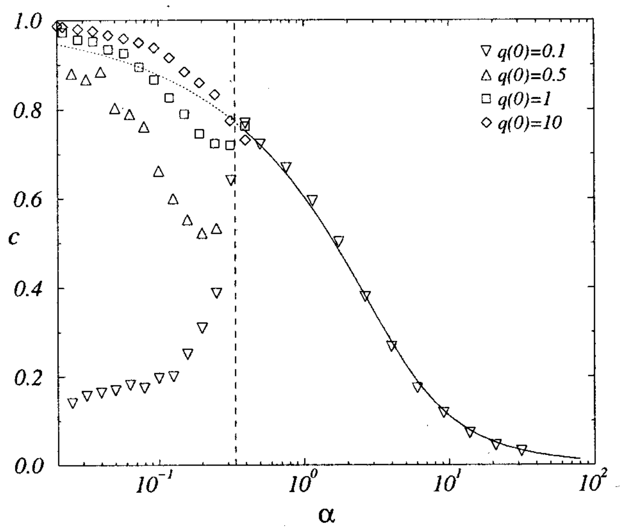

FIG. 1. Asymptotic average $c=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leq\tau}C(\tau)$ of the stationary covariance. The markers are obtained from individual simulation runs performed with a system of $N=4000$ agents and various homogeneous initial valuations [where $q_i(0)=q(0)$], and in excess of 1000 iteration steps. The solid curve to the right of the critical point is the theoretical prediction, given by the solution of Eq. (26). The dotted curve to the left is its continuation into the $\alpha<\alpha_c$ regime (where it should no longer be correct).

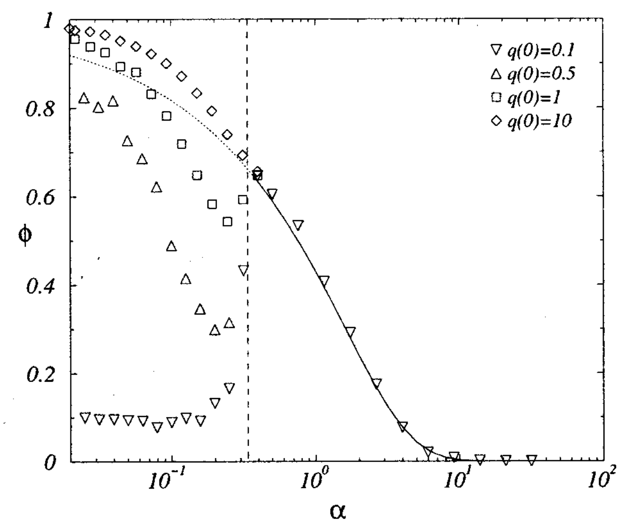

FIG. 2. Fraction $\phi=1-\mathrm{erf}[\sqrt{\alpha/2(1+c)}]$ of frozen agents in the stationary state. The markers are obtained from individual simulation runs performed with a system of $N=4000$ agents and various homogeneous initial conditions, where $q_i(0)=q(0)$, and in excess of 1000 iteration steps. The solid line to the right of the critical point is the theoretical prediction, obtained from the solution of Eq. (26). The dotted curve to the left is its continuation into the $\alpha<\alpha_c$ regime (where it should no longer be correct).

From this equation the value of c can be obtained numerically. For large α the solution behaves as $c\sim\alpha^{-1}$. In Figs. 1 and 2 we show the solution of Eq. (26) and the fraction ϕ of frozen agents, given according to the theory by $\phi=\langle\theta[|\eta|-\sqrt{\alpha}/(1+k)]\rangle=1-\mathrm{erf}[\sqrt{\alpha/2(1+c)}]$, as functions of α, together with the values for c and ϕ as obtained by carrying out numerical simulations of the minority game. One observes excellent agreement between theory and experiment above a critical value α_c, which we will calculate below.

From the time-averaged asymptotic correlation c we next move on to calculate the integrated response $k=\lim_{\tau\to\infty}\Sigma_{t\leq\tau}G(t)$. Since the occurrence of the Gaussian noise term $\eta(t)$ in Eq. (17) is (apart from a factor α) similar to that of an external field, we can write the response function as $G_{tt'}=\alpha^{-1/2}\langle\partial\,\mathrm{sgn}[q(t)]/\partial\eta(t')\rangle_\star$. Integration by parts in this expression generates

$$\langle\partial\,\mathrm{sgn}[q(t)]/\partial\eta(t')\rangle_\star=\sum_{t''}\Sigma^{-1}_{t't''}\langle\mathrm{sgn}[q(t)]\,\eta(t'')\rangle_\star$$

and hence,

$$\sqrt{\alpha}\sum_{t''}\langle\eta(t)\,\eta(t'')\rangle G^T_{t''t'}=\langle\mathrm{sgn}[q(t)]\,\eta(t')\rangle_\star. \tag{27}$$

Averaging over the two times t and t' now gives, in a stationary state, upon using again the assumption of absent anomalous response and the familiar notational conventions $s=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leq\tau}\mathrm{sgn}[q(t)]$ and $\eta=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leq\tau}\eta(t)$

$$\langle s\,\eta\rangle=\sqrt{\alpha}\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t'\leq\tau}\sum_{t''}\langle\eta\,\eta(t'')\rangle G^T_{t''t'}$$
$$=k\sqrt{\alpha}\langle\eta^2\rangle. \tag{28}$$

The variance $\langle\eta^2\rangle$ is given in Eq. (25). We calculate the remaining object $\langle s\,\eta\rangle$ similarly to our calculation of c, by distinguishing between frozen and nonfrozen agents and by using the two identities $s=\mathrm{sgn}[\eta]$ (for frozen agents) and $s=\eta(1+k)/\sqrt{\alpha}$ (for the nonfrozen ones), both of which follow immediately from Eq. (24). This results in

$$\langle s\,\eta\rangle=\left\langle\theta\left[|\eta|-\frac{\sqrt{\alpha}}{1+k}\right]|\eta|\right\rangle+\left\langle\theta\left[\frac{\sqrt{\alpha}}{1+k}-|\eta|\right]\frac{\eta^2(1+k)}{\sqrt{\alpha}}\right\rangle$$
$$=\frac{1+c}{(1+k)\sqrt{\alpha}}\,\mathrm{erf}\left[\sqrt{\frac{\alpha}{2(1+c)}}\right].$$

Insertion into Eq. (28), together with Eq. (25), then gives the desired expression for the integrated response:

$$\frac{1}{k}=\frac{\alpha}{\mathrm{erf}[\sqrt{\alpha/2(1+c)}]}-1 \tag{29}$$

with the value of c to be determined by solving Eq. (26). Equivalently, using $\phi=1-\mathrm{erf}[\sqrt{\alpha/2(1+c)}]$, we get

$$k=\frac{1-\phi}{\alpha-1+\phi}. \tag{30}$$

The integrated response k is positive and finite, and hence our solution (based on this property) is exact, for $\alpha>\alpha_c$.

Here α_c is the point at which k diverges, which is found to happen when the fraction of fickle agents equals α. According to Eqs. (26) and (29), we can write α_c as $\alpha_c=\mathrm{erf}[x]$, where x is the solution of the transcendental equation

$$\mathrm{erf}[x]=2-\frac{1}{x\sqrt{\pi}}e^{-x^2}. \tag{31}$$

This equation is identical to that derived in [4] (for a stochastic version of the game) using replica calculations. The resulting value is $\alpha_c\approx 0.337\,40$. Below α_c there might well be multiple ergodic components, i.e., more than one stationary solution of our fundamental order parameter equations (19).

VIII. STATIONARY VOLATILITY FOR $\alpha>\alpha_c$

In contrast to the persistent order parameter c and its relative k, the volatility matrix (5), to be calculated within our theory from expressions (18) and (20) and in a stationary state of the Toeplitz form $\Xi_{tt'}=\Xi(t-t')$, generally involves both long-term and short-term fluctuations. This becomes apparent when we work out $\Xi(t)$ using Eq. (18) and the results of Appendix B. We separate in the functions C and G the persistent from the nonpersistent terms, i.e., $C(t)=c+\tilde{C}(t)$ and $G(t)=\tilde{G}(t)$ (there is no persistent response for $\alpha>\alpha_c$), and find

$$2\Xi(t)=\frac{1+c}{(1+k)^2}+\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{u\leq\tau}\sum_{t't''}(\mathbb{1}+\tilde{G})^{-1}_{u+t\,t'}\tilde{C}_{t't''}\times(\mathbb{1}+\tilde{G}^T)^{-1}_{t''u}. \tag{32}$$

Clearly, the asymptotic (stationary) value of the volatility $\sigma^2=\Xi(0)$ cannot be expressed in terms of persistent order parameters only. It requires solving our coupled saddle-point equations (19) for $C_{tt'}$ and $G_{tt'}$ for large times but finite temporal separations $t-t'$. The persistent market correlations, however, are found to be expressible in terms of persistent order parameters:

$$\Xi(\infty)=\frac{1+c}{2(1+k)^2}. \tag{33}$$

Above α_c, this quantity can be recognized as the "energy" per agent H/N used in the replica calculations [4]. In order to find the volatility we separate the correlations at stationarity into a frozen and a fickle contribution:

$$C(t-t')=\phi\langle\mathrm{sgn}[\tilde{q}(t)\tilde{q}q(t')_{\mathrm{fr}}+(1-\phi)\langle\mathrm{sgn}[\tilde{q}(t)\tilde{q}(t')]\rangle_{\mathrm{fi}}$$
$$=\phi+(1-\phi)\langle\mathrm{sgn}[\tilde{q}(t)]\mathrm{sgn}[\tilde{q}(t')]\rangle_{\mathrm{fi}}$$

and hence

$$\tilde{C}(t-t')=\phi-c+(1-\phi)\langle\mathrm{sgn}[\tilde{q}(t)]\mathrm{sgn}[\tilde{q}(t')]\rangle_{\mathrm{fi}}.$$

Insertion into Eq. (32) and putting $t=0$ then gives

$$2\sigma^2=\frac{1+\phi}{(1+k)^2}+(1-\phi)\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t\leq\tau}\sum_{t't''}(\mathbb{1}+\tilde{G})^{-1}_{tt'}\times\langle\mathrm{sgn}[\tilde{q}(t')]\mathrm{sgn}[\tilde{q}(t'')]\rangle_{\mathrm{fi}}(\mathbb{1}+\tilde{G}^T)^{-1}_{t''t}$$
$$=\frac{1+\phi}{(1+k)^2}+(1-\phi)\lim_{\tau\to\infty}\frac{1}{\tau}\times\sum_{t\leq\tau}\left\langle\left\{\sum_{t'\leq t}(\mathbb{1}+\tilde{G})^{-1}_{tt'}\mathrm{sgn}[\tilde{q}(t')]\right\}^2\right\rangle_{\mathrm{fi}}. \tag{34}$$

We note that the sum $\Sigma_{t'<t}(\mathbb{1}+\tilde{G})^{-1}_{tt'}\mathrm{sgn}[\tilde{q}(t')]$ is the retarded self-interaction term in Eq. (17). Such a term is a familiar ingredient of disordered systems with "glassy" dynamics (see, e.g., [20]), and generally acts as the mechanism that drives the system to a frozen state. Hence, self-consistency of the distinction between frozen and fickle traders dictates that the retarded self-interaction term can be large for frozen traders, but must be small (if not absent) for fickle ones. Our approximation now consists in consequently disregarding the retarded self-interaction for the fickle traders:

$$\sum_{t'<t}(\mathbb{1}+\tilde{G})^{-1}_{tt'}\mathrm{sgn}[\tilde{q}(t')]\approx 0\quad\text{for }|\eta|<\frac{\sqrt{\alpha}}{1+k}.$$

Thus we retain for fickle traders only the instantaneous $t'=t$ term in $\Sigma_{t'\leq t}(\mathbb{1}+\tilde{G})^{-1}_{tt'}\mathrm{sgn}[\tilde{q}(t')]$, and find the (exact) expression (34) being replaced by the approximation

$$\sigma^2=\frac{1+\phi}{2(1+k)^2}+\frac{1}{2}(1-\phi). \tag{35}$$

This turns out to be a surprisingly accurate approximation of the volatility for $\alpha>\alpha_c$, as can be observed in Fig. 3.

Only in the limit $\alpha\to\infty$ can we expect to be able to go beyond Eqs. (33) and (35), and work out expressions (32) and (34) exactly. This requires calculating the response function $\tilde{G}(\tau)$ for small τ, which we will set out to do next. Since we assume absent anomalous response we may choose trivial initial conditions. We also choose the perturbation fields $\theta(t)$ to be nonzero only for a given time $t-\tau$, where $\tau>0$. From Eq. (17) we now derive

$$\mathrm{sgn}[q(t)]=\mathrm{sgn}\left[\frac{\theta(t-\tau)}{t\sqrt{\alpha}}+\frac{1}{t}\sum_{t'\leq t}\eta(t')-\frac{\sqrt{\alpha}}{t}\sum_{t't''\leq t}(\mathbb{1}+G)^{-1}_{t't''}\mathrm{sgn}[q(t'')]\right]. \tag{36}$$

Hence, for vanishingly small perturbations $\theta(t-\tau)$, and upon taking the $t\to\infty$ limit,

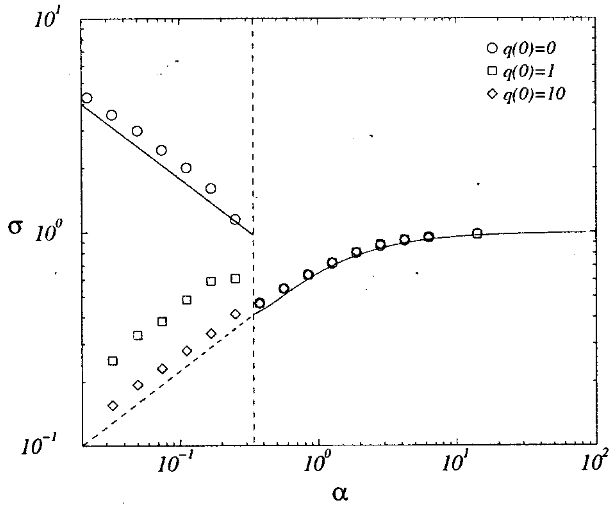

FIG. 3. The volatility σ as a function of the relative number $\alpha = p/N$ of possible values for the external information. The markers are obtained from individual simulation runs performed with a system of $N=4000$ agents and various homogeneous initial conditions, where $q_i(0)=q(0)$, and in excess of 1000 iteration steps. The solid curve for $\alpha>\alpha_c$ is the approximate expression (35). Below α_c the approximate asymptotic solutions of Eqs. (61) (solid) and (62) (dashed) are drawn.

$$\tilde{G}(\tau) = -\frac{2\sqrt{\alpha}}{1+k}\lim_{t\to\infty}\frac{1}{t}\sum_{t'\leq t}\left\langle \delta\left[\eta - \frac{s\sqrt{\alpha}}{1+k}\right]\left[\frac{\partial\,\mathrm{sgn}[q(t')]}{\partial\theta(t'-\tau)}\right]\right\rangle + 2\left\langle \delta\left[\eta - \frac{s\sqrt{\alpha}}{1+k}\right]\left[\lim_{t\to\infty}\frac{1}{t}\sum_{t'\leq t}\frac{\partial\eta(t')}{\partial\theta(t-\tau)}\right]\right\rangle.$$

We observe that $\eta = s\sqrt{\alpha}/(1+k)$ is precisely the condition for a trader to be fickle, in the language of the effective single agent. Secondly, from causality it follows that $\lim_{t\to\infty} t^{-1}\sum_{t'\leq t}\partial\eta(t')/\partial\theta(t-\tau) = \lim_{t\to\infty} t^{-1}\sum_{t'=t-\tau+1}^{t} \times \partial\eta(t')/\partial\theta(t-\tau)=0$. Hence our result can in a stationary state be written as

$$\tilde{G}(\tau) = -\frac{2\sqrt{\alpha}(1-\phi)}{1+k}\lim_{t\to\infty}\left\langle \frac{\partial\,\mathrm{sgn}[q(t)]}{\partial\theta(t-\tau)}\right\rangle_{\mathrm{fi}}. \quad (37)$$

For $\alpha\to\infty$ our stationary order parameter equations give $(1-\phi)/(1+k)\to 1$. Furthermore, for $\alpha\to\infty$ all traders will become fickle, so $\langle \partial\,\mathrm{sgn}[q(t)]/\partial\theta(t-\tau)\rangle_{\mathrm{fi}} \to \tilde{G}(\tau)$. This leaves for $\alpha\to\infty$ only the trivial solution for Eq. (37): $\lim_{\alpha\to\infty}\tilde{G}(\tau)=0$ for all τ. Insertion into our exact expression (32) for the stationary volatility matrix gives

$$\lim_{\alpha\to\infty}\Xi(t) = \frac{1}{2}+\frac{1}{2}\lim_{\alpha\to\infty}\tilde{C}(t)$$

and hence

$$\lim_{\alpha\to\infty}\lim_{t\to\infty}\sigma = 1. \quad (38)$$

This is the random trading limit.

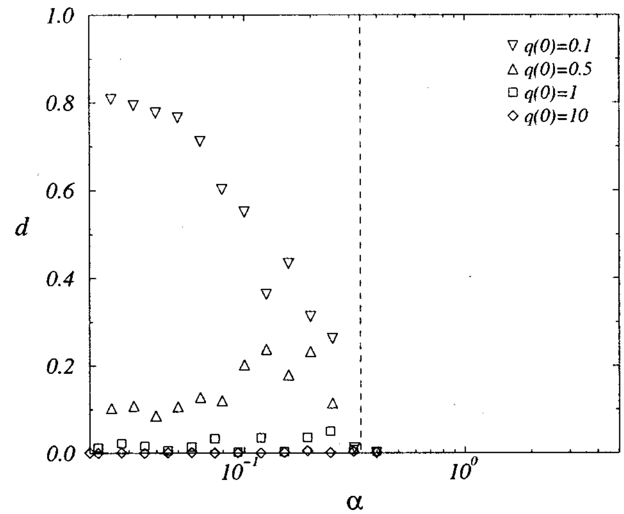

FIG. 4. The oscillatory component d of the covariance [see Eq. (40)]. The markers represent the results of individual simulations, performed with $N=4000$ agents and various homogeneous initial conditions, where $q_i(0)=q(0)$, and after in excess of 1000 iteration steps.

IX. THE STATIONARY STATE FOR $\alpha<\alpha_c$

When the amount of external information available for agents to base their actions upon (i.e., the value of α) becomes small, the behavior of the market is found to become strongly dependent on initial conditions. Numerical simulations show that below α_c the sequence $\Sigma_{t'}G_{tt'}$ is unbounded, and that within the limits of experimental accuracy:

$$\lim_{t\to\infty}\sum_{t'}(\mathbb{1}+G)^{-1}_{tt'}=0, \quad (39)$$

$$C_{t+\tau,t} = c + d(-1)^{\tau} \quad \text{for} \quad \tau\neq 0 \quad (40)$$

(with $C_{tt}=1$, by definition). Figure 4 shows the asymptotic values of d as measured during numerical simulations, for different values of α and $q(0)$. One clearly observes the dependence on initial conditions, as already seen in e.g., simulations of Ref. [13].

We will now use Eqs. (39) and (40) as *ansätze*, i.e., we will construct special self-consistent stationary state solutions of the fundamental order parameter equations (19) which obey Eqs. (39) and (40), as well as the stationary state conditions $C_{tt'}=C(t-t')$ and $G_{tt'}=G(t-t')$. First we analyze the statistical properties of the Gaussian noise $\eta(t)$ in the single-agent equation (17). From Eqs. (39) and (40) it follows that the noise covariance matrix (18) obeys

$$\lim_{t\to\infty}\langle \eta(t+\tau)\eta(t)\rangle = (-1)^{\tau}d\gamma^2 + (1-c-d) \times \sum_t (\mathbb{1}+G)^{-1}(t+\tau)(\mathbb{1}+G)^{-1}(t), \quad (41)$$

in which

$$\gamma = \sum_t (\mathbb{1}+G)^{-1}(t)(-1)^t. \quad (42)$$

From Eq. (41) one can derive, in turn, that the noise variables must asymptotically take the form

$$\eta(t)=(-1)^t\gamma z\sqrt{d}+\xi(t)\sqrt{1-c-d},\quad t\to\infty, \tag{43}$$

where z and $\{\xi(t)\}$ are zero-average Gaussian variables, with $\langle z^2\rangle=1$, $\langle z\xi(t)\rangle=0$, and

$$\lim_{t\to\infty}\langle\xi(t+\tau)\xi(t)\rangle=\sum_t (\mathbb{1}+G)^{-1}(t+\tau)(\mathbb{1}+G)^{-1}(t).$$

From Eq. (39) we know that $\lim_{\tau\to\infty}\lim_{t\to\infty}\langle\xi(t+\tau)\xi(t)\rangle=0$, i.e., in the stationary state the $\xi(t)$ decorrelate for large temporal separations. For sufficiently large t, and without external perturbations, Eq. (17) now acquires the form

$$q(t+1)=q(t)+\gamma z\sqrt{\alpha d}(-1)^t+\xi(t)\sqrt{\alpha(1-c-d)}$$
$$-\alpha\sum_{t'\leq t}(\mathbb{1}+G)^{-1}_{tt'}\,\text{sgn}[q(t')]. \tag{44}$$

Frozen agents are those for which sgn$[q(t)]$ is independent of time; due to Eq. (39) these will not experience the last term in Eq. (44). However, due to the properties of the noise in the $\alpha<\alpha_c$ regime (and in contrast to the situation with $\alpha>\alpha_c$), even frozen agents will now have $\lim_{t\to\infty}q(t)/t=0$. Insertion into Eq. (44) shows that frozen solutions of the following form exist:

$$q(t)=q-\tfrac{1}{2}\gamma z\sqrt{\alpha d}(-1)^t \tag{45}$$

provided sgn$[q(t)]$=sgn$[q]$ for all t, so q and d must obey

$$d=1-c,\quad |q|>|\tfrac{1}{2}\gamma z\sqrt{\alpha d}|. \tag{46}$$

Oscillating agents, on the other hand, are those for which sgn$[q(t)]=\hat{\sigma}(-1)^t$, with $\hat{\sigma}=\pm1$. Insertion into Eq. (44) shows that oscillating solutions of the following form exist:

$$q(t)=q+\tfrac{1}{2}\gamma\hat{\sigma}[\alpha-z\hat{\sigma}\sqrt{\alpha d}](-1)^t \tag{47}$$

provided sgn$[q(t+1)]=-$sgn$[q(t)]$ for all t, so q and d must obey

$$d=1-c,\quad \gamma[\alpha-z\hat{\sigma}\sqrt{\alpha d}]>0,\quad |q|<\tfrac{1}{2}\gamma[\alpha-z\hat{\sigma}\sqrt{\alpha d}]. \tag{48}$$

Note that, if rigorously frozen and/or rigorously oscillating agents were asymptotic solutions of Eq. (44), then the correlations would come out as $C(\tau)=\phi+(1-\phi)(-1)^\tau$ (with ϕ, as before, denoting the fraction of frozen agents), and we would find $c+d=1$. Figures 1 and 4, however, show that this simple relation holds only near $\alpha=0$. Away from $\alpha=0$ there will therefore be solutions describing fickle agents that change strategy at intervals intermediate between 1 (oscillating) and infinity (frozen). This can be understood on the basis of Eq. (44), where due to the noise term $\xi(t)$ (with a finite temporal correlation length) there will for $c+d<1$ always be a nonzero probability of nearly frozen agents changing strategy occasionally, and of nearly oscillating agents not changing strategy occasionally.

X. THE LIMIT $\alpha\to 0$

Let us finally investigate the situation near $\alpha=0$ more closely, where we may use the experimental observation that $c+d\approx1$, which implies that all agents will be either frozen or oscillating. We put $c=\phi$ (the fraction of frozen agents) and $d=1-\phi$, and choose homogeneous initial conditions with $q(0)>0$. We now find $\eta(t)=(-1)^t\gamma z\sqrt{(1-\phi)}$ and our two solution types are given by

$$q(t)=\begin{cases} q-\tfrac{1}{2}\gamma z\sqrt{\alpha(1-\phi)}(-1)^t, & \text{frozen,}\\ |q|<\tfrac{1}{2}\gamma[\alpha-z\hat{\sigma}\sqrt{\alpha(1-\phi)}], & \text{oscillating,}\end{cases}$$

provided the following conditions for existence are met:

$$|q|>|\tfrac{1}{2}\gamma z\sqrt{\alpha(1-\phi)}|,\quad\text{frozen,}$$
$$|q|<\tfrac{1}{2}\gamma[\alpha-z\hat{\sigma}\sqrt{\alpha(1-\phi)}],\quad\text{oscillating,} \tag{49}$$
$$\gamma\sqrt{\alpha}>\gamma z\hat{\sigma}\sqrt{1-\phi}. \tag{50}$$

Near $\alpha=0$ we also know, due to $c+d=1$, that

$$\langle\eta(t+\tau)\eta(t)\rangle=(-1)^\tau(1-\phi)\gamma^2,\quad t\to\infty, \tag{51}$$

$$\eta(t)=(-1)^t\gamma z\sqrt{1-\phi},\quad t\to\infty, \tag{52}$$

and that $\lim_{t\to\infty}\sigma^2=\tfrac{1}{2}(1-\phi)\gamma^2$. In order to eliminate the remaining parameters γ and ϕ we note that time translation invariance guarantees the validity of the relation $\Sigma_t(G^n)(t)(-1)^t=[\Sigma_t G(t)(-1)^t]^n$, and hence

$$\gamma=(1+\Gamma)^{-1},\quad \Gamma=\sum_t G(t)(-1)^t. \tag{53}$$

The quantity Γ can, in turn, be expressed in terms of γ upon inserting Eqs. (51) and (52) into Eq. (27). We obtain

$$\sqrt{\alpha}(1-\phi)\gamma(1-\gamma)(-1)^\tau=\lim_{t\to\infty}\langle\text{sgn}[q(t+\tau)]\eta(t)\rangle_\star.$$

Working out the average on the right-hand side, by separating frozen from fickle solutions, gives for large t

$$\langle\text{sgn}[q(t+\tau)]\eta(t)\rangle_\star=\phi\langle\text{sgn}[q(t+\tau)]\eta(t)\rangle_{\text{fr}}+(1-\phi)\times\langle\text{sgn}[q(t+\tau)]\eta(t)\rangle_{\text{fi}}$$
$$=\gamma\sqrt{(1-\phi)}(-1)^\tau\{\phi(-1)^t\times\langle\text{sgn}[q]z\rangle_{\text{fr}}+(1-\phi)\langle\hat{\sigma}z\rangle_{\text{fi}}\}.$$

Since in a stationary state the correlation function $\langle\text{sgn}[q(t)]\eta(t')\rangle_\star$ can only depend on $t-t'$, we must conclude that $\langle\text{sgn}[q]z\rangle_{\text{fr}}=0$ and that either

$$\lim_{\alpha\to0}\gamma(1-\phi)=0\quad\text{or}\quad\gamma=1-\sqrt{(1-\phi)/\alpha}\langle\hat{\sigma}z\rangle_{\text{fi}} \tag{54}$$

(in leading order for $\alpha\to0$). Multiplication of both sides of the second equation in (54) by $\gamma\sqrt{\alpha}$ shows that it automatically ensures the validity of the second condition of Eq. (50). The first equation of (54) will satisfy the second condition of Eq. (50) as long as $\gamma>0$.

In order to proceed we need to calculate the persistent term q in the proposed solutions, which can be seen as representing their effective initial conditions. It incorporates both the true initial conditions and the effects of the transients of the dynamics, which initially will not be of the simple form (44). Exact evaluation would require solving our order parameter equations for arbitrary times, which is not feasible. However, one can proceed for now on the basis of the postulate that the properties of the long-term attractors (viz., the Gaussian variable z) are uncorrelated with the value of q. The conditions (49) and (50) then simply state whether a value of q, generated independently of z according to some distribution $P(q)$, is compatible with a given attractor. Although we will not be able to generate all possible stationary solutions of the process (17), we will show how two qualitatively different solutions, one with a diverging volatility for $\alpha\to0$ and one with a vanishing volatility for $\alpha\to0$, can both be extracted from our equations.

The first type of solution is obtained for $\lim_{\alpha\to0}\phi=\phi_0<1$. Now one finds, in leading order in α, that $\hat{\sigma}=-\mathrm{sgn}[\gamma z]$ and that $\gamma=\langle|z|\rangle_{\mathrm{fi}}\sqrt{(1-\phi_0)/\alpha}$. The conditions (49) and (50) reduce in leading order to the complementary pair

$$|q|>\tfrac{1}{2}\gamma|z|\sqrt{\alpha(1-\phi_0)},\quad \text{frozen}, \tag{55}$$

$$|q|<\tfrac{1}{2}\gamma|z|\sqrt{\alpha(1-\phi_0)},\quad \text{oscillating}. \tag{56}$$

This, in turn, allows us to calculate ϕ_0 and $\langle|z|\rangle_{\mathrm{fi}}$:

$$\phi_0=\int dq\,P(q)\int\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\theta[|q|-\tfrac{1}{2}\gamma|z|\sqrt{\alpha(1-\phi)}]$$

$$=\int dq\,P(q)\mathrm{erf}\left[\frac{\sqrt{2}|q|}{\gamma\sqrt{\alpha(1-\phi)}}\right],$$

$$\langle|z|\rangle_{\mathrm{fi}}=\int\frac{dq\,P(q)}{1-\phi_0}\int\frac{dz|z|}{\sqrt{2\pi}}e^{-z^2/2}$$

$$\times\theta[\tfrac{1}{2}\gamma|z|\sqrt{\alpha(1-\phi_0)}-|q|]$$

$$=\frac{\sqrt{2}}{(1-\phi_0)\sqrt{\pi}}\int dq\,P(q)e^{-2q^2/\gamma^2\alpha(1-\phi_0)}.$$

We eliminate γ in favor of $\sigma=\frac{1}{2}\sqrt{2}\gamma\sqrt{1-\phi_0}$ and end up with the following simple closed equation for σ:

$$\sigma=\int dq\,P(q)\frac{e^{-q^2/\sigma^2\alpha}}{\sqrt{\alpha\pi}}. \tag{57}$$

The associated value for ϕ_0 then follows from

$$\phi_0=\int dq\,P(q)\mathrm{erf}\left[\frac{|q|}{\sigma\sqrt{\alpha}}\right]. \tag{58}$$

Finally, we can use our observations regarding the first few time steps (Sec. VI) of the process in order to obtain an estimate for $P(q)$. These showed for small α that initially (i) for small $|q(0)|=O(\sqrt{\alpha})$ the system is driven toward the oscillating state, (ii) for large $|q(0)|=O(\alpha^0)$ the system tends to freeze, (iii) the transient processes are dominated by the (Gaussian) noise term in Eq. (17), and (iv) the noise term is automatically being ''amplified'' (either via a diverging response function, or via accumulation over time) to an effective $O(\alpha^0)$ contribution. Note that (i) and (ii) confirm that q can indeed be seen as the sum of $q(0)$ and the net effect of the transient processes, and that (iii) and (iv) subsequently suggest representing the transient processes by adding a single effective Gaussian variable. Hence for small α it would appear sensible to write $P(q)=(\Lambda\sqrt{2\pi})^{-1}e^{-[q-q(0)]^2/2\Lambda^2}$, which converts Eqs. (57) and (58) into

$$\sigma^2\alpha+2\Lambda^2=\frac{1}{\pi}e^{-2q^2(0)/(\sigma^2\alpha+2\Lambda^2)}.$$

We conclude that σ can be written in terms of the solution y of a transcendental equation

$$\sigma=\frac{1}{\sqrt{\alpha}}\left[\frac{2q^2(0)}{y}-2\Lambda^2\right]^{1/2},\quad 2q^2(0)=\frac{y}{\pi}e^{-y}. \tag{59}$$

For $|q(0)|\to0$ we find that $\sigma=(\alpha\pi)^{-1/2}\sqrt{1-2\pi\Lambda^2}$; hence we must obviously require $\Lambda^2<1/2\pi$. The associated value for ϕ_0 then follows from

$$\phi_0=\int Dx\,\mathrm{erf}\left[\frac{|q(0)+\Lambda x|}{\sigma\sqrt{\alpha}}\right]. \tag{60}$$

Since we cannot calculate or estimate the width Λ of the effective Gaussian noise term without solving our order parameter equations for short times [Λ could even depend on $q(0)$], it is quite satisfactory that several interesting properties of the solution are found to be independent of Λ. For instance, one always finds a diverging volatility of the form $\sigma=O(\alpha^{-1/2})$, and there is a critical value $q_c=(2\pi e)^{-1/2}\approx0.242$ such that for $|q(0)|>q_c$ the solution no longer exists. This solution is clearly the type of volatile state that has been reported regularly (see, e.g., [8,9]) upon observing numerical simulations. We have now found, however, that whether or not it will appear depends critically on the choice made for the initial conditions. Numerical simulations indeed appear to support the existence and predicted magnitude of a critical value $q_c\approx0.242$ (see Fig. 5); fully conclusive experiments, however (with even smaller values of α), would require impractical amounts of CPU time and/or memory in order to meet the requirements $p\to\infty$ and $N\to\infty$ for increasingly small values of α, and are presently ruled out. In the limit $q(0)\to0$ one can easily carry out the integrals in Eq. (60), giving $\Lambda=(2\pi)^{-1/2}\sin[\frac{1}{2}\pi\phi_0]$. Elimination of Λ via insertion into $\sigma=(\alpha\pi)^{-1/2}\sqrt{1-2\pi\Lambda^2}$ then leads to the simple relation

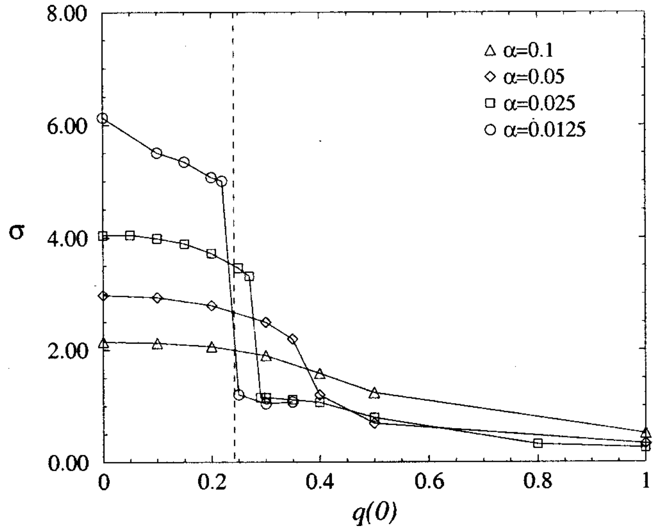

FIG. 5. Experimental evidence in support of the existence of a critical value for the initial strategy valuation $q(0)$ below which a high-volatility solution exists. The connected markers represent the results of measuring the volatility in individual simulations, performed with $N=4000$ agents and initial conditions where $q_i(0)=q(0)$, and after in excess of 1000 iteration steps. CPU time and memory limitations prevent us from doing reliable and conclusive experiments for $\alpha<0.0125$; the available data, however, are clearly not in conflict with our theoretical prediction $q_c\approx 0.242$ (vertical dashed line), which follows from Eq. (59).

$$\sigma=\frac{\cos[\frac{1}{2}\pi\phi_0]}{\sqrt{\alpha\pi}}+O(\alpha^0),\quad \alpha,q(0)\to 0. \tag{61}$$

This is the high-volatility solution shown in the $\alpha<\alpha_c$ regime of Fig. 3, with ϕ_0 as measured in simulations (see, e.g., Fig. 2). The power of α in Eq. (61) is observed to be correct. The observed difference between theory and experiment with regard to the prefactor can be understood as a reflection of our approximation $c+d\approx 1$; this amounts to disregarding deviations from the idealized purely frozen or purely oscillating behavior, which can indeed be expected to give an approximate theory that (even for small α) slightly underestimates the volatility.

We note that the condition $\lim_{\alpha\to 0}\phi<1$ for the above reasoning to apply can in fact be weakened to $\lim_{\alpha\to 0}\alpha/(1-\phi)=0$. The above solution ceases to hold, however, at the point where the fraction ϕ of frozen agents scales as $\phi=1-\kappa\alpha+O(\alpha^2)$, in which case we have to turn to the first option in Eq. (54), rather than the second. This is consistent with our previous observation that small values of $|q(0)|$ lead to a relatively small fraction of frozen agents (and a large volatility), whereas for large $|q(0)|$ such a solution will break down in favor of states with a larger fraction of frozen agents. Since we can now no longer use the second equation in (54) to determine γ and hence find the volatility $\sigma=\frac{1}{2}\sqrt{2}\gamma\sqrt{1-\phi}$, we have to return to Eq. (53). A fully frozen state, which for $\alpha\to 0$ will indeed be described by this second type of solution (since $\lim_{\alpha\to 0}\phi=1$), must necessarily have $G(t>0)=g$. This is consistent with our *ansätze*, since it gives

$$(\mathbb{1}+G)^{-1}(t)=-g(1-g)^{t-1},\quad t>0,$$

which implies $\Sigma_{t\geq 0}(\mathbb{1}+G)^{-1}(t)=0$, provided $0<g<2$. We can now calculate γ from Eq. (53) and find $\lim_{\alpha\to 0}\gamma=2/(2-g)$. Thus we obtain, provided $2-g=O(\alpha^0)$,

$$\sigma=\frac{\sqrt{2\kappa}}{2-g}\sqrt{\alpha}+O(\alpha),\quad \kappa=\lim_{\alpha\to 0}\frac{1-\phi}{\alpha}.$$

We also note that the scaling property $\phi=1-O(\alpha)$ implies that $P(0)=\lim_{q\to 0}P(q)=O(\sqrt{\alpha})$, since all q values of order $q=O(\sqrt{\alpha})$ will contribute to the fraction $1-\phi$ of fickle agents, giving $1-\phi=O(P(0)\sqrt{\alpha})$. We can now calculate $\lim_{\alpha\to 0}g$ upon explicitly inspecting the effect of a perturbation of a frozen state. Since $G(t>0)=g$ we may restrict ourselves to considering the effect on sgn$[q(t+1)]$ of a perturbation at time t, giving in leading order for $\alpha\to 0$

$$\begin{aligned}\lim_{\alpha\to 0} g&=\lim_{\alpha\to 0}\lim_{\theta\to 0}\left\langle\frac{\partial}{\partial\theta}\,\mathrm{sgn}[q+\tfrac{1}{2}\alpha\gamma z\sqrt{\kappa}(-1)^t+\theta]\right\rangle\\&=2\lim_{\alpha\to 0}\langle\delta[q+\tfrac{1}{2}\alpha\gamma z\sqrt{\kappa}(-1)^t]\rangle\\&=2\lim_{\alpha\to 0}P(0)=0.\end{aligned}$$

Hence, since the frozen state has $q=O(\alpha^0)$, we find $\lim_{\alpha\to 0}\gamma=1$ and

$$\sigma=\tfrac{1}{2}\sqrt{2\kappa\alpha}+O(\alpha),\quad \alpha\to 0. \tag{62}$$

Explicit calculation of the prefactor in Eq. (62) would require taking our calculations beyond the leading order in α, in order to find κ. Equation (62) is the low-volatility solution shown in the $\alpha<\alpha_c$ regime of Fig. 3, with κ as measured in simulations (see, e.g., Fig. 6). Again the power of α in Eq. (62) is observed to be correct. The remaining difference between theory and experiment with regard to the prefactor can again be understood as a reflection of our approximation $c+d\approx 1$, which induces a structural underestimation of the volatility.

XI. DISCUSSION

In this paper we have solved a ''batch'' version of the minority game with random external information, using generating functional analysis (or dynamic mean field theory) as introduced by De Dominicis, which allows one to carry out the disorder averages in a dynamical context. Since the dynamics of the game is not described by a detailed balance type of stochastic process, equilibrium statistical mechanical tools cannot be applied directly. Phase transitions (if present) must be of a dynamical nature. The disorder in the minority game consists of the microscopic realization of the repertoire of randomly drawn trading strategies of the N agents. Upon taking the limit $N\to\infty$ one ends up with an exact non-Markovian stochastic equation describing the dynamics of an effective single agent (17), whose statistical properties are identical to those of the original system (averaged over all realizations of the disorder). The key control parameter in

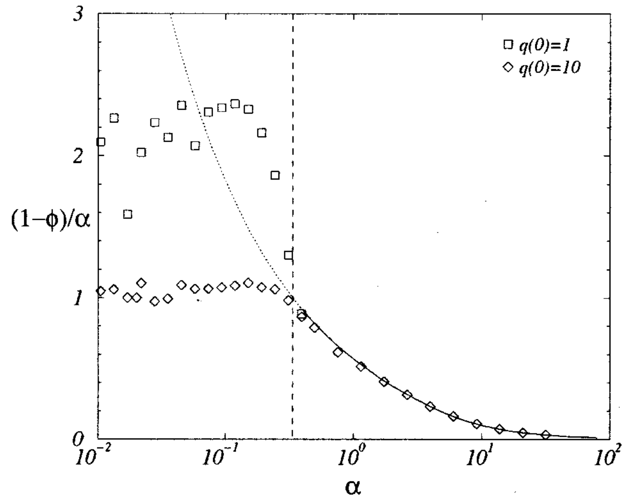

FIG. 6. Experimental evidence for the existence of the limit $\kappa = \lim_{\alpha\to 0}(1-\phi)/\alpha$ for the low-volatility solution. The markers are obtained from individual simulation runs performed with a system of $N=4000$ agents and initial valuations of the form $q_i(0)=q(0)>q_c$ (to evoke the low-volatility state), and in excess of 1000 iteration steps. The solid curve to the right of the critical point is the theoretical prediction, obtained from the exact equations (26) and $\phi=1-\mathrm{erf}[\sqrt{\alpha/2(1+c)}]$ describing the $\alpha>\alpha_c$ regime. The dotted curve to the left is its continuation into the $\alpha<\alpha_c$ regime (where it should indeed no longer be correct).

this problem is the ratio $\alpha=p/N$ of the number of possible values of the external information over the number of agents.

We find a phase transition at $\alpha_c\approx 0.33740$, signaled by the onset of anomalous response, in agreement with the value reported recently in [4]. The method used in [4] depends on the fact that for their stochastic version of the minority game a Lyapunov function exists. Our approach does not have this constraint and can be easily applied to those variations of the game where a Lyapunov function is not available, thus opening up a wider range of models for analysis (see, e.g., [3]). Above α_c (where anomalous response is absent) we can solve the stationary state of the system exactly, giving exact expressions for quantities such as the fraction of frozen agents (which is zero for $\alpha\to\infty$ but increases with decreasing α), the persistent two-time correlations, and the persistent correlations in the total bid. The volatility (which is itself not an order parameter of the system) can be calculated to a very good approximation. Above α_c, our method and that of [6,4] are likely to describe the same behavior [21]. Below α_c, i.e., in the region of complex dynamics (inaccessible by the replica approach [15]), our present method still applies. In this region we demonstrate the existence of multiple stationary states, and derive expressions for the relevant observables in leading order in α as $\alpha\to 0$. We show, more specifically, that the occurrence and practical observability of a diverging volatility for $\alpha\to 0$ (as reported in, e.g., [8,9]) is crucially dependent on the overall degree of *a priori* preference for specific strategies exhibited by the agents at $t=0$, which may explain the different observations regarding the $\alpha\to 0$ behavior that have been reported in the literature [13]. More specifically, our theory points at the existence of a critical value for the initial strategy valuations, above which the system will revert to a state with vanishing volatility. Our theoretical predictions find quite satisfactory confirmation in numerical simulations.

The fact that we can analyze the stationary state of Eq. (17), in spite of it describing a non-Markovian stochastic process, suggests that the present method should also be suitable to deal with models where the external information depends on time, or on the previous behavior of the agents, as in the original model [1,22].

ACKNOWLEDGMENTS

We would like to thank Andrea Cavagna and David Sherrington for introducing us to the minority games and Matteo Marsili for a discussion of previous work done. J.A.F.H. wishes to thank King's College London Association for financial support.

APPENDIX A: EXPRESSIONS FOR AVERAGE BID AND VOLATILITY

First we calculate $\lim_{N\to\infty}\overline{\langle A_t\rangle}$ using expression (3). We note that we obtain $\langle A_t\rangle$ simply by making the replacement $\exp[i\Sigma_{ti}\psi_i(t)q_i(t)]\to(\tau/\alpha N)\Sigma_\mu x_t^\mu$ in the right-hand side of Eq. (6). The disorder average is carried out as before, but instead of Eq. (7) we now obtain

$$\overline{\langle A_t\rangle}=\tau\int[DC\,D\hat{C}][DK\,D\hat{K}][DL\,D\hat{L}]\times e^{N[\Psi+\Phi+\Omega]+O(N^0)}e^{-\Phi/\alpha}\int Dw\,D\hat{w}\,Dx\,D\hat{x}\,x_t\times\exp\left(i\sum_s[\hat{w}_s w_s+\hat{x}_s x_s+w_s x_s]\right)\times\exp\left(-\frac{1}{2}\sum_{ss'}[w_s w_{s'}+\hat{w}_s L_{ss'}\hat{w}_{s'}+2\hat{x}_s K_{ss'}\hat{w}_{s'}+\hat{x}_s C_{ss'}\hat{x}_{s'}]\right),$$

where we have used permutation invariance with respect to μ (after the disorder average). The integral is dominated by the familiar saddle point. Since the $O(N^0)$ term in the exponent is identical to that in Eq. (7), we can now simply use the identity $\overline{Z[\mathbf{0}]}=1$ to show that

$$\lim_{N\to\infty}\overline{\langle A_t\rangle}=\tau e^{-\Phi/\alpha}\int Dw\,D\hat{w}\,Dx\,D\hat{x}\,x_t\times\exp\left(i\sum_s[\hat{w}_s w_s+\hat{x}_s x_s+w_s x_s]\right)\times\exp\left(-\frac{1}{2}\sum_{ss'}[w_s w_{s'}+2i\hat{x}_s G_{ss'}\hat{w}_{s'}+\hat{x}_s C_{ss'}\hat{x}_{s'}]\right)=0. \tag{A1}$$

The last step follows immediately from the antisymmetry of the integrand under overall reflection.

To determine the disorder-averaged volatility matrix, which for $N\to\infty$ becomes identical to $\overline{\langle A_t A_{t'}\rangle}$ due to Eq. (A1) and the self-averaging property, we first work out the dominant terms in Eq. (5). Using $\lim_{N\to\infty}(\alpha N)^{-1}\Sigma_\mu \Omega_\mu^2 = \frac{1}{2}$, we obtain the relatively simple expression

$$\lim_{N\to\infty}\langle A_t A_{t'}\rangle = \lim_{N\to\infty}\frac{1}{2\alpha N}\sum_\mu \langle [x_t^\mu + \Omega^\mu/\tau][x_{t'}^\mu + \Omega^\mu/\tau]\rangle .$$

We calculate this average by making the replacement $\exp[i\Sigma_{ti}\,\psi_i(t)q_i(t)]\to(2\alpha N)^{-1}\Sigma_\mu\langle[x_t^\mu+\Omega^\mu/\tau][x_{t'}^\mu+\Omega^\mu/\tau]\rangle$ on the right-hand side of Eq. (6). Repeated integration by parts over the w_t^μ shows that we may equivalently put $\exp[i\Sigma_{ti}\,\psi_i(t)q_i(t)]\to(2\alpha N)^{-1}\Sigma_\mu \hat{w}_t^\mu \hat{w}_{t'}^\mu$. Following the steps we also took in calculating $\overline{\langle A\rangle}$ now gives

$$\begin{aligned}\lim_{N\to\infty}\overline{\langle A_t A_{t'}\rangle} &= \frac{1}{2}e^{-\Phi/\alpha}\int Dw\,D\hat{w}\,Dx\,D\hat{x}\;\hat{w}_t\hat{w}_{t'}\exp\left(i\sum_s [\hat{w}_s w_s + \hat{x}_s x_s + w_s x_s]\right)\\ &\quad\times\exp\left(-\frac{1}{2}\sum_{ss'}[w_s w_{s'} + 2i\hat{x}_s G_{ss'}\hat{w}_{s'} + \hat{x}_s C_{ss'}\hat{x}_{s'}]\right)\\ &= \frac{1}{2}\frac{\int D\hat{w}\;\hat{w}_t\hat{w}_{t'}\exp\left(-\frac{1}{2}\Sigma_{ss'}\hat{w}_s[(\mathbb{1}+G)^T D^{-1}(\mathbb{1}+G)]_{ss'}\hat{w}_{s'}\right)}{\int D\hat{w}\exp\left(-\frac{1}{2}\Sigma_{ss'}[(\mathbb{1}+G)^T D^{-1}(\mathbb{1}+G)]_{ss'}\hat{w}_{s'}\right)}\\ &= \tfrac{1}{2}[(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^T)^{-1}]_{tt'}. \end{aligned} \tag{A2}$$

APPENDIX B: CONSEQUENCES OF ABSENCE OF ANOMALOUS RESPONSE

Lemma 1. Consider two bounded sequences of real numbers A_t and b_t. Because b_t is bounded, there exists a number b such that $\lim_{\tau\to\infty}(1/\tau)\Sigma_{t\leq\tau}b_t = b$. Define $a_\tau = \Sigma_{t\leq\tau}A_t$, and assume that $\lim_{\tau\to\infty}a_\tau = a$. Then

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t\leq\tau}\sum_{t'\leq t}A_{t-t'}b_{tt'} = ab.$$

Proof. Upon substituting $t\to t+t'$ we find

$$\frac{1}{\tau}\sum_{t\leq\tau}\sum_{t'\leq t}A_{t-t'}b_{t'} = \frac{1}{\tau}\sum_{t'\leq\tau}b_{t'}\sum_{t\leq\tau-t'}A_t = \frac{1}{\tau}\sum_{t\leq\tau}a_{\tau-t}b_t .$$

The sequences $\{a\}$ and $\{b\}$ are bounded, so there exist numbers C_a and C_b such that $|a_t|<C_a$ and $|b_t|<C_b$ for all $t\geq 0$. The sequence $\{a\}$ converges to a, so for any $\epsilon>0$ there exists a K such that for all $t>K$ $|a_t-a|<\epsilon/3C_b$. We now choose M such that for all $\tau>M$ $|(1/\tau)\Sigma_{t\leq\tau}b_t - b|<\epsilon/3|a|$ and $KC_aC_b/\tau<\epsilon/3$. Then we find for all $\tau>M$

$$\begin{aligned}&\left|\frac{1}{\tau}\sum_{t\leq\tau}a_{\tau-t}b_t - ab\right|\\ &= \left|\frac{1}{\tau}\sum_{t=\tau-K}^{\tau}a_{\tau-t}b_t + \sum_{t<\tau-K}a_{\tau-t}b_t - ab\right|\\ &\leq \left|\frac{1}{\tau}\sum_{t=\tau-K}^{\tau}a_{\tau-t}b_t\right| + \left|\frac{1}{\tau}\sum_{t<\tau-K}(a_{\tau-t}-a)b_t\right.\\ &\quad\left. -a\left(b-\frac{1}{\tau}\sum_{t<\tau-K}b_t\right)\right|\\ &\leq \frac{KC_aC_b}{\tau} + \left|\frac{1}{\tau}\sum_{t<\tau-N}(a_{\tau-t}-a)b_t\right|\\ &\quad + |a|\left|b-\frac{1}{\tau}\sum_{t<\tau-K}b_t\right| \leq \epsilon.\end{aligned}$$

Hence the limit is as claimed. ■

Lemma 2. Suppose $G_{st}=G(s-t)\in\mathfrak{R}$, where $G(t)=0$ for all $t<0$ and with $\lim_{\tau\to\infty}\Sigma_{t\leq\tau}G(t)=k$, and suppose $\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leq\tau}s(t)=s$. Then for all $n\in\mathbb{N}$

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{t'}(G^n)_{tt'}s(t') = k^n s.$$

Proof. The proof proceeds by induction. For $n=0$, the statement is trivially true. Suppose now that it is true for all $n \leqslant m$. Then

$$\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{t'} (G^{m+1})_{tt'}s(t')$$

$$=\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{t''\leqslant t} G(t-t'')\sum_{t'\leqslant t''} (G^{m})_{t''t'}s(t').$$

The sequence $b_t=\Sigma_{t'\leqslant t}(G^m)_{tt'}s(t')$ satisfies the conditions of Lemma 1, application of which gives

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}\sum_{t'} (G^{m+1})_{tt'}s(t')=kk^m s=k^{m+1}s.$$

Hence the claim holds for $m+1$, and by induction it is now proved for all n. ■

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