CM111A – Calculus I
Compact Lecture Notes

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1. Introduction

1.1. A bit of history ...

1.1.1. Birth of modern science and of calculus
Stage I, 1500–1630: from speculation to science ...

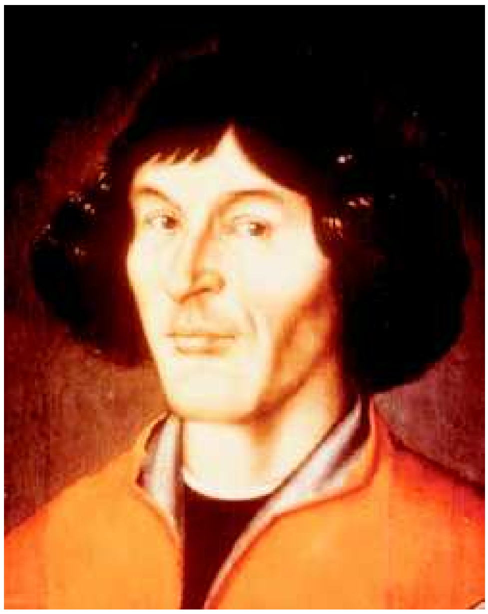
Ptolemy of Alexandria , 2nd century AD:

Style of the ancient Greeks: no experiments, just logical thought and elegance

published 'Almagest' (summary of astronomy, based on 500 years of Greek astronomical and cosmological thinking)

earth is centre of the universe
complicated model of spheres carrying heavenly bodies, moving themselves in circles

Nicolaus Copernicus , 1473–1543:



problems with the motion of the moon ...

published 'De Revolutionibus', sun-centred universe, with moon orbiting around the earth

Catholic Church:
put 'De Revolutionibus' on the Index of banned books
(stayed on the Index until 1835!)

Tycho Brahe , 1546–1601:

The genius observer...

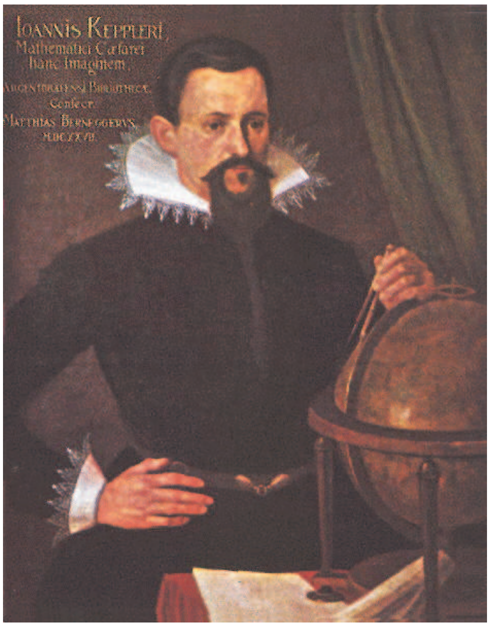
First systematic and comprehensive measurement of the trajectories of the moon, the planets, the comets, and the stars, over many years and with unrivaled precision!

Compiled *huge* amounts of data

Did *not* himself believe Copernicus' ideas ...

(lost his nose while a student in a duel in 1566)

Johannes Kepler , 1571–1630:



The genius in analyzing data ...

Believed Copernicus, but could not observe anything himself (poor eyesight ...)

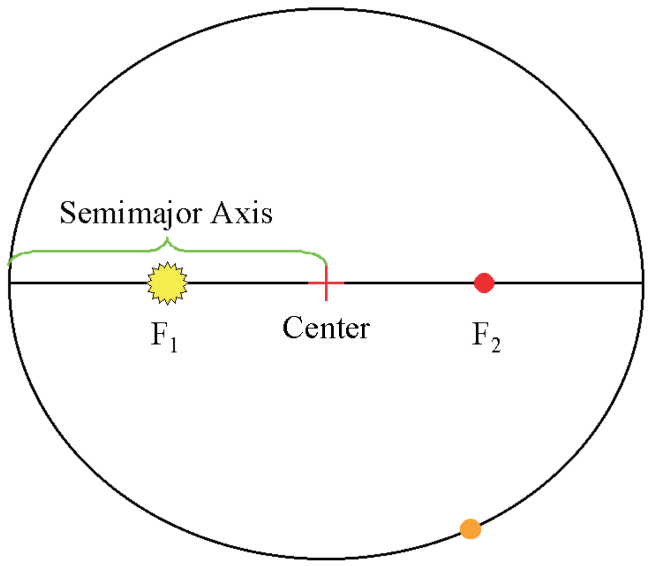
Developed further models of sun-centered universe, with spheres within spheres

Became Brahe's assistant in 1599, discovered *quantitative* laws, based on Tycho Brahe's data

published 'Astronomia Nova' in 1609, 'Harmonice Mundi' in 1619, 'Epitome of Copernican Astronomy' (3 volumes) 1618–1621

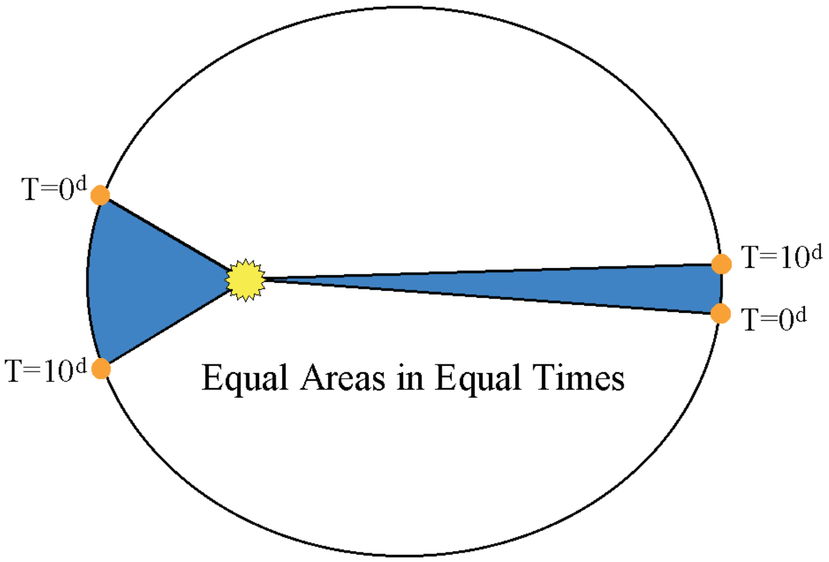
Kepler's First Law (1605):

the orbit of each planet is an ellipse, with the sun at one of the two foci



Kepler's Second Law (1602):

a line joining the sun to an orbiting planet sweeps out equal areas in equal times



Kepler's Third Law (1618):

the square of a planet's orbit period is proportional to cube of its distance to the sun

1.1.2. Birth of modern science and of calculus
Stage II, 1630–1680: science is written in the language of mathematics!

Galileo Galilei , 1564–1642:

'the wrangler', loved arguments ...

the first real scientist:
(i) state a hypothesis,
(ii) devise an experiment to test it,
(iii) carry out the experiment,
(iv) accept or reject the hypothesis

always worried about money (sisters' dowries ...)
worked on inventions to get rich
(thermometer, calculator)

interested in movement of objects
constructed improved telescope in 1609: new observations all supported Copernicus ...

published 'Dialogue on the Two Chief World Systems' in 1632
(Salviati vs Simplicio, with Sagredo as impartial commentator)

it was suggested that Pope Urban VIII was the 'simpleton' ...
1633: show trial by the Inquisition, Galileo (69 and fearing torture):
'I abjure, curse and detest my errors'

published 'Discourses and Mathematical Demonstrations Concerning Two New Sciences'
(first modern scientific textbook), smuggled out of Italy, published in 1638

René Descartes , 1596–1650:

1637: 'Discours de la Méthode pour bien conduire
la raison et chercher la Vérité dans les Sciences'

invented 'Cartesian coordinates':
each position in space represented by three numbers

introduced letters x, y, z to denote
unknown quantities in mathematical problems

published 'Principia Philosophiae' (1644)

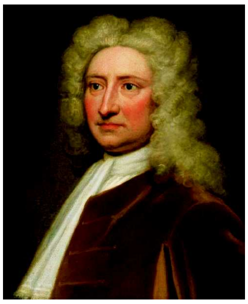
1.1.3. Birth of modern science and of calculus
Stage III, around 1680: how to speak the language of mathematics!

(i) state a hypothesis,
(ii) devise an experiment to test it,
(iii) carry out the experiment,
(iv) accept or reject the hypothesis

The problem in making it work in practice:
To test hypotheses on forces and movements of objects, one needs to be able to *calculate the trajectories* that would be caused by the assumed forces ...

Huygens

Hooke



Halley

Wren

1673
Christiaan Huygens: outward force on object in circular orbit of radius R is proportional to R^{-2}

1674
Robert Hooke: object that feels no force will move along a straight line (Newton's first law of motion ...)

1684 at the Royal Society ...
Edmond Halley, Christopher Wren, Robert Hooke

hypothesis: the sun attracts planets at distance R with a force proportional to R^{-2}

Is it possible to derive the observed motion of the planets from this inverse square law?

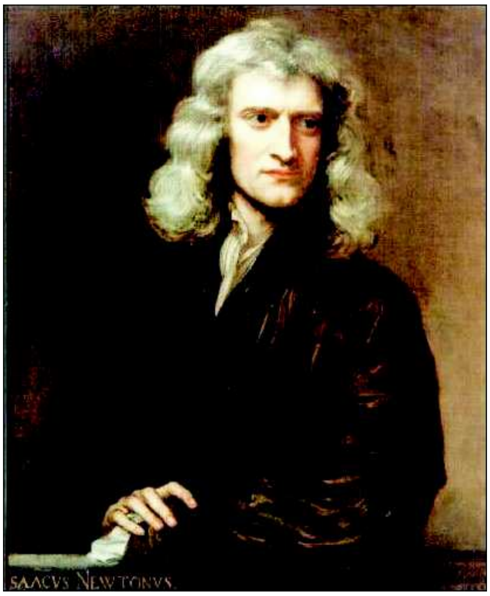
1684 somewhat later ... Halley visits Isaac Newton

according to Newton's friend De Moivre:

'Dr Halley came to visit him in Cambridge, after they had been some time together the Dr asked him what he thought the Curve would be that would be described by the Planets supposing the force of attraction towards the Sun to be reciprocal to the square of their distance from it. Sir Isaac replied immediately it would be an Ellipsis, the Dr struck with joy & amazement asked him how he knew it, why saith he, I have calculated it, whereupon Dr Halley asked him for the calculation without any further delay, Sr Isaac looked among his papers, but could not find it, but he promised him to renew it, & then send it him.'

The two parents of Calculus:

Isaac Newton , 1642–1727:



developed mechanics, calculus, theory of light
before the age of 30 ...
then spent 20 years of his life on alchemy ...

1687: publishes
'Philosophiae Naturalis Principia Mathematica'

1704: published 'Opticks'

brilliant but obsessive and nasty piece of work ...
great re-writer of history (in his own favour ...)

e.g. Hooke
(no references in Principia or Opticks!
'... by standing on the shoulders of Giants ...'
move of the Royal Society and a missing portrait)

or Leibniz
('independent commission of the Royal Society')

Gottfried Leibniz , 1646–1716:

invented calculus independently of Newton
(although slightly later)

Leibniz' notation more transparent
it is in fact what we use today!

Newton and his successors established the principles and the mode of work for *all* quantitative sciences (physics, biology, economics, etc):

- science: no longer descriptive, but aimed at finding the (usually mathematical) laws underlying the observed phenomena
- one's degree of understanding of an area of science is measured by the extent to which one can *predict* new phenomena from the discovered laws
- Galileo's principles define the procedure for finding the underlying laws.
  They now (i.e. after Newton and Leibniz) take the form:

  (i) state a hypothesis,
  (ii) devise an experiment to test it,
  (iii) calculate the predicted outcome of the experiment from the hypothesis
  (iv) carry out the experiment,
  (v) accept or reject the hypothesis
- When there are several distinct hypotheses, that are all consistent with the available data: select the simplest hypothesis ('Occam's Razor')

Side effects of the scientific revolution:

- industrial revolution
- 'mechanistic' view of the universe: nature is governed by differential equations
  (i) solution depends only on initial conditions
  (ii) no free will
  (iii) no divine intervention required to keep the world going ...

  of course that all changed around 1920 ...

1.2. *Style of the course*

Description of contents:

- Calculus: the related areas of differentiation, integration, sequences and series, that are united in their reliance on the idea of limits.
- Additional topics: complex numbers and trigonometric functions. This simplifies the discussion of the 'classical' functions of calculus (i.e. trigonometric, hyperbolic, exponential, logarithmic) and their relations.

Relation between calculus and analysis:

- Calculus:
  intuitive and operational ideas, no emphasis on strict step-by-step logical derivation
  e.g. derivative as limit of a ratio, integral as limit of a sum
  initially (Newton, Leibniz) without rigorous definition of 'limit'.
- Analysis:
  logical, rigorous proofs of the intuitive ideas of calculus.

stage 1 (calculus): find a method to crack the problem
stage 2 (analysis): determine carefully why and when the method works
(this order of developing maths continues today: see e.g. path integrals in particle physics)

Rationale behind this division of work:

- Hard to prove a theorem without being already familiar with the unproven (but strongly believed in) result.
- Hard to understand the need for the rigorous style of analysis until one has sufficient experience with calculus to realize the need to prove theorems, and to appreciate the beauty and elegance of such a logical formal treatment.
- Too much initial attention to the details of proofs while learning a subject often conceals the relative simplicity of the result.

Variation in approaches to calculus:

- there are different but mathematically equivalent routes via which to develop calculus, e.g. alternative definitions of trigonometric functions:
  (i) introduce series,
  (ii) define sin and cos as power series
  (iii) proof of the properties of sin cos through study of the series
- since all are equivalent, we will jump between definitions, dependent on the problem
- you should, however, never be satisfied by results without any derivation; we will try to give as many (full or partial) proofs as feasible within the time constraints of the course

1.3. Revision of some elementary mathematics

1.3.1. Numbers

First we start with some terminology and definitions:

definition : set : non−ordered collection of objects (elements)
e.g. : $S = \{a, b, c, d\}$
no ordering : $\{a, b, c, d\} = \{b, a, d, c, \}$ etc.
set membership : $a \in S$ (a belongs to the set S)

definition : $\emptyset = \{\ \}$ (the empty set)

definition : $\mathbb{N} = \{1, 2, 3, \ldots\}$ (natural numbers)

definition : $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integer numbers)

definition : $\mathbb{Z}^+ = \{1, 2, 3, \ldots\} = \mathbb{N}$ (positive integers)

definition : $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$ (negative integers, China $\pm$ 1500 BC)

definition : $\mathbb{Q} =$ set of all numbers of the form p/q with $p, q \in \mathbb{Z}$ (rational numbers)
e.g. $\frac{1}{2} \in \mathbb{Q},\ \frac{1}{2} \notin \mathbb{Z} \quad -\frac{27}{3} \in \mathbb{Q},\ -\frac{27}{3} \in \mathbb{Z}$

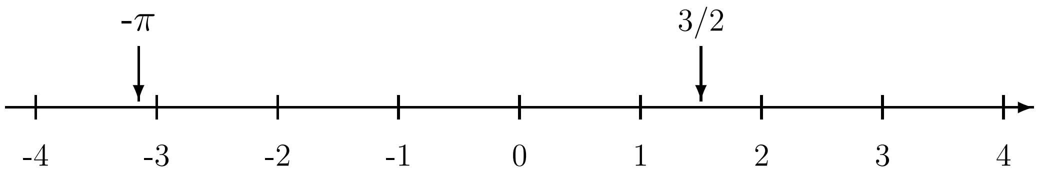
definition : $\mathbb{R} =$ set of all rational and irrational numbers (real numbers)
e.g. $\pi \in \mathbb{R},\ \pi \notin \mathbb{Q} \quad \sqrt{2} \in \mathbb{R},\ \sqrt{2} \notin \mathbb{Q} \quad \sqrt{64} \in \mathbb{R},\ \sqrt{64} \in \mathbb{N}$

definition : subsets of sets : $A \subseteq B$ if and only if every $a \in A$ obeys $a \in B$
note : $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Logical symbols: $\wedge$ (AND), $\vee$ (OR)
$S_1 \wedge S_2$: both statements S_1 and S_2 are true
$S_1 \vee S_2$: either S_1 is true, or S_2 is true, or both are true

Logical consequences: $S_1 \Rightarrow S_2$ means ‘if statement $S1$ is true then also statement $S2$ is true’

Every element $x \in \mathbb{R}$ can be represented by a point on the number line:



The set $\mathbb{R}$ is an ordered set. Let $x, y \in \mathbb{R}$, then the ordering symbols are defined as:

$x < y$: x is smaller than y , i.e. x to the left of y on the number line

$x > y$: x is larger than y , i.e. x to the right of y on the number line

$x \leq y$: x is smaller than or equal to y

$x \geq y$: x is larger than or equal to y

Note (should be obvious):

$$x < y \ \wedge \ x > y \quad \Rightarrow \quad x, y \in \emptyset \quad \text{(i.e. no such } x, y \in \mathbb{R} \text{ exist)}$$
$$x < y \ \wedge \ x \geq y \quad \Rightarrow \quad x, y \in \emptyset \quad \text{(i.e. no such } x, y \in \mathbb{R} \text{ exist)}$$
$$x > y \ \wedge \ x \leq y \quad \Rightarrow \quad x, y \in \emptyset \quad \text{(i.e. no such } x, y \in \mathbb{R} \text{ exist)}$$
$$x \leq y \ \wedge \ x \geq y \quad \Rightarrow \quad x = y$$

Interval: segment of the number line

Closed interval: line segment that includes both end points

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Open interval: line segment that does not include end points

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

Semi-open (or semi-closed) interval, exactly one endpoint is included

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\} \quad \text{or} \quad (a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

Unions and intersections:

union of A and B : $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$

intersection of A and B : $A \cap B = \{x \mid x \in A \ \wedge \ x \in B\}$

Example:

$$(-5, 4) \cup [2, 5] = (-5, 5] \qquad (-5, 4) \cap [2, 5] = [2, 4)$$

1.3.2. Powers of real numbers

In an expression of the form x^n we call x the *base* and n the *power*.
First define *natural* powers of real numbers. Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $x \neq 0$:

<u>definition</u> $x^0 = 1$

<u>definition</u> $x^n = x.x.\ldots.x$ $\quad$ ($n-$fold product, for $n > 0$)

Generalize to *integer* powers, by giving a definition for negative powers. Let $n \in \mathbb{N}$, $n > 0$:

<u>definition</u> $x^{-n} = \dfrac{1}{x.x.\ldots.x}$ $\quad$ ($n-$fold product in denominator)

Three basic laws of manipulation, let $x \in \mathbb{R}$ and $n, m \in \mathbb{Z}$:

first law : $\quad x^m.x^n = x^{n+m}$

second law : $\quad x^m/x^n = x^{m-n}$

third law : $\quad (x^m)^n = x^{mn}$

(i) prove these laws from the definitions, by checking the different case for the signs of m and n
(ii) note that (ii) can be derived from (i) and (iii)

Generalize to *fractional* powers of positive real numbers $x \in \mathbb{R}^+$, by giving a definition for powers of the form $1/n$ with $n \in \mathbb{N}$, $n > 0$:

<u>definition</u> $x^{1/n} = \sqrt[n]{x}$ $\quad$ ($n-$th root)

Here $\sqrt[n]{x}$ is the number $y \in \mathbb{R}^+$ with the property that $y^n = x$.
Verify that the above laws of manipulation still hold in the case of fractional powers.
For example, let $m, n, \ell \in \mathbb{N}^+$:

$$a^{\ell/n} = (\sqrt[n]{a})^\ell \qquad a^{q+\ell/n} = a^q.a^{\ell/n} = a^q.(\sqrt[n]{a})^\ell$$

Generalize to *real* powers of positive real numbers $x \in \mathbb{R}^+$. Each real number $y \in \mathbb{R}$ can be approximated to arbitrary accuracy by fractions n/m, where $m \in \mathbb{N}^+$, $n \in \mathbb{Z}$. One then defines x^y similarly by substituting for y this fraction approximation. Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$:

if $\dfrac{n}{m}$ is the best approximation of y by a fraction with denominator $\leq m$

then $x^{n/m} = (\sqrt[m]{x})^n$ is our associated approximation of x^y

Negative real powers are again defined via $x^{-y} = 1/x^y$, and our laws of manipulation still hold!

1.3.3. Solving quadratic equations

Quadratic equations are equations of the following form (or can be reduced to this form), where x is the unknown quantity to be determined, and $a, b, c \in \mathbb{R}$ (the coefficients) are given:

$$ax^2 + bx + c = 0$$

Assume $a \neq 0$, otherwise equation reduces to a linear one.
Note: solutions $x \in \mathbb{R}$ do not always exist!

Methods for solution:

- solution by factorization: find $d, f, g, h \in \mathbb{R}$ such that

  $$ax^2 + bx + c = (dx + f)(gx + h)$$

  (not always possible!)
  New problem involves two *linear* expressions: $(dx + f)(gx + h) = 0$.
  Solutions: $x = -f/d$ and $x = -h/g$
- solution by completing a square: find $d, f \in \mathbb{R}$ such that

  $$ax^2 + bx + c = a[(x + d)^2 - f]$$

  (always possible!)
  New problem is solved using square root: $(x + d)^2 = f$, so $x + d = \pm\sqrt{f}$, so $x = -d \pm \sqrt{f}$.
  No solution $x \in \mathbb{R}$ exists if $f < 0$.
- solution via a the general formula:

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  Solutions $x \in \mathbb{R}$ exist only if $b^2 \geq 4ac$.

Try all three methods on the following quadratic equations:

$$x^2 + 3x - 10 = 0 \qquad 6x^2 + 5x = 4 \qquad x^2 - 8x = 0 \qquad x^2 = 4x - 5$$

1.3.4. Functions, inverse functions, and graphs

A *function* f is a rule (or 'recipe') that assigns a unique output number $f(x)$ to each input number x. The set of input values x for which the function is defined is called the *domain* of f. The full set of output values that the function can generate when we choose values of x from its domain is called the *range* of f. Domain D and range R are indicated in our notation via $f : D \to R$.

Example 1:
let the recipe of a function f be: 'take any input $x \in \mathbb{R}$, add 2 to this input number'
We write

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x + 2$$

Example 2:
let the recipe of a function g be: 'take any input $x \in [-1, 1]$ and square it, then subtract 7'
We write

$$g : [-1, 1] \to [-7, -6] \qquad g(x) = x^2 - 7$$

Note:
a recipe is allowed to take different forms on different intervals, e.g.
Example 3:

$$f : [0, \infty) \to [0, 12] \cup (14, 16) \cup \{9\} \qquad f(x) = \begin{cases} 3x & \text{for } x \in [0, 4] \\ 2x + 6 & \text{for } x \in (4, 5) \\ 9 & \text{for } x \geq 5 \end{cases}$$

<u>definition</u>:
The inverse f^{-1} of a function $f : D \to R$ is defined by the following:

$$f^{-1} : R \to D \qquad f^{-1}(f(x)) = x \ \text{ for all } x \in D$$

In words: f^{-1} restores the original number x after the action of the function f.

Claim:
f^{-1} *can not not exist* if there exist two *different* numbers $x_1, x_2 \in D$ with $f(x_1) = f(x_2)$

Intuitively: how would f^{-1} select from the two candidates x_1, x_2 which one to restore?
Formal proof: call $f(x_1) = y$ and substitute x_1 and x_2 into the above definition of f^{-1}, one then finds the simultaneous requirements

$$f^{-1}(f(x_1)) = x_1 \quad \Rightarrow \quad f^{-1}(y) = x_1$$
$$f^{-1}(f(x_2)) = x_2 \quad \Rightarrow \quad f^{-1}(y) = x_2$$

The assumption of the existence of f^{-1} would thus lead to $x_1 = x_2$, in contradiction with the starting point $x_1 \neq x_2$. Hence f^{-1} cannot exist.

<u>definition</u>:
A function $f : D \to R$ is *invertible* if and only if $f(x_1) \neq f(x_2)$ for any two values $x_1, x_2 \in D$ with $x_1 \neq x_2$

How to find the inverse of a given function f?

(i) write $y = f(x)$
(ii) transpose this formula, to make x the subject (i.e. obtain $x =$ some recipe on y)
(iii) interchange x and y
(iv) result: $y = f^{-1}(x)$, then verify ...

Work out the inverse functions for

$$x \in \mathbb{R},\ f(x) = x + 4 \qquad x \in \mathbb{R},\ f(x) = 6x + 4 \qquad x \in \mathbb{R}^+,\ f(x) = \sqrt{x}$$

1.3.5. Exponential function, logarithm, laws for logarithms

definition:
the exponential function is $f(x) = e^x$, where e is a special irrational number ('exponential growth')

Properties:
(i) $e^x > 0$ for all $x \in \mathbb{R}$
(ii) e^x increases monotonically
(iii) from the value 0 as $x \to -\infty$, via $e^0 = 1$ at $x = 0$, to unbounded growth as $x \to \infty$

Define $n! = n(n-1)(n-2)\ldots.3.2.1$ ('n factorial')
Equivalent expressions for $e = 2.71828182\ldots$ (more about this later):

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \qquad e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$$

Similarly, 'exponential decay' is described by $f(x) = e^{-x} = 1/e^x$
(same shape of curve, just exchange $x \leftrightarrow -x$)

Let $a \in \mathbb{R}^+$:

definition:
the logarithmic function to the base a, written as $\log_a(x)$, is the inverse of the function $f(x) = a^x$
in words: '$\log_a(y)$ gives the power to which I must raise a to get y'

definition:
the *natural* logarithmic is defined as the logarithmic function to the base e, i.e. $\ln(x) = \log_e(x)$
in words: '$\ln(y)$ gives the power to which I must raise e to get y'

Properties (direct consequences of the concept of 'inverse'):

$$a^{\log_a(y)} = y \qquad \log_a(a^x) = x \qquad e^{\ln(y)} = y \qquad \ln(e^x) = x$$

Manipulation identities for logarithms:

$$\text{switching base}: \qquad \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \tag{1}$$

$$\text{products, fractions}: \qquad \log_a(x.y) = \log_a(x) + \log_a(y) \tag{2}$$

$$\log_a(x/y) = \log_a(x) - \log_a(y) \tag{3}$$

$$\text{powers}: \qquad \log_a(x^y) = y \log_a(x) \tag{4}$$

Proof of (1):

Strategy: we prove the equivalent statement $\log_a(x).\log_b(a) = \log_b(x)$
using the manipulation identities for powers
Show that the left-hand side (LHS) of the latter equation has the property $b^{\text{LHS}} = x$:

$$b^{\text{LHS}} = b^{\log_a(x).\log_b(a)} = (b^{\log_b(a)})^{\log_a(x)} = a^{\log_a(x)} = x$$

By the definition of $\log_b(x)$ this implies that LHS= $\log_b(x)$, which is exactly the right-hand side. This completes the proof.

Proof of (2):

Strategy: we use the manipulation identities for powers
We show that the right-hand side (RHS) of (2) has the property $a^{\text{RHS}} = xy$:

$$a^{\text{RHS}} = a^{\log_a(x)+\log_a(y)} = a^{\log_a(x)}.a^{\log_a(y)} = x.y$$

By the definition of $\log_a(x)$ this implies that RHS= $\log_a(xy)$, which is exactly the left-hand side of (2). This completes the proof.

Proof of (3):

Strategy: we use the manipulation identities for powers
We show that the right-hand side (RHS) of (3) has the property $a^{\text{RHS}} = x/y$:

$$a^{\text{RHS}} = a^{\log_a(x)-\log_a(y)} = a^{\log_a(x)}.a^{-\log_a(y)} = x/y$$

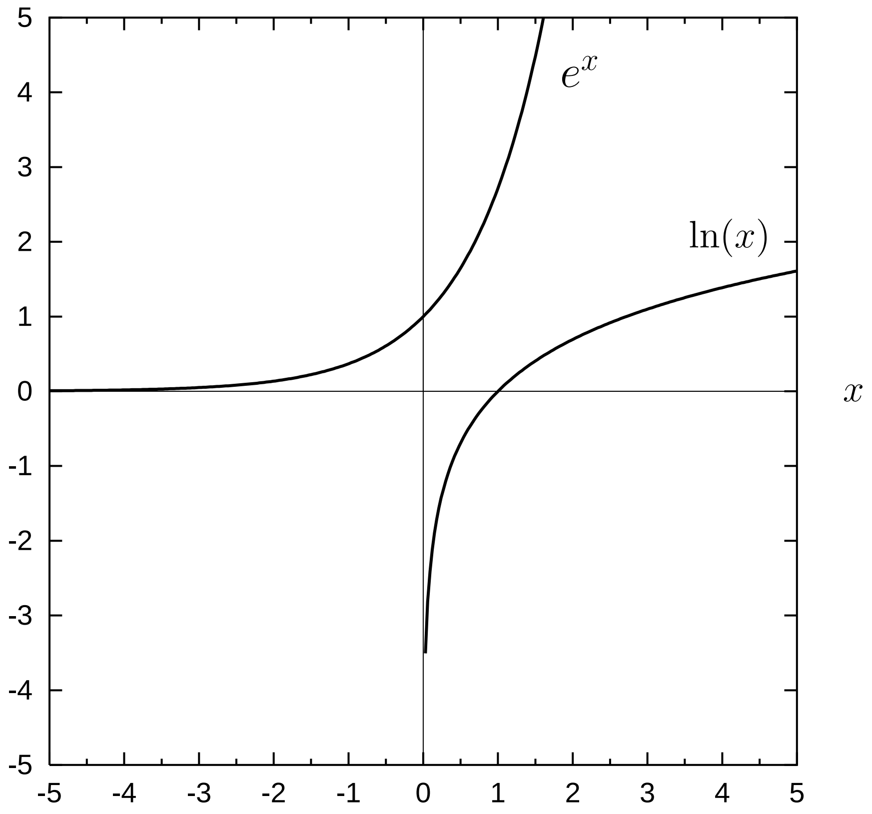
By the definition of $\log_a(x)$ this implies that RHS= $\log_a(x/y)$, which is exactly the left-hand side of (3). This completes the proof.

Proof of (4):

Strategy: we use the various manipulation identities for powers
We show that the right-hand side (RHS) of (4) has the property $a^{\text{RHS}} = x^y$:

$$a^{\text{RHS}} = a^{y\log_a(x)} = (a^{\log_a(x)})^y = x^y$$

By the definition of $\log_a(x)$ this implies that RHS= $\log_a(x^y)$, which is exactly the left-hand side of (4). This completes the proof.



1.3.6. Trigonometric functions

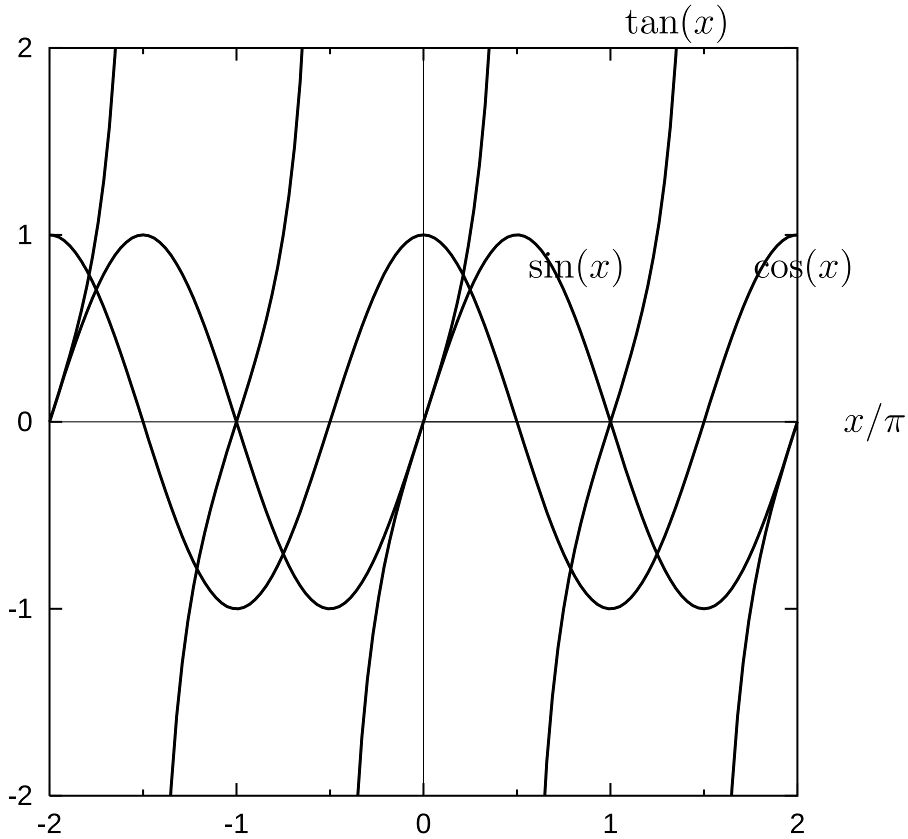
Many equivalent definitions possible (more follows in this course).

definition:
Consider rotations around the origin. 1 radian is the magnitude of a rotation angle such that it 'cuts' a segment of the unit circle of length 1.

Consequence: going round once implies an angle of 2π, i.e. 2π radians $= 360^0$
(since circumference of a radius-R circle equals $2\pi R$)

definition:
Consider a half-line with its one end-point in the origin. Choose it initially to lie along the positive x-axis, and then rotate it anti-clockwise around the origin; call the rotation angle α (measured in radians). Find the coordinates (X, Y) of the point where the half-line intersects the unit circle: now call $\cos(\alpha) = X$, $\sin(\alpha) = Y$.

definition: $\tan(\alpha) = \sin(\alpha)/\cos(\alpha)$



Consequences:

(i) $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$
(definition of the unit circle!)

(ii) $\cos(x)$ and $\sin(x)$ are periodic, with period 2π
(since 2π rotation gives a complete turn)

(iii) special values of $\sin(x)$ and $\cos(x)$ follow immediately,
e.g. for $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \ldots$

(iv) more general than definition in terms of ratios of sides in triangles
(the latter do come out for $x \in [0, \frac{\pi}{2}]$, here we have a definition for *any* value of x)

(v) zero points:
$\sin(x) = 0$ for $x = n\pi$ with $n \in \mathbb{Z}$, $\cos(x) = 0$ for $x = \frac{\pi}{2} + n\pi$ with $n \in \mathbb{Z}$

(vi) $\tan(x) = 0$ for $x = n\pi$ with $n \in \mathbb{Z}$, doesn't exist for $x = \frac{\pi}{2} + n\pi$ with $n \in \mathbb{Z}$

2. Proof by induction

Induction is a method that allows us to prove an *infinite* number of statements, by proving just *two* statements. The basic ideas are

- Let a set $S \subseteq \mathbb{N}$ have the following properties:
  (i) $1 \in S$, (ii) if $n \in S$ then also $n+1 \in S$
  this construction generates *all* natural numbers: $S = \mathbb{N}$
- If we prove
  (i) that a statement is true for $n = 1$ (the 'basis')
  (ii) that if it is true for a given n, it must also be true for $n+1$ (the 'induction step')
  then we will have proven that the statement is true for all $n \in \mathbb{N}$

definition: the summation symbol $\sum$

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n \qquad (n, m \in \mathbb{Z},\ n \geq m)$$

definition: binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{with the convention} \qquad 0! = 1$$

meaning:
the number of distinct ways to select k elements from a set of n elements
(where permutations are not counted separately)

Example 1:
Let $n \in \mathbb{N}$. Use induction to prove that

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

Proof:
We subtract the two sides and define $A_n = \sum_{k=1}^{n} k - \frac{1}{2}n(n+1)$.
We must now prove that $A_n = 0$ for all $n \in \mathbb{N}$

(i) the basis:
for $n = 1$ one has $A_1 = \sum_{k=1}^{1} k - \frac{1}{2}1.2 = 1 - 1 = 0$ (so claim is true for $n = 1$)

(ii) induction step:
now suppose that $A_n = 0$ for some $n \in \mathbb{N}$, i.e. $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$
It follows that

$$A_{n+1} = \sum_{k=1}^{n+1} k - \frac{1}{2}(n+1)(n+2)$$

$$= \sum_{k=1}^{n} k + (n+1) - \frac{1}{2}(n+1)(n+2) \qquad \text{(next use } A_n = 0 \text{ !)}$$
$$= \frac{1}{2}n(n+1) + (n+1) - \frac{1}{2}(n+1)(n+2)$$
$$= (n+1)[\frac{1}{2}n + 1 - \frac{1}{2}(n+2)] = 0$$

We have shown: if $A_n = 0$ then also $A_{n+1} = 0$. Knowing already that $A_1 = 0$ (the basis), this completes the proof that $A_n = 0$ for all $n \in \mathbb{N}$.

Example 2:
use induction to prove Newton's binomial formula for all integer $n \geq 0$:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof:
We subtract the two sides and define $A_n = (a+b)^n - \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.
We must now prove that $A_n = 0$ for all integer $n \geq 0$
(i) the basis:
for $n = 0$ one has $A_0 = (a+b)^0 - \sum_{k=0}^{0} \binom{0}{k} a^{-k} b^k = 1 - \binom{0}{0} = 0$, so claim is true for $n = 0$
(ii) induction step:
now suppose that $A_n = 0$ for some $n \in \mathbb{N}$, i.e. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
It follows that

$$A_{n+1} = (a+b)^{n+1} - \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$
$$= (a+b)(a+b)^n - \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k \qquad \text{(next use } A_n = 0\text{)}$$
$$= (a+b) \left\{ \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \right\} - \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} - \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

In the middle term we substitute $k = \ell - 1$, so $\ell = 1, \ldots, n+1$. Then we separate terms with b^0 and with b^{n+1} from the rest. This gives

$$A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} a^{n+1-\ell} b^\ell - \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$
$$= \binom{n}{0} a^{n+1-0} b^0 - \binom{n+1}{0} a^{n+1-0} b^0 + \binom{n}{n+1-1} a^{n+1-n-1} b^{n+1} - \binom{n+1}{n+1} a^{n+1-n-1} b^{n+1}$$
$$+ \sum_{k=1}^{n} a^{n+1-k} b^k \left[\binom{n}{k} + \binom{n}{k-1} - \binom{n+1}{k} \right]$$

$$= \sum_{k=1}^{n} a^{n+1-k} b^k \left[\binom{n}{k} + \binom{n}{k-1} - \binom{n+1}{k}\right]$$

Finally we must work out the combinatorial terms, for $k \in 1, \ldots, n\}$:

$$\binom{n}{k} + \binom{n}{k-1} - \binom{n+1}{k} = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} - \frac{(n+1)!}{(n+1-k)!k!}$$
$$= \frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{k} + \frac{1}{n-k+1} - \frac{n+1}{k(n+1-k)}\right]$$
$$= \frac{n!}{(k-1)!(n-k)!}\left[\frac{(n+1-k)+k-(n+1)}{k(n+1-k)}\right] = 0$$

Insertion into our previous intermediate result gives $A_{n+1} = 0$.
Thus we have shown: if $A_n = 0$ then also $A_{n+1} = 0$. Knowing already that $A_0 = 0$ (the basis), this completes the proof that $A_n = 0$ for all integer $n \geq 0$.

Example 3:
Let $x \in \mathbb{R},\ x \neq 1$ and $n \in \mathbb{N}$. Use induction to prove the following ('geometric series')

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$$

Proof:
We subtract the two sides and define $A_n = \sum_{k=0}^{n-1} x^k - \frac{1-x^n}{1-x}$.
We must now prove that $A_n = 0$ for all integer $n \in \mathbb{N}$
(i) the basis:
for $n = 1$ one has $A_1 = \sum_{k=0}^{0} x^k - \frac{1-x}{1-x} = 1 - 1 = 0$, so claim is true for $n = 1$
(ii) induction step:
now suppose that $A_n = 0$ for some $n \in \mathbb{N}$, i.e. $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$. It follows that

$$A_{n+1} = \sum_{k=0}^{n} x^k - \frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n-1} x^k + x^n - \frac{1-x^{n+1}}{1-x} \qquad \text{(next use } A_n = 0\text{)}$$
$$= \frac{1-x^n}{1-x} + x^n - \frac{1-x^{n+1}}{1-x} =$$
$$= \frac{1-x^n+(1-x)x^n-(1-x^{n+1})}{1-x} = 0$$

Thus we have shown: if $A_n = 0$ then also $A_{n+1} = 0$. Knowing already that $A_1 = 0$ (the basis), this completes the proof that $A_n = 0$ for all $n \in \mathbb{N}$.

Exercises:
Similarly, prove the following statements for $n \in \mathbb{N}$ by induction:

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \qquad\qquad \sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$$

3. Complex numbers

3.1. Introduction and definition

definition:
The number i is defined as a solution of the equation $z^2+1=0$
(note: this eqn had no solutions $z \in \mathbb{R}$)

definition:
The set $\mathbb{C}$ of 'complex numbers' consists of all expressions of the form $a+ib$ with $a,b \in \mathbb{R}$,

$$\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$$

definition:
Addition and multiplication of numbers in $\mathbb{C}$ is defined as follows. Let $a,b,c,d \in \mathbb{R}$:

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i \\ (a+bi)(c+di) &= (ac-bd)+(ad+bc)i\end{aligned}$$

(i.e. calculate as if with real numbers, and put $i^2=-1$)

note:
$i^2=-1$ can alternatively be taken as a *consequence* of the multiplication definition

definition: let $a,b \in \mathbb{R}$
The 'real part' of $z=a+ib$ is defined as $\text{Re}(z)=a$
The 'imaginary part' of $z=a+ib$ (where $a,b \in \mathbb{R}$) is defined as $\text{Im}(z)=b$

definition: let $a,b \in \mathbb{R}$
The complex conjugate of a complex number $z=a+ib$ is defined as $\overline{z}=a-ib$
(i.e. obtained from z by replacing i by $-i$)
z and $\overline{z}$ are called a 'complex conjugate pair'

Notes:

(i) also $z=-i$ is a solution of $z^2+1=0$
proof: $z^2+1=(-i)^2+1=(-1)^2 i^2+1=i^2+1=0$

(ii) sometimes i is written as $i=\sqrt{-1}$

(iii) sometimes $\overline{z}$ is written as $z^\star$

(iv) unlike $\mathbb{R}$, it is impossible to 'order' the numbers of $\mathbb{C}$ in terms of 'larger' and 'smaller'
(just postulate $i<0$ or $i>0$ and see what happens ...)

3.2. *Elementary properties of complex numbers*

- Every quadratic equation $az^2+bz+c=0$ can be solved in $\mathbb{C}$, giving the solutions
$$z_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
(where for $b^2-4ac<0$ one has $\sqrt{b^2-4ac}=\sqrt{-1}\sqrt{4ac-b^2}=i\sqrt{4ac-b^2}$)

  proof:

$$a(z-z_+)(z-z_-) = a\Big(z^2-z(z_++z_-)+z_+z_-\Big)$$
$$= az^2 - az\left(\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}\right)+a\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$
$$= az^2-\frac{1}{2}z\,(-2b)+\frac{1}{4a}\left(-b+\sqrt{b^2-4ac}\right)\left(-b-\sqrt{b^2-4ac}\right)$$
$$= az^2+bz+\frac{1}{4a}\left(b^2-(\sqrt{b^2-4ac})^2\right)$$
$$= az^2+bz+\frac{1}{4a}\left(b^2-b^2+4ac\right) = az^2+bz+c$$

  It now follows immediately that $az^2+bz+c=0$ for $z=z_\pm$. This completes the proof.

- Every n-th order polynomial with real-valued or complex coefficients,
$P(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, can be factorized into linear factors to give
$$P(z)=(z-z_1)(z-z_2)\ldots(z-z_{n-1})(z-z_n)$$
with n complex numbers $z_1\ldots z_n$ (the zeros or 'roots' of the polynomial)
(the proof will not be given here)

- For all $z,w\in\mathbb{C}$: $\overline{z+w}=\overline{z}+\overline{w}$ (proof in tutorials)
For all $z,w\in\mathbb{C}$: $\overline{zw}=\overline{z}.\overline{w}$ (proof in tutorials)

- For every $z\in\mathbb{C}$: $z\overline{z}\in\mathbb{R}$, with $z\overline{z}\geq 0$ and where $z\overline{z}=0$ if and only if $z=0$
(proof in tutorials)

- If z is a root of a polynomial $P(z)$ with real coefficients, then also $\overline{z}$ will be a root

  proof:

  We know that $P(z)=0$, i.e. $z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0=0$.
  Now, since $\overline{z+w}=\overline{z}+\overline{w}$ and $\overline{zw}=\overline{z}.\overline{w}$, also
$$\overline{0}=\overline{z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0}$$
$$0=\overline{z^n}+\overline{a_{n-1}}\,\overline{z^{n-1}}+\ldots+\overline{a_1}\,\overline{z}+\overline{a_0}$$
$$0=\overline{z^n}+a_{n-1}\overline{z^{n-1}}+\ldots+a_1\overline{z}+a_0$$
  Thus $P(\overline{z})=0$. This completes the proof.

3.3. Absolute value and division

definition:
The absolute value (or 'modulus') of a complex number is defined as $|z| = \sqrt{z.\overline{z}}$

Consequences:

- If $z = a + ib$ with $a = \text{Re}(z)$ and $b = \text{Im}(z)$, then $|z| = \sqrt{a^2 + b^2}$

  proof:
  $$|z| = \sqrt{z.\overline{z}} = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2 - (ib)^2} = \sqrt{a^2 + b^2}$$

- $\overline{(\overline{z})} = z$

  proof: let $z = a + ib$,
  $$\overline{(\overline{z})} = \overline{(a - ib)} = \overline{a + i(-b)} = a - i(-b) = a + ib = z$$

- $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$
  (proof in tutorials)

- $|z.w| = |z||w|, \quad |\overline{z}| = |z|$
  (proof in tutorials)

- The above definition of $|z|$ obeys the triangular inequality $|z + w| \leq |z| + |w|$

  proof:
  $$\begin{aligned} |z+w|^2 &= (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) \\ &= z.\overline{z} + z.\overline{w} + w.\overline{z} + w.\overline{w} \\ &= |z|^2 + 2\text{Re}(z.\overline{w}) + |w|^2 \qquad (\text{due to } z + \overline{z} = 2\text{Re}(z)) \\ &\leq |z|^2 + 2|z.\overline{w}| + |w|^2 \qquad (\text{due to } |\text{Re}(z)| \leq |z|) \\ &= |z|^2 + 2|z||\overline{w}| + |w|^2 \qquad (\text{due to } |zw| = |z||w| \text{ and } |\overline{w}| = |w|) \\ &= (|z| + |w|)^2 \end{aligned}$$

  Taking the square root of both sides then completes the proof.

The property $|z| \in \mathbb{R}$ makes it easy to work out the ratio of complex numbers, and write it in the standard form $v = \text{Re}(v) + i\text{Im}(v)$. Method: multiply numerator and denominator by the complex conjugate of the denominator.

Let $z = a + ib$ and $w = c + id$, with $a, b, c, d \in \mathbb{R}$ and $w \neq 0$:
$$\begin{aligned} \frac{a+ib}{c+id} &= \frac{a+ib}{c+id}\frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{|c+id|^2} \\ &= \frac{ac + ibc - iad + bd}{c^2 + d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2} \end{aligned}$$

In particular: $\quad 1/i = -i, \qquad 1/z = \overline{z}/|z|^2$

3.4. The complex plane (Argand diagram)

3.4.1. Complex numbers as points in a plane

underlying idea:

there is a one-to-one correspondence between complex numbers $z \in \mathbb{C}$ and points in the ordinary plane $\mathbb{R}^2$, namely

$$z = a + ib \in \mathbb{C} \quad \longleftrightarrow \quad (a, b) \in \mathbb{R}^2$$

where a is taken as the x-coordinate and b is taken as the y-coordinate.

one-to-one:
(i) with every $z \in \mathbb{C}$ corresponds exactly one point (a, b) in the plane
(ii) with every point (a, b) in the plane corresponds exactly one $z \in \mathbb{C}$

examples:

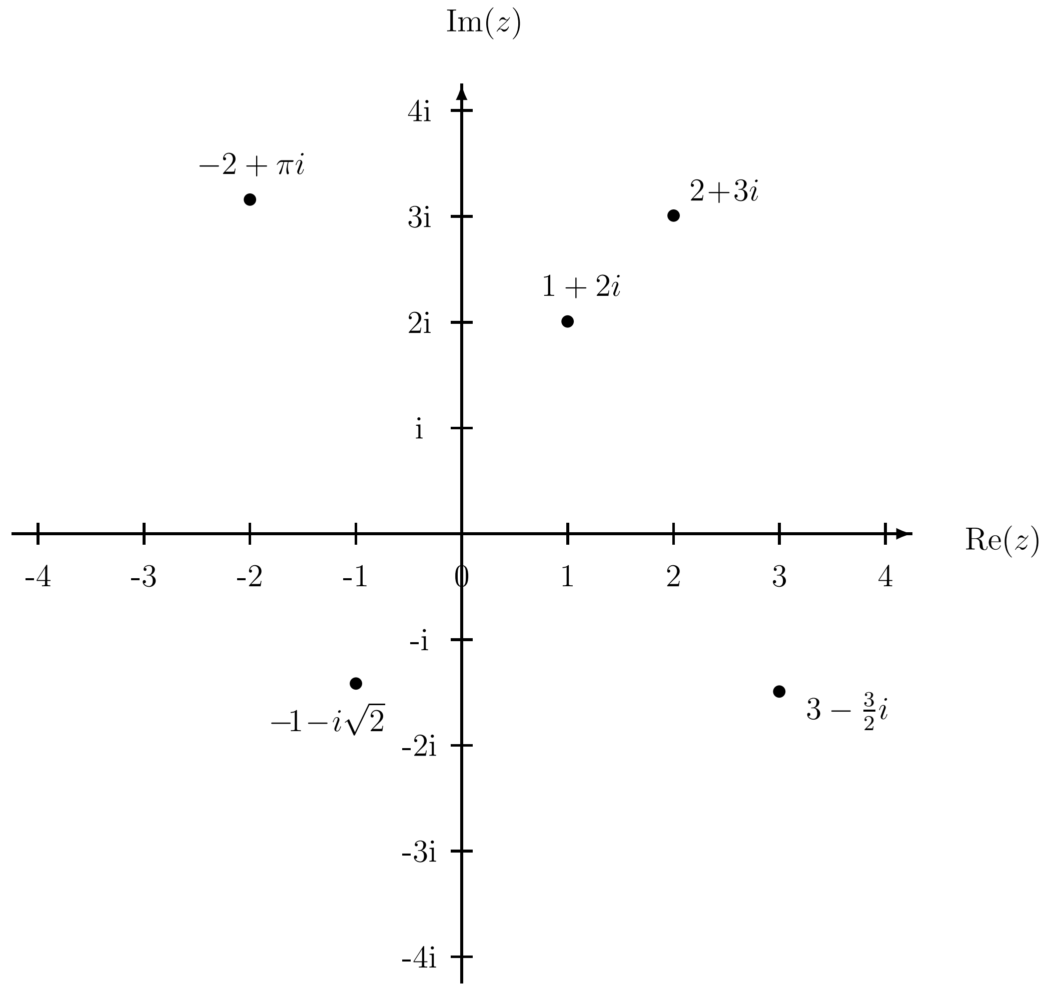
$z \in \mathbb{C}$:		$(a, b) \in \mathbb{R}^2$:
$1 = 1 + 0.i$	$\longleftrightarrow$	$(1, 0)$
$i = 0 + 1.i$	$\longleftrightarrow$	$(0, 1)$
$-1 = -1 + 0.i$	$\longleftrightarrow$	$(-1, 0)$
$-i = 0 - 1.i$	$\longleftrightarrow$	$(0, -1)$
$2 + 3i$	$\longleftrightarrow$	$(2, 3)$
$-1 - i\sqrt{2}$	$\longleftrightarrow$	$(-1, -\sqrt{2})$
$1 + 2i$	$\longleftrightarrow$	$(1, 2)$
$3 - \frac{3}{2}i$	$\longleftrightarrow$	$(3, -\frac{3}{2})$
$-2 + \pi i$	$\longleftrightarrow$	$(-2, \pi)$

definition:
The 'complex plane' (or Argand diagram) is defined as the plane in which complex numbers $z = a + ib$ (with $a, b \in \mathbb{R}$) are represented by points with coordinates $(a, b) = (\mathrm{Re}(z), \mathrm{Im}(z))$. The horizontal axis is called the 'real axis', and the vertical axis the 'imaginary axis'.

notes:

(i) The real axis is the set of all *real* numbers,
(as it contains all $z = a + ib \in \mathbb{C}$ for which $b = 0$, i.e. of the form $z = a$)

(ii) The imaginary axis is the set of all 'purely imaginary' numbers,
(it contains all $z = a + ib \in \mathbb{C}$ for which $a = 0$, i.e. of the form $z = ib$)



3.4.2. Polar coordinates

Recall the definition of the trigonometric functions:

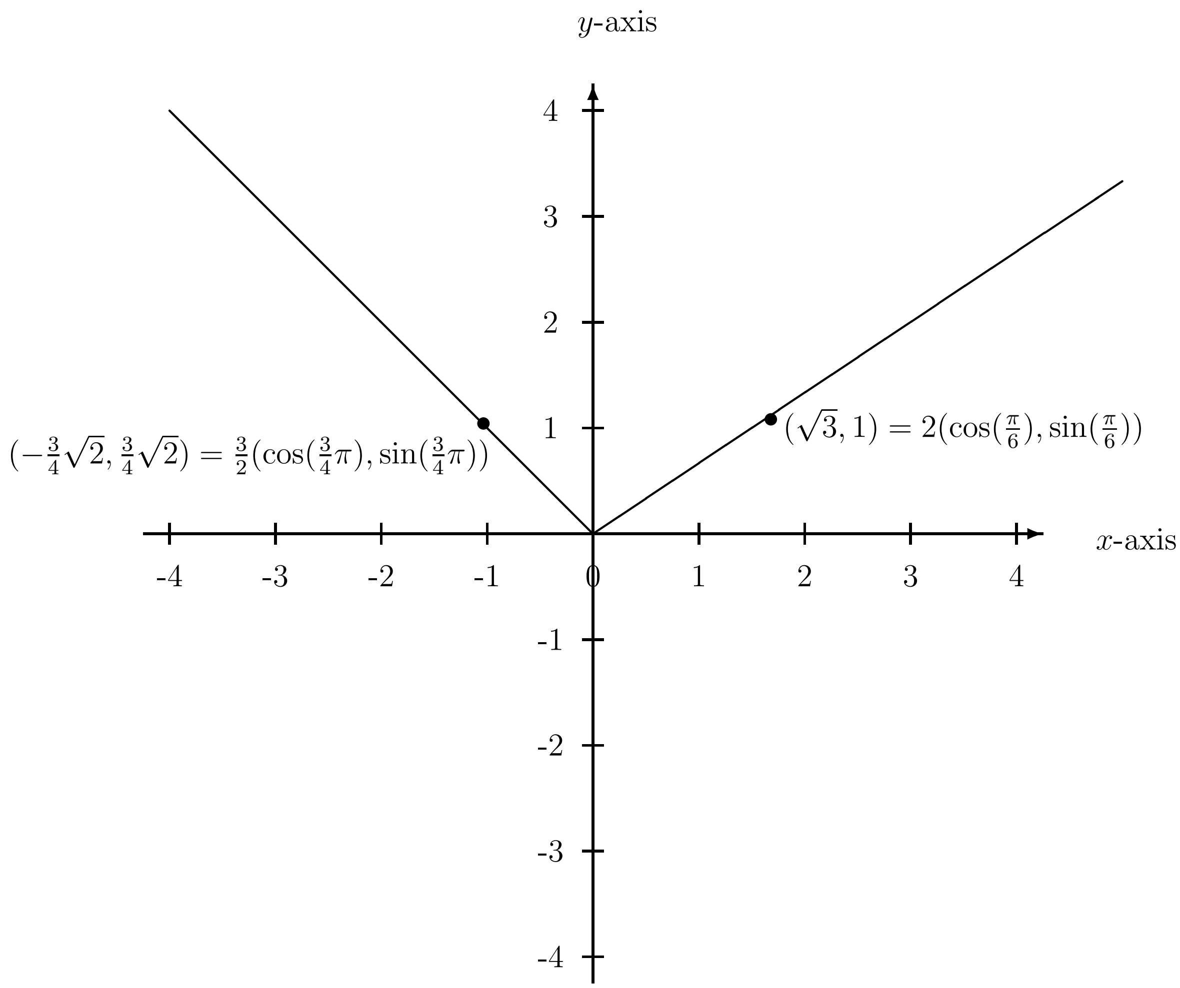
(i) each point on the unit circle around the origin,
with Cartesian coordinates (x, y) such that $x^2 + y^2 = 1$,
can be written in the form $(x, y) = (\cos(\theta), \sin(\theta))$

(ii) Here θ denotes the angle with the x-axis of a half-line through the origin and (x, y)

This can be generalized easily to *any* circle around the origin:

(i) each point on the circle of radius r around the origin,
with Cartesian coordinates (x, y) such that $x^2 + y^2 = r^2$,
can be written in the form $(x, y) = (r\cos(\theta), r\sin(\theta))$

(ii) Here θ denotes the angle with the x-axis of a half-line through the origin and (x, y)



underlying idea:

We can represent each point in the plane with Cartesian coordinates (x, y) alternatively by so-called polar coordinates (r, θ).
The same is then also true for each complex number $z \in \mathbb{C}$.

examples:

polar coordinates:	Cartesian coordinates:	complex number:
(r, θ)	$(x, y) = (r\cos(\theta), r\sin(\theta))$	$z = r\cos\theta + ir\sin\theta$
$(r, \theta) = (1, 0)$	$(x, y) = (1, 0)$	$z = 1 + 0.i = 1$
$(r, \theta) = (1, \frac{\pi}{2})$	$(x, y) = (0, 1)$	$z = 0 + 1.i = i$
$(r, \theta) = (1, \pi)$	$(x, y) = (-1, 0)$	$z = -1 + 0.i = -1$
$(r, \theta) = (1, -\frac{\pi}{2})$	$(x, y) = (0, -1)$	$z = 0 - 1.i = -i$
$(r, \theta) = (\frac{3}{2}, \frac{3\pi}{4})$	$(x, y) = (-\frac{3}{4}\sqrt{2}, \frac{3}{4}\sqrt{2})$	$z = -\frac{3}{4}\sqrt{2} + \frac{3}{4}\sqrt{2}i$
$(r, \theta) = (2, \frac{\pi}{6})$	$(x, y) = (\sqrt{3}, 1)$	$z = \sqrt{3} + 1.i = \sqrt{3} + i$

Notes:

(i) each complex number can thus be written as $z = r(\cos(\theta) + i\sin(\theta))$,
with $r \geq 0$ and $\theta \in \mathbb{R}$

(ii) due to periodicity of sin() and cos():
for all $n \in \mathbb{N}$ also $r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi) = z$

(iii) If $z = r(\cos(\theta) + i\sin(\theta)$ then $r = |z|$
Proof:

$$\begin{aligned} |z|^2 = \overline{z} &= r^2(\cos(\theta) + i\sin(\theta)(\cos(\theta) - i\sin(\theta) \\ &= r^2(\cos^2(\theta) - i^2\sin^2(\theta)) = r^2 \end{aligned}$$

3.4.3. The exponential form of numbers on the unit circle

<u>definition</u>:
The unit circle in the complex plane is the set $\{z \in \mathbb{C} \mid \text{Re}^2(z) + \text{Im}^2(z) = 1\}$
Alternatively, using polar coordinates: $\{z \in \mathbb{C} \mid z = \cos(\theta) + i\sin(\theta) \text{ for some } \theta \in \mathbb{R}\}$

We now proceed, for numbers on the unit circle, to one of the main statements in complex number theory. It impacts on all complex numbers. Whether it is a definition or a theorem depends on one's starting point. We have so far only defined exponential functions with real arguments, so here it has the status of an expansion of the definition of the exponential function:

<u>definition</u>:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

<u>rationale</u>:

Let us show why this is the natural extension of the exponential function. Define $f(\theta) = \cos(\theta) + i\sin(\theta)$, then (recall definition of derivative from school, and remember that $\frac{d}{d\theta}\cos(\theta) = -\sin(\theta)$ and $\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$):

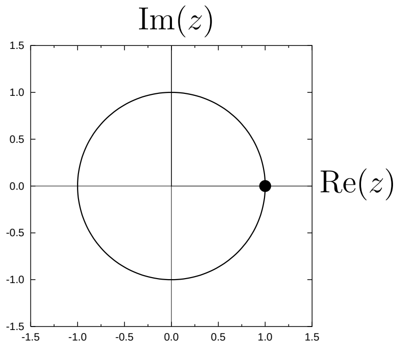
$$\begin{aligned} \frac{d}{d\theta}f(\theta) &= \frac{d}{d\theta}\cos(\theta) + i\frac{d}{d\theta}\sin(\theta) = -\sin(\theta) + i\cos(\theta) \\ &= i\Big(\cos(\theta) + i\sin(\theta)\Big) = ie^{i\theta} = if(\theta) \\ f(0) &= 1 \end{aligned}$$

Whereas for *real* numbers a we would have had, with $f(\theta) = e^{b\theta}$:

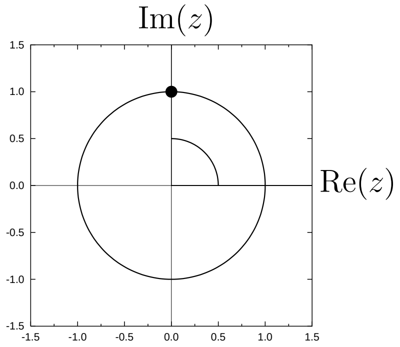
$$\begin{aligned} \frac{d}{d\theta}f(\theta) &= \frac{d}{d\theta}e^{b\theta} = be^{b\theta} = bf(\theta) \\ f(0) &= 1 \end{aligned}$$

We see that the two situations (real exponentials versus complex exponentials) connect if and only if we choose $b = i$, i.e. if we define $f(\theta) = \cos(\theta) + i\sin(\theta) = e^{i\theta}$

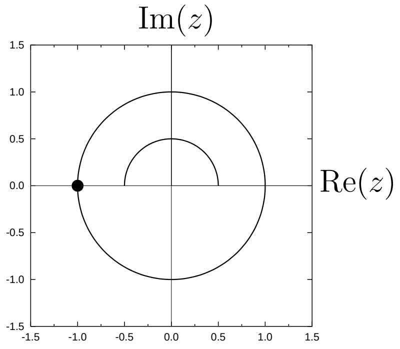
Some examples:



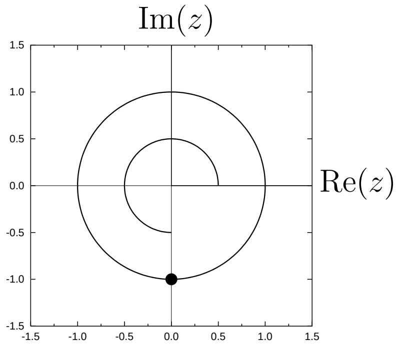
$$z = e^{i0} = \cos(0) + i\sin(0) = 1 + 0.i = 1$$



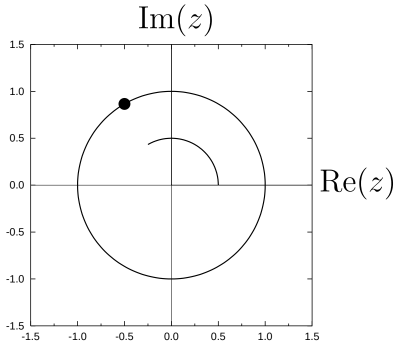
$$z = e^{i\pi/2} = \cos(\tfrac{\pi}{2}) + i\sin(\tfrac{\pi}{2}) = 0 + 1.i = i$$



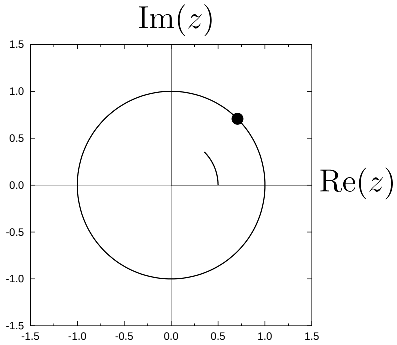
$$z = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0.i = -1$$



$$z = e^{3i\pi/2} = \cos(\tfrac{3\pi}{2}) + i\sin(\tfrac{3\pi}{2}) = 0 - 1.i = -i$$



$$z = e^{2\pi i/3} = \cos(\tfrac{2\pi}{3}) + i\sin(\tfrac{2\pi}{3}) = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i$$



$$z = e^{i\pi/4} = \cos(\tfrac{\pi}{4}) + i\sin(\tfrac{\pi}{4}) = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i$$

3.5. Complex numbers in exponential notation

3.5.1. Definition and general properties

Upon combining the polar coordinate representation $z = r\cos(\theta) + ir\sin(\theta)$ with the new identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$:

claim:

Each complex number can be written in the form $z = re^{i\theta}$ with $r, \theta \in \mathbb{R}$ and $r \geq 0$

Notes:

(i) $|z| = r$
proof: $|z| = \sqrt{z\overline{z}} = \sqrt{re^{i\theta}.re^{-i\theta}} = \sqrt{r^2} = r$

(ii) $\mathrm{Re}(z) = \mathrm{Re}(r\cos(\theta) + ir\sin(\theta)) = r\cos(\theta)$
$\mathrm{Im}(z) = \mathrm{Im}(r\cos(\theta) + ir\sin(\theta)) = r\sin(\theta)$

(iii) $\overline{z} = re^{-i\theta}$

proof: $\overline{z} = \overline{r\cos(\theta) + i\sin(\theta)} = r\cos(\theta) - i\sin(\theta) = re^{-i\theta}$

(iv) $1/z = \frac{1}{r}e^{-i\theta}$

proof: multiply numerator and denominator by $\overline{z}$:

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{re^{i\theta}}\frac{re^{-i\theta}}{re^{-i\theta}} = \frac{re^{-i\theta}}{r^2} = \frac{1}{r}e^{-i\theta}$$

(v) $1/\overline{z} = \frac{1}{r}e^{i\theta}$

proof: multiply numerator and denominator by z:

$$\frac{1}{\overline{z}} = \frac{1}{re^{-i\theta}} = \frac{1}{re^{-i\theta}}\frac{re^{i\theta}}{re^{i\theta}} = \frac{re^{i\theta}}{r^2} = \frac{1}{r}e^{i\theta}$$

(vi) The angle θ in $z = re^{i\theta}$ is as yet not uniquely defined,

changing $\theta \to \theta + 2n\pi$ with $n \in \mathbb{Z}$ leaves one with the *same* number z

proof:

$$\begin{aligned} re^{i(\theta+2n\pi)} &= re^{i(\theta+2n\pi)} = re^{i\theta}e^{2ni\pi} \\ &= re^{i\theta}\left(\cos(2n\pi) + i\sin(2n\pi)\right) = re^{i\theta}.1 = re^{i\theta} \end{aligned}$$

3.5.2. Multiplication and division in exponential notation

claim:

Multiplication of two complex numbers $z = re^{i\theta}$ and $w = \rho e^{i\phi}$,
with real $r, \rho \geq 0$ and real θ, ϕ, implies
(i) *multiplication* of the absolute values, and
(ii) *addition* of the arguments

proof:

$$z.w = re^{i\theta}.\rho e^{i\phi} = r\rho\; e^{i\theta+i\phi} = r\rho\; e^{i(\theta+\phi)}$$

claim:

Division of two complex numbers $z = re^{i\theta}$ and $w = \rho e^{i\phi}$,
with real $r, \rho \geq 0$ and real θ, ϕ, implies
(i) *division* of the absolute values, and
(ii) *subtraction* of the arguments

proof:

$$z/w = \frac{re^{i\theta}}{\rho e^{i\phi}} = \frac{r}{\rho}\; e^{i\theta}e^{-i\phi} = \frac{r}{\rho}\; e^{i(\theta-\phi)}$$

Examples:

$$\begin{aligned}
\frac{1}{2e^{i\pi/4}} &= \frac{1}{2}e^{-i\pi/4} \\
3e^{3i\pi/2}.\frac{1}{2}e^{-i\pi/2} &= \frac{3}{2}e^{i\pi} \\
\frac{6e^{-i\pi/6}}{3e^{-i\pi/4}} &= 2e^{i\pi(1/4-1/6)} = 2e^{i\pi/12} \\
i.re^{i\theta} &= e^{i\pi/2}.re^{i\theta} = re^{i(\theta+\pi/2)}
\end{aligned}$$

In a nutshell:
adding or subtracting complex numbers is easier in standard notation $z = a + ib$
multiplying or dividing is easier in exponential notation $z = re^{i\theta}$

3.5.3. The argument of a complex number

<u>motivation</u>

The angle θ in $z = re^{i\theta}$ is not unique, we could add multiples of 2π
This also makes it impossible to define $\ln(z)$ in $\mathbb{C}$ as the inverse of e^z
(see conditions for inverse: demand $e^z \neq e^{z'}$ if $z \neq z'$)

<u>definition</u>:
The argument $\arg(z)$ of a complex number z is the angle θ such that
(i) $z = re^{i\theta}$ with $r, \theta \in \mathbb{R}$ and $r \geq 0$
(ii) $-\pi < \theta \leq \pi$

Notes:

- One always has, by construction: $z = |z|\ e^{i\arg(z)}$
- The new condition that $\theta \in (-\pi, \pi]$ removes the previous ambiguity, leaving only *one* unique angle $\theta = \arg(z)$ to represent z

<u>definition</u>:
The natural logarithm $\ln(z)$ of a complex number $z \in \mathbb{C}$ can now be defined as follows

$$\ln(z) = \ln\left(|z|e^{i\arg(z)}\right) = \ln(|z|) + \ln\left(e^{i\arg(z)}\right) = \ln(|z|) + i\ \arg(z)$$

A common task is to write a complex number from standard into exponential form, i.e. to find $r = |z|$ and $\arg(z)$ when $z = a + ib$ is given

Three-step method for finding $|z|$ and $\arg(z)$ when z is given:

(i) calculate $|z|=r$ using $r^2=z\overline{z}$
(ii) calculate $e^{i\theta}$ using $e^{i\theta}=z/r$,
and find all solutions θ using $e^{i\theta}=\cos(\theta)+i\sin(\theta)$ (draw diagram)
(iii) determine which one obeys $-\pi<\theta\leq\pi$: this must be $\arg(z)$

Examples:

$z=i:$

$$r^2=z\overline{z}=i.(-i)=-i^2=1 \;\text{ hence }\; r=\sqrt{1}=1$$
$$e^{i\theta}=z/r=i/1=i, \;\text{ hence}$$
$$\cos(\theta)+i\sin(\theta)=i \;\Rightarrow\; \cos(\theta)=0 \text{ and } \sin(\theta)=1$$
$$\text{solutions}: \;\theta=\pi/2+2n\pi \;\; (n\in\mathbb{Z})$$
$$\theta\in(-\pi,\pi]: \;\; \theta=\pi/2 \;\text{ (i.e. } n=0), \;\text{ thus }\; \arg(z)=\pi/2$$

$z=-3+3i:$

$$r^2=(-3+3i)(-3-3i)=9^2+9^2=18 \;\text{ hence }\; r=\sqrt{18}=3\sqrt{2}$$
$$e^{i\theta}=z/r=\frac{1}{3\sqrt{2}}(-3+3i)=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, \;\text{ hence}$$
$$\cos(\theta)+i\sin(\theta)=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}} \;\Rightarrow\; \cos(\theta)=-\frac{1}{\sqrt{2}},\; \sin(\theta)=\frac{1}{\sqrt{2}}$$
$$\text{solutions}: \;\theta=3\pi/4+2n\pi \;\; (n\in\mathbb{Z})$$
$$\theta\in(-\pi,\pi]: \;\; \theta=3\pi/4 \;\text{ (i.e. } n=0), \;\text{ thus }\; \arg(z)=3\pi/4$$

$z=-\sqrt{3}-i:$

$$r^2=(-\sqrt{3}-i)(-\sqrt{3}+i)=3+1=4 \;\text{ hence }\; r=\sqrt{4}=2$$
$$e^{i\theta}=z/r=-\frac{1}{2}\sqrt{3}-\frac{1}{2}i, \;\text{ hence}$$
$$\cos(\theta)+i\sin(\theta)=-\frac{\sqrt{3}}{2}-\frac{i}{2} \;\Rightarrow\; \cos(\theta)=-\frac{\sqrt{3}}{2},\; \sin(\theta)=-\frac{1}{2}$$
$$\text{solutions}: \;\theta=7\pi/6+2n\pi \;\; (n\in\mathbb{Z})$$
$$\theta\in(-\pi,\pi]: \;\; \theta=-5\pi/6 \;\text{ (i.e. } n=-1), \;\text{ thus }\; \arg(z)=-5\pi/6$$

$z=-e^{2\pi i/3}:$

$$r^2=(-e^{2\pi i/3})(-e^{-2\pi i/3})=1, \;\text{ hence }\; r=\sqrt{1}=1$$
$$e^{i\theta}=z/r=-e^{2\pi i/3}=-\cos(2\pi/3)-i\sin(2\pi/3)=\frac{1}{2}-\frac{i\sqrt{3}}{2}, \;\text{ hence}$$
$$\cos(\theta)+i\sin(\theta)=\frac{1}{2}-\frac{i\sqrt{3}}{2} \;\Rightarrow\; \cos(\theta)=\frac{1}{2},\; \sin(\theta)=-\frac{\sqrt{3}}{2}$$

$$\text{solutions}: \quad \theta = -\pi/3 + 2n\pi \quad (n \in \mathbb{Z})$$

$$\theta \in (-pi, \pi]: \quad \theta = -\pi/3 \quad (\text{i.e. } n = 0), \quad \text{thus} \quad \arg(z) = -\pi/3$$

common pitfalls
(see last example):

- If $z = -e^{2\pi i/3}$ this does *not* imply that $r = |z| = -1$ and $\arg(z) = 2\pi/3$
  note that always $|z| \geq 0$
- If you arrive at $e^{i\theta} = -e^{2\pi i/3}$ this does *not* imply that there has been a mistake,
  note that $-1 = e^{i\pi}$, so we may write $-e^{2\pi i/3} = e^{i\pi}e^{2\pi i/3} = e^{5\pi i/3}$

3.6. De Moivre's Theorem

3.6.1. Statement and proof

theorem:

For all $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$\Big(\cos(\theta) + i\sin(\theta)\Big)^n = \cos(n\theta) + i\sin(n\theta)$$

proof 1: (via induction)

We will use $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.
Define $A_n = \Big(\cos(\theta) + i\sin(\theta)\Big)^n - \cos(n\theta) - i\sin(n\theta)$.
We have to prove that $A_n = 0$ for all $n \in \mathbb{N}$.

(i) Induction basis: $A_1 = \cos(\theta) + i\sin(\theta) - \cos(\theta) - i\sin(\theta) = 0$, so claim is true for $n = 1$

(ii) Induction step. We assume that $A_n = 0$ for some $n \in \mathbb{N}$. Now

$$\begin{aligned} A_{n+1} &= \Big(\cos(\theta) + i\sin(\theta)\Big)^{n+1} - \cos(n\theta + \theta) - i\sin(n\theta + \theta) \quad \text{use } A_n = 0: \\ &= (\cos(\theta) + i\sin(\theta)\Big)\Big(\cos(n\theta) + i\sin(n\theta)\Big) - \cos(n\theta + \theta) - i\sin(n\theta + \theta) \\ &= \cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta) + i\sin(\theta)\cos(n\theta) + i\cos(\theta)\sin(n\theta) \\ &\quad - \Big(\cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)\Big) - i\Big(\sin(n\theta)\cos(\theta) + \cos(n\theta)\sin(\theta)\Big) \\ &= 0 \end{aligned}$$

If $A_n = 0$ for some n, then also $A_{n+1} = 0$. Hence, in combination with the basis,
we have now shown that $A_n = 0$ for all $n \in \mathbb{N}$. This completes the proof.

proof 2:

$\Big(\cos(\theta) + i\sin(\theta)\Big)^n = (e^{i\phi})^n = e^{in\phi} = \cos(n\theta) + i\sin(n\theta)$

3.6.2. Applications

First application:
easy derivation of identities for trigonometric functions of multiple angles

- $n = 2$:

$$\cos(2\theta) + i\sin(2\theta) = [\cos(\theta) + i\sin(\theta)]^2 = \cos^2(\theta) - \sin^2(\theta) + 2i\sin(\theta)\cos(\theta)$$

  Real and imaginary parts on both sides must be equal, so

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \end{aligned}$$

- $n = 3$:

$$\begin{aligned} \cos(3\theta) + i\sin(3\theta) &= [\cos(\theta) + i\sin(\theta)]^3 \\ &= \Big(\cos^2(\theta) - \sin^2(\theta) + 2i\sin(\theta)\cos(\theta)\Big)\,(\cos(\theta) + i\sin(\theta)) \\ &= \cos^3(\theta) - \cos(\theta)\sin^2(\theta) + 2i\sin(\theta)\cos^2(\theta) \\ &\quad + i\Big(\cos^2(\theta)\sin(\theta) - \sin^3(\theta) + 2i\cos(\theta)\sin^2(\theta)\Big) \\ &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + 3i\sin(\theta)\cos^2(\theta) - i\sin^3(\theta) \end{aligned}$$

  Real and imaginary parts on both sides must be equal, so

$$\begin{aligned} \cos(3\theta) &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) \\ \sin(3\theta) &= 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta) \end{aligned}$$

  Using $\sin^2(\theta) + \cos^2(\theta) = 1$, this can also be written as

$$\begin{aligned} \cos(3\theta) &= \cos(\theta)\Big[4\cos^2(\theta) - 3\Big] \\ \sin(3\theta) &= \sin(\theta)\Big[4\cos^2(\theta) - 1\Big] \end{aligned}$$

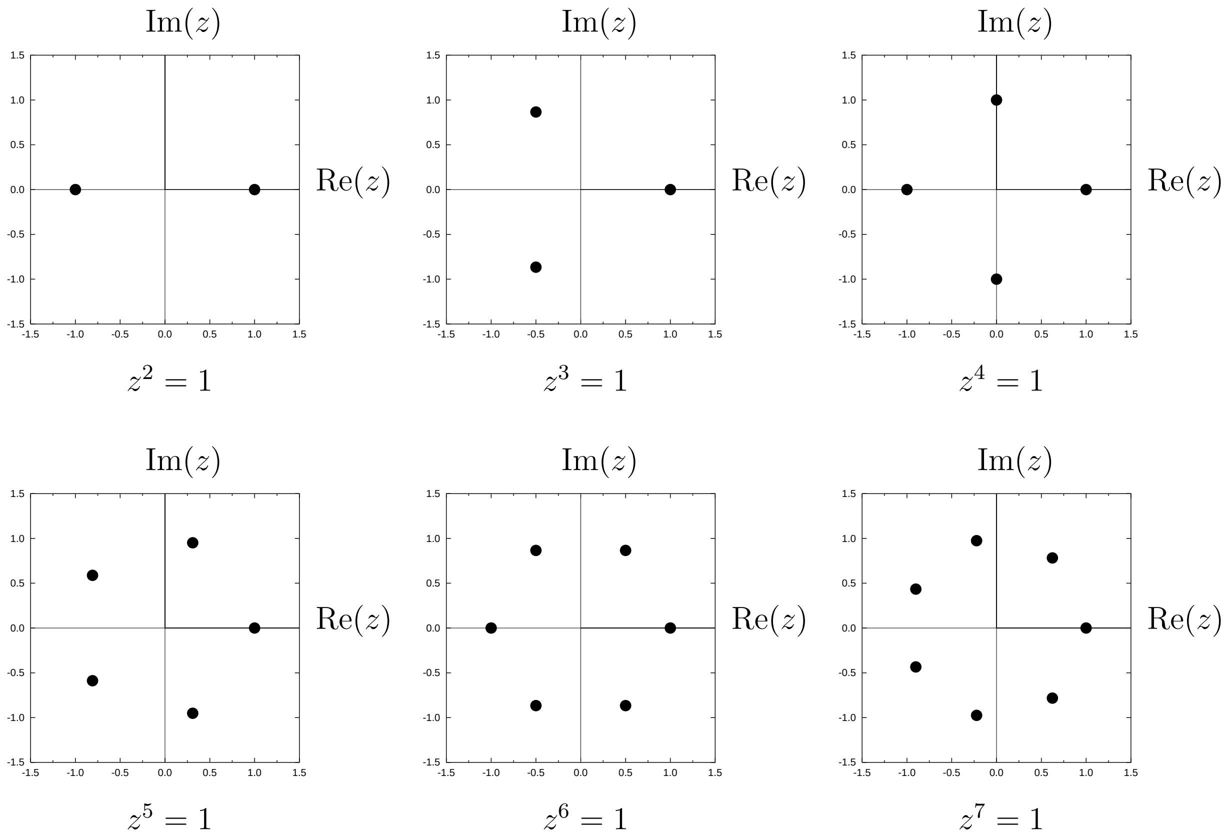
Second application:
finding the roots of unity, i.e. the n solutions of $z^n = 1$ (integer n)

- We know that $|z| = 1$, due to $1 = |z^n|^n = |z|^n$, so we may put $z = e^{i\theta}$
  For any integer $m = 0, 1, 2, \ldots$ we may write $1 = \cos(2m\pi) + i\sin(2m\pi)$
  Our equation then becomes $(e^{i\theta})^n = \cos(2m\pi) + i\sin(2m\pi)$, or $e^{in\theta} = e^{2mi\pi}$
- It follows that for *every* integer m we have a solution $\theta = 2\pi m/n$,
  i.e. a complex root $z = e^{2\pi i m/n}$
  Note, finally: for $m \geq n$ we will generate solutions already found earlier

Hence: the n solutions of $z^n = 1$ are given by

$$z = e^{2\pi i m/n} \qquad \text{for} \quad m = 0, 1, 2, \ldots, n-1$$

For example:



3.7. Complex equations

Complex equations are equations that can be reduced to the general form $F(z,\overline{z})=0$, where F denotes some function of z and $\overline{z}$. Note: F can involve $|z|$, since $|z|^2=z\overline{z}$. Solving a complex equation means finding all $z\in\mathbb{C}$ with the property $F(z,\overline{z})=0$.

We have already encountered examples:

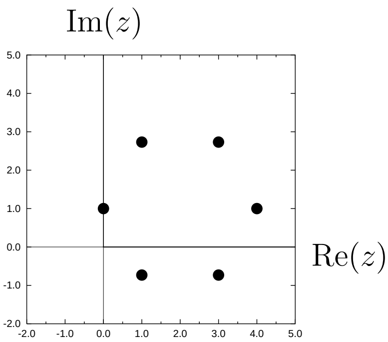
(i) $z\overline{z}-1=0$:
  Here the solution set is the unit circle in the complex plane (i.e. infinitely many)

(ii) $z^n-1=0$:
  Here the solution set is a discrete set of n points (see previous section)

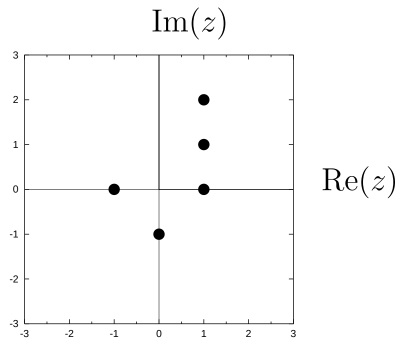
Note:
the solution sets in the complex plane of complex equations
can be more diverse than those of real equations, or than the previous examples

- Lines in the complex plane:
  these are found as solutions of linear equations, i.e.
  $uz + v\overline{z} + w = 0$ with $u, v, w \in \mathbb{C}$ (check this)

- Discrete sets of points that are not arranged around the origin:



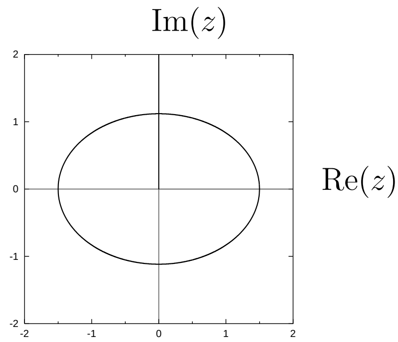
$$(z - 2 - i)^6 - 2 = 0$$



$$z^5 - 2(1+i)z^4 + (1+i)z^3 + (i-1)z^2 - (2+i)z + i + 3 = 0$$

- Ellipses:
  these are solutions of equations of the type
  $|z - u| + |z - w| - R = 0$
  with $u, w \in \mathbb{C}$ and $R \in \mathbb{R}^+$

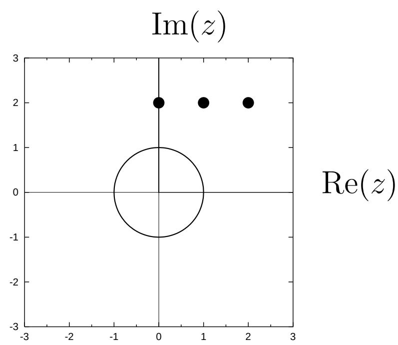
  (u and w will be the foci of the ellipse)



$$|z - 1| + |z + 1| - 3 = 0$$

- But also unions of isolated points and curves

- And more ...



$$(z\overline{z} - 1)(z^3 - 2iz^2 - 2(2i+1)z - 20 - 8i) = 0$$

4. Trigonometric and hyperbolic functions

4.1. Definitions of trigonometric functions

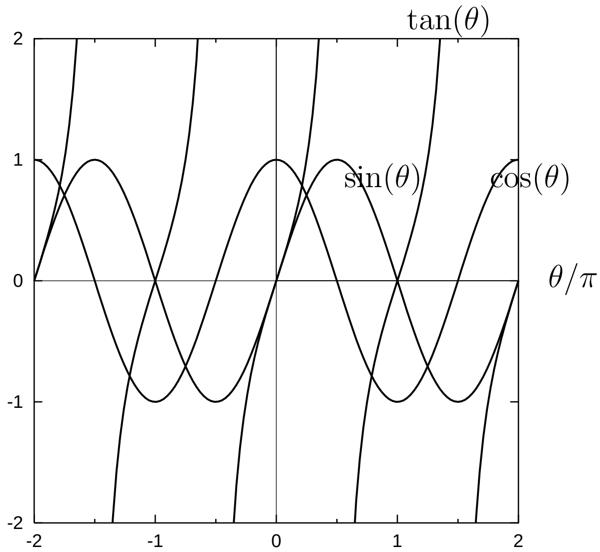
4.1.1. Definition of sine and cosine

We need only define $\sin(\theta)$ and $\cos(\theta)$, since all other trigonometric functions are simply combinations of these elementary two.
There are different but mathematically equivalent options:

Option I. geometric definition:

Consider a half-line with its one end-point in the origin. Choose it first to lie along the positive x-axis, then rotate it anti-clockwise around the origin; call the rotation angle θ (in radians). Find the coordinates $(X(\theta), Y(\theta))$ of the point where the half-line intersects the unit circle. Now define

$$\cos(\theta) = X(\theta)$$
$$\sin(\theta) = Y(\theta)$$



Option II. definition via differential equations:

We could also define trigonometric functions as the solutions of the following equations, with specific initial values:

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$$
$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta)$$
$$\cos(0) = 1, \quad \sin(0) = 0$$

Option III. analytic definition:

Here we start by defining the function e^z for any $z \in \mathbb{C}$ via a series, and subsequently define trigonometric functions via

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) \qquad \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$

(note: this will also generalize trigonometric functions to complex arguments!)
The exponential function takes the form

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

(note: one defines $0^0 = 1$). We turn later in detail to existence and convergence questions for infinite series. For now we assume (rightly, as will turn out) that this infinite sum is always finite and well-behaved.
One then finds

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

proof:

Let us start with $\sin(z)$:

$$\sin(z) = \frac{1}{2i}\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \frac{1}{2i}\sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} = \frac{1}{i}\sum_{n=0}^{\infty} \frac{i^n z^n}{n!}\frac{1}{2}\left(1-(-1)^n\right)$$

Note:

(i) that $\frac{1}{2}(1-(-1)^n) = 0$ for even n, and $\frac{1}{2}(1-(-1)^n) = 1$ for odd n
hence in the sum we only retain term with odd n

(ii) that the odd values of n can be written as $n = 2k+1$, with $k = 0, 1, 2, 3, \ldots$

$$\sin(z) = \frac{1}{i}\sum_{n\ \mathrm{odd}}^{\infty} \frac{i^n z^n}{n!} = \frac{1}{i}\sum_{k=0}^{\infty} \frac{i^{2k+1} z^{2k+1}}{(2k+1)!}$$
$$= \sum_{k=0}^{\infty} \frac{(i^2)^k z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

Next we turn to $\cos(z)$:

$$\cos(z) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \frac{1}{2i}\sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!}\frac{1}{2}\left(1+(-1)^n\right)$$

Note:

(i) that $\frac{1}{2}(1+(-1)^n) = 1$ for even n, and $\frac{1}{2}(1+(-1)^n) = 0$ for odd n
hence in the sum we only retain term with even n

(ii) that the even values of n can be written as $n = 2k$, with $k = 0, 1, 2, 3, \ldots$

$$\cos(z) = \sum_{n\ \mathrm{even}}^{\infty} \frac{i^n z^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k} z^{2k}}{(2k)!}$$
$$= \sum_{k=0}^{\infty} \frac{(i^2)^k z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

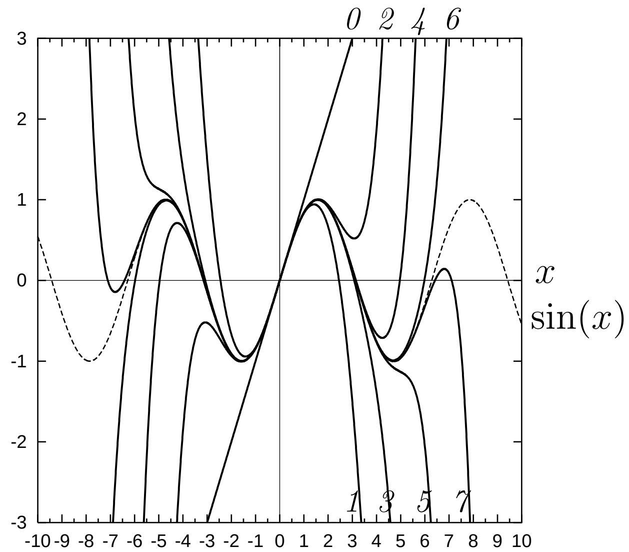


Figure 1. Building $\sin(x)$ as a power series, by taking more and more terms in the summation. Dashed: $\sin(x)$. Solid: $f(x) = \sum_{k=0}^{N}(-1)^k x^{2k+1}/(2k+1)!$ for different choices of N (values of N are indicated in italics). As N increases: $f(x)$ starts to resemble $\sin(x)$ more and more, but $f(x)$ is only fully *identical* to $\sin(x)$ for $N \to \infty$.

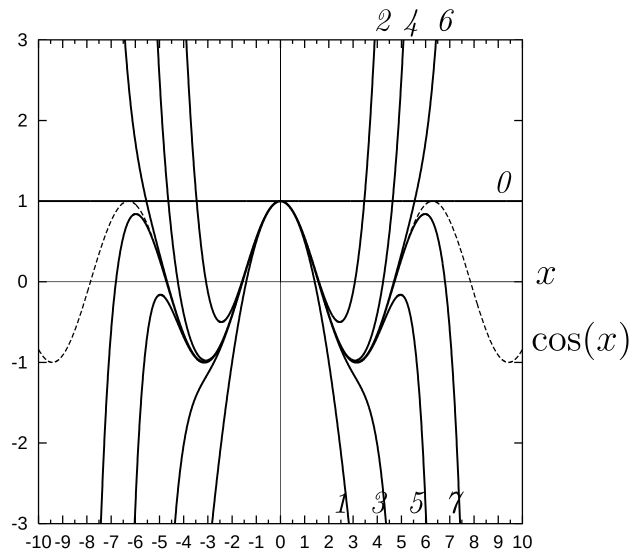


Figure 2. Building $\cos(x)$ as a power series, by taking more and more terms in the summation. Dashed: $\cos(x)$. Solid: $f(x) = \sum_{k=0}^{N}(-1)^k x^{2k}/(2k)!$ for different choices of N (values of N are indicated in italics). As N increases: $f(x)$ starts to resemble $\cos(x)$ more and more, but $f(x)$ is only fully *identical* to $\sin(x)$ for $N \to \infty$.

4.1.2. Elementary values

These can all be extracted from either (i) suitably chosen triangles, and/or (ii) projections of special points on the unit circle onto (x, y)-axes, or (iii) simple transformations to reduce to one of the previous two classes:

- $\theta = 0$, inspect projections of point on unit circle,
  $\cos(0) = 1$, $\sin(0) = 0$
- $\theta = \pi/4$, inspect inspect projections of point on unit circle, use $\sin^2(\theta) + \cos^2(\theta) = 1$
  $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$
- $\theta = \pi/6$: cut a triangle with three equal sides in half and pick a suitable corner,
  $\sin(\pi/6) = \frac{1}{2}$, $\cos(\pi/6) = \frac{1}{2}\sqrt{3}$
- $\theta = \pi/3$: cut a triangle with three equal sides in half and pick a suitable corner,
  $\sin(\pi/3) = \frac{1}{2}\sqrt{3}$, $\cos(\pi/3) = 1/2$
- $\theta = \pi/2$: inspect projections of point on unit circle,
  $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$

4.1.3. Related functions

These are just *short-hands* for frequently occurring combinations of sine and cosine:

- <u>tangent</u>: $\tan(\theta) = \sin(\theta)/\cos(\theta)$

  (i) not defined for $\theta = \pi/2 + n\pi$ with $n \in \mathbb{Z}$, where $\cos(\theta) = 0$
  (ii) $\tan(\theta + \pi) = \tan(\theta)$ for all $\theta \in \mathbb{R}$
  since $\sin(\theta + \pi) = -\sin(\theta)$ and $\cos(\theta + \pi) = -\cos(\theta)$

- <u>cotangent</u>: $\cot(\theta) = \cos(\theta)/\sin(\theta)$

  (i) not defined for $\theta = n\pi$ with $n \in \mathbb{Z}$, where $\sin(\theta) = 0$
  (ii) $\cot(\theta + \pi) = \cot(\theta)$ for all $\theta \in \mathbb{R}$
  since $\sin(\theta + \pi) = -\sin(\theta)$ and $\cos(\theta + \pi) = -\cos(\theta)$

- <u>secant</u>: $\sec(\theta) = 1/\cos(\theta)$

  (i) not defined for $\theta = \pi/2 + n\pi$ with $n \in \mathbb{Z}$, where $\cos(\theta) = 0$
  (ii) $\sec(\theta + 2\pi) = \sec(\theta)$ for all $\theta \in \mathbb{R}$
  since $\cos(\theta + 2\pi) = \cos(\theta)$

- <u>cosecant</u>: $\mathrm{cosec}(\theta) = 1/\sin(\theta)$

  (i) not defined for $\theta = n\pi$ with $n \in \mathbb{Z}$, where $\sin(\theta) = 0$
  (ii) $\mathrm{cosec}(\theta + 2\pi) = \mathrm{cosec}(\theta)$ for all $\theta \in \mathbb{R}$
  since $\sin(\theta + 2\pi) = \sin(\theta)$

4.1.4. Inverse trigonometric functions

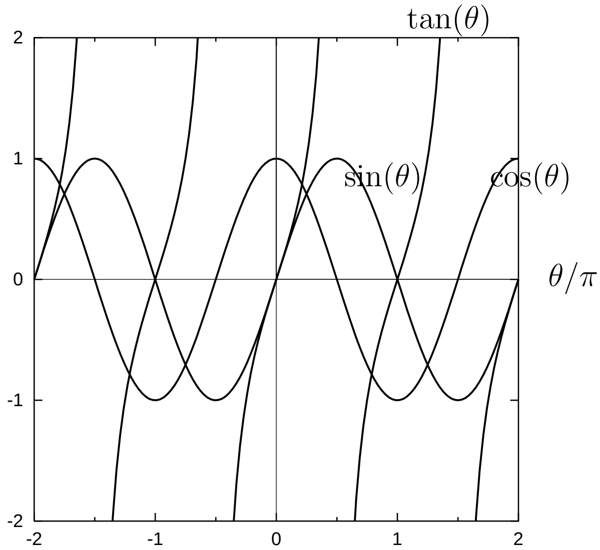
Recall our definitions:

- The inverse f^{-1} of a function $f : D \to R$ is defined by:

$$f^{-1} : R \to D$$
$$f^{-1}(f(x)) = x \quad \text{for all } x \in D$$
$$f(f^{-1}(x)) = x \quad \text{for all } x \in R$$

- A function $f : D \to R$ is invertible if and only if:

  $f(x_1) \neq f(x_2)$ for any two $x_1, x_2 \in D$ with $x_1 \neq x_2$



Inspect graphs of trigonometric functions:

(i) problem:
there are *many* $\theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1 \neq \theta_2$ such that $\sin(\theta_1) = \sin(\theta_2)$ ...
(same is true for cosine and tangent)

(ii) hence:

one can only define inverse trigonometric functions by limiting their domains to sets where any two distinct angles will give *different* function values!

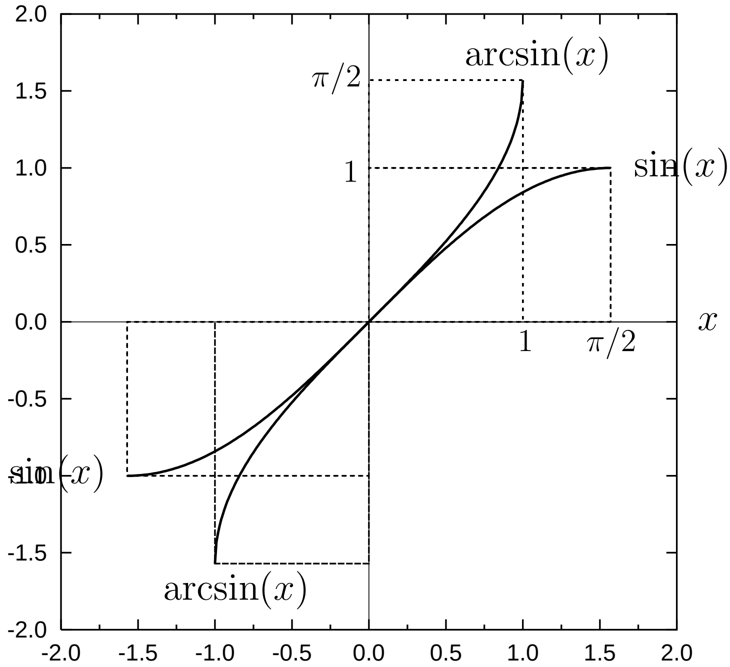
- <u>definition of the inverse of $\sin(\theta)$</u>: $\arcsin(x)$

need interval D that satisfies
(i) $\sin(\theta_1) \neq \sin(\theta_2)$ for all $\theta_1, \theta_2 \in D$ with $\theta_1 \neq \theta_2$
(ii) the range corresponding to D covers all possible values of sine, $R = [-1, 1]$

Answer: $D = [-\pi/2, \pi/2]$, so

$$\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$$
$$\arcsin(\sin(\theta)) = \theta \quad \text{for all } \theta \in [-\pi/2, \pi/2]$$
$$\sin(\arcsin(x)) = x \quad \text{for all } x \in [-1, 1]$$

$\arcsin(x)$ in words:
gives the angle $\theta \in [-\pi/2, \pi/2]$ such that $\sin(\theta) = x$



Special values:

$\arcsin(0) = 0$, $\arcsin(1/\sqrt{2}) = \pi/4$, $\arcsin(\frac{1}{2}) = \pi/6$, $\arcsin(\frac{1}{2}\sqrt{3}) = \pi/3$, $\arcsin(1) = \pi/4$
negative values via: $\arcsin(-x) = -\arcsin(x)$

- <u>definition of the inverse of $\cos(\theta)$</u>: $\arccos(x)$

need interval D that satisfies
(i) $\cos(\theta_1) \neq \cos(\theta_2)$ for all $\theta_1, \theta_2 \in D$ with $\theta_1 \neq \theta_2$
(ii) the range corresponding to D covers all possible values of cosine: $R = [-1, 1]$

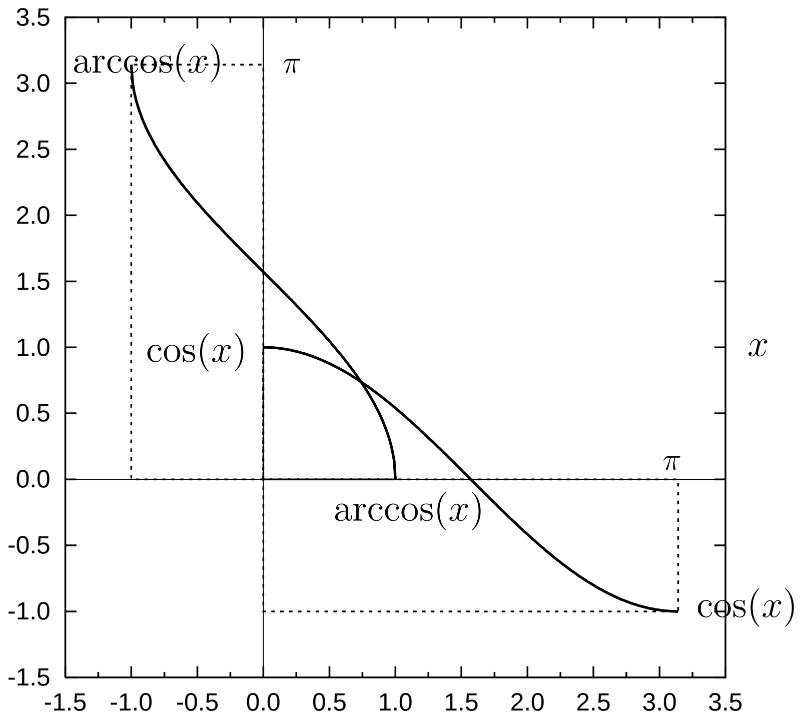
Answer: $D = [0, \pi]$, so

$$\arccos : [-1, 1] \to [0, \pi]$$

$$\arccos(\cos(\theta)) = \theta \quad \text{for all } \theta \in [0, \pi]$$

$$\cos(\arccos(x)) = x \quad \text{for all } x \in [-1, 1]$$

$\arccos(x)$ in words:
gives the angle $\theta \in [0, \pi]$ such that $\cos(\theta) = x$



Special values:

$\arccos(0) = \pi/2$, $\arccos(\frac{1}{2}) = \pi/3$, $\arccos(1/\sqrt{2}) = \pi/4$, $\arccos(\frac{1}{2}\sqrt{3}) = \pi/6$, $\arccos(1) = 0$
negative values of x via: $\arccos(-x) = \pi - \arccos(x)$

- <u>definition of the inverse of $\tan(\theta)$</u>: $\arctan(x)$

need interval D that satisfies
(i) $\tan(\theta_1) \neq \tan(\theta_2)$ for all $\theta_1, \theta_2 \in D$ with $\theta_1 \neq \theta_2$
(ii) the range corresponding to D covers all possible values of tangent, $R = \mathbb{R}$

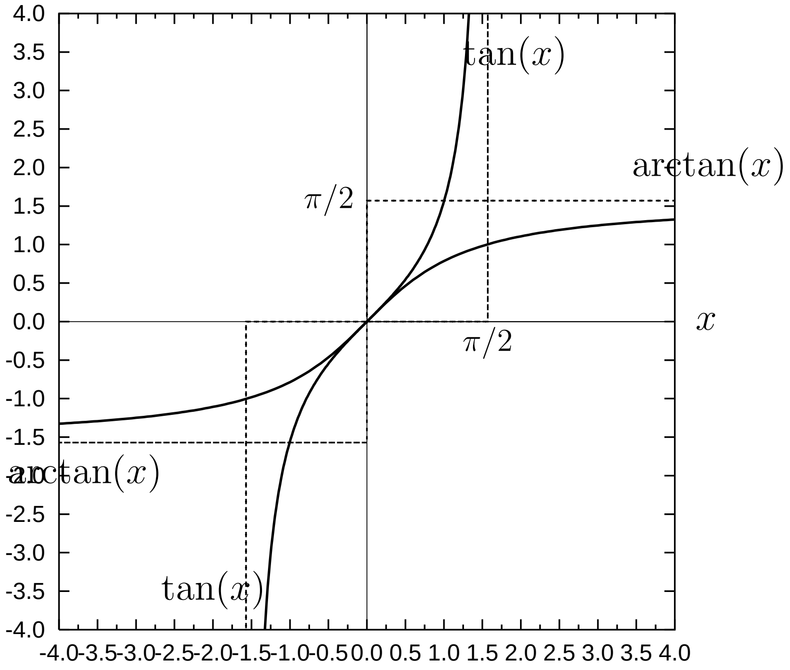
Answer: $D = (-\pi/2, \pi/2)$, so

$$\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$$

$$\arctan(\tan(\theta)) = \theta \quad \text{for all } \theta \in (-\pi/2, \pi/2)$$

$$\tan(\arctan(x)) = x \quad \text{for all } x \in \mathbb{R}$$

$\arctan(x)$ in words:
gives the angle $\theta \in (-\pi/2, \pi/2)$ such that $\tan(\theta) = x$



Special values:

$\arctan(0) = 0$, $\arctan(1/\sqrt{3}) = \pi/6$, $\arctan(1) = \pi/4$, $\arctan(\sqrt{3}) = \pi/3$
negative values via: $\arctan(-x) = -\arctan(x)$

4.2. Elementary properties of trigonometric functions

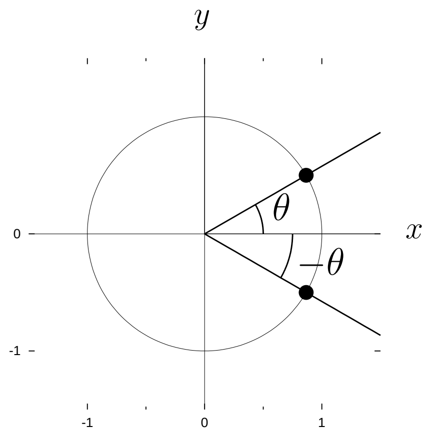
4.2.1. Symmetry properties

The unit circle is invariant under:

(i) reflection in the y-axis, i.e. $(x, y) \to (-x, y)$
(ii) reflection in the x-axis, i.e. $(x, y) \to (x, -y)$
(iii) reflection in the origin, i.e. $(x, y) \to (-x, -y)$
(iv) reflection in the line $x = y$, i.e. $(x, y) \to (y, x)$

Symmetries have implications for the values of trigonometric functions
(note the geometric definition of sine and cosine):

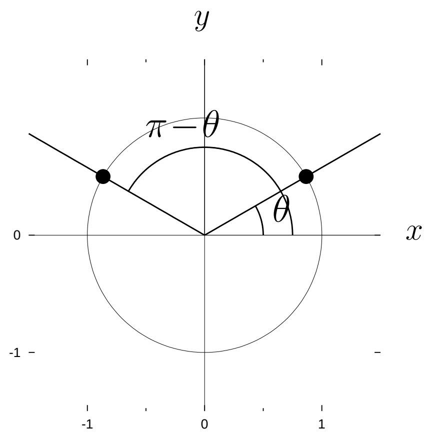
- reflection in x-axis, i.e. $\theta \to -\theta$:



We see that:

$\sin(-\theta) = -\sin(\theta)$
$\cos(-\theta) = \cos(\theta)$

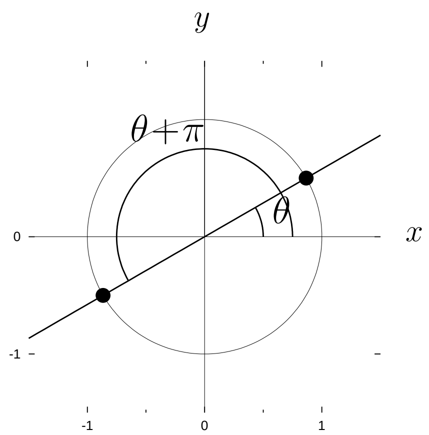
- reflection in y-axis, i.e. $\theta \to \pi - \theta$:



We see that:

$\sin(\pi - \theta) = \sin(\theta)$
$\cos(\pi - \theta) = -\cos(\theta)$

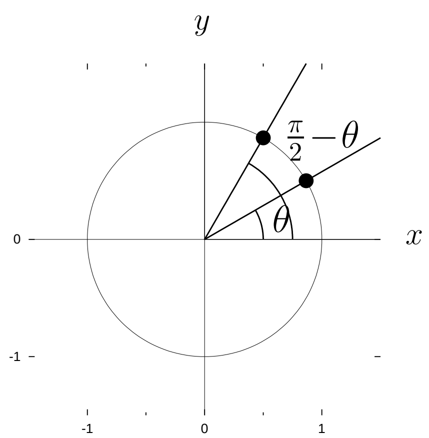
- reflection in origin, i.e. $\theta \to \theta + \pi$:



We see that:

$\sin(\theta + \pi) = -\sin(\theta)$
$\cos(\theta + \pi) = -\cos(\theta)$

- reflection in line $x = y$, i.e. $\theta \to \pi/2 - \theta$:



We see that:

$\sin(\pi/2 - \theta) = \cos(\theta)$
$\cos(\pi/2 - \theta) = \sin(\theta)$

4.2.2. Addition formulae

Trigonometric functions of sums or differences of angles:

claim: for all $\theta, \phi \in \mathbb{R}$ one has

$$\begin{aligned}\sin(\theta+\phi) &= \sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)\\ \cos(\theta+\phi) &= \cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)\end{aligned}$$

proofs:

subtract left- and right-hand sides of the two identities,
and show that the result is zero
use definitions in terms of exponentials:

$$\begin{aligned}\text{LHS1}-\text{RHS1} &= \sin(\theta+\phi)-\sin(\theta)\cos(\phi)-\cos(\theta)\sin(\phi)\\ &= \frac{1}{2i}(e^{i(\theta+\phi)}-e^{-i(\theta+\phi)})-\frac{1}{4i}(e^{i\theta}-e^{-i\theta})(e^{i\phi}+e^{-i\phi})-\frac{1}{4i}(e^{i\theta}+e^{-i\theta})(e^{i\phi}-e^{-i\phi})\\ &= \frac{1}{2i}e^{i(\theta+\phi)}-\frac{1}{2i}e^{-i(\theta+\phi)}-\frac{1}{4i}\left(e^{i(\theta+\phi)}+e^{i(\theta-\phi)}-e^{i(\phi-\theta)}-e^{-i(\theta+\phi}\right)\\ &\quad -\frac{1}{4i}\left(e^{i(\theta+\phi)}-e^{i(\theta-\phi)}+e^{i(\phi-\theta)}-e^{-i(\theta+\phi)}\right)\\ &= \frac{1}{2i}e^{i(\theta+\phi)}-\frac{1}{2i}e^{-i(\theta+\phi)}-\frac{1}{4i}\left(2e^{i(\theta+\phi)}-2e^{-i(\theta+\phi}\right)=0\end{aligned}$$

$$\begin{aligned}\text{LHS2}-\text{RHS2} &= \cos(\theta+\phi)-\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi)\\ &= \frac{1}{2}(e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}-\frac{1}{4}(e^{i\theta}+e^{-i\theta})(e^{i\phi}+e^{-i\phi})-\frac{1}{4}(e^{i\theta}-e^{-i\theta})(e^{i\phi}-e^{-i\phi})\\ &= \frac{1}{2}e^{i(\theta+\phi)}+\frac{1}{2}e^{-i(\theta+\phi)}-\frac{1}{4}\left(e^{i(\theta+\phi)}+e^{i(\theta-\phi)}+e^{-i(\theta-\phi)}+e^{-i(\theta+\phi)}\right)\\ &\quad -\frac{1}{4}\left(e^{i(\theta+\phi)}-e^{i(\theta-\phi)}-e^{-i(\theta-\phi)}+e^{-i(\theta+\phi)}\right)\\ &= \frac{1}{2}e^{i(\theta+\phi)}+\frac{1}{2}e^{-i(\theta+\phi)}-\frac{1}{4}\left(2e^{i(\theta+\phi)}+2e^{-i(\theta+\phi)}\right)=0\end{aligned}$$

This completes the proofs.

From the formulae for sine and cosine follow also:
(don't memorize, but derive when needed!)

$$\begin{aligned}\tan(\theta+\phi) &= \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}=\frac{\sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)}{\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)}\\ &= \frac{\sin(\theta)/\cos(\theta)\ \ +\ \ \sin(\phi)/\cos(\phi)}{1-\sin(\theta)\sin(\phi)/\cos(\theta)\cos(\phi)}\\ &= \frac{\tan(\theta)+\tan(\phi)}{1-\tan(\theta)\tan(\phi)}\end{aligned}$$

<u>observation</u>:

we rely on the property $e^{z+w} = e^z.e^w$
interesting to prove this property
from the series representation of e^z alone!

use Newton's binomial formula and $(\sum_m a_m)(\sum_n b_n) = \sum_{n,m} a_m b_n$:

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

Inspect these summations closer:
we have ultimately $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots,,$
but we restrict ourselves to those combinations (n, k) with $k \leq n$
Hence we may also write :

$$e^{z+w} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n!} \binom{n}{k} z^k w^{n-k} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n!} \frac{n!}{k!(n-k)!} z^k w^{n-k}$$
$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} z^k w^{n-k}$$

Finally, switch from the index n to $\ell = n - k$, so $\ell = 0, 1, 2, \ldots$:

$$e^{z+w} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{k!} \frac{1}{\ell!} z^k w^\ell = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \right) \left(\sum_{\ell=1}^{\infty} \frac{w^\ell}{\ell!} \right) = e^z.e^w$$

4.2.3. Applications of addition formulae

- writing products of trigonometric functions as sums:

$$\cos(\theta)\cos(\phi) = \frac{1}{2}\Big(\cos(\theta+\phi) + \cos(\theta-\phi) \Big)$$
$$\sin(\theta)\sin(\phi) = \frac{1}{2}\Big(\cos(\theta-\phi) - \cos(\theta+\phi) \Big)$$
$$\sin(\theta)\cos(\phi) = \frac{1}{2}\Big(\sin(\theta+\phi) + \sin(\theta-\phi) \Big)$$

  proofs: trivial
  just insert the appropriate addition formulae in the right-hand sides

- recovering formulae for double angles:
  just choose $\phi = \theta$ in the addition formulae

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
$$\tan(2\theta) = 2\tan(\theta)/[1 - \tan^2(\theta)]$$

- half-angle formulae:

$$
\begin{aligned}
\cos(\alpha)+\cos(\beta) =&\ \cos(\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha-\beta))+\cos(\frac{1}{2}(\alpha+\beta)-\frac{1}{2}(\alpha-\beta))\\
=&\Big(\cos(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))-\sin(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
&+\Big(\cos(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))+\sin(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
=&\,2\cos(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha)+\sin(\beta) =&\ \sin(\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha-\beta))+\sin(\frac{1}{2}(\alpha+\beta)-\frac{1}{2}(\alpha-\beta))\\
=&\Big(\sin(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))+\cos(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
&+\Big(\sin(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))-\cos(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
=&\,2\sin(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))
\end{aligned}
$$

$$
\begin{aligned}
\cos(\alpha)-\cos(\beta) =&\ \cos(\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha-\beta))-\cos(\frac{1}{2}(\alpha+\beta)-\frac{1}{2}(\alpha-\beta))\\
=&\Big(\cos(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))-\sin(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
&-\Big(\cos(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))+\sin(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
=&\ -2\sin(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha)-\sin(\beta) =&\ \sin(\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha-\beta))-\sin(\frac{1}{2}(\alpha+\beta)-\frac{1}{2}(\alpha-\beta))\\
=&\Big(\sin(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))+\cos(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
&-\Big(\sin(\frac{1}{2}(\alpha+\beta))\cos(\frac{1}{2}(\alpha-\beta))-\cos(\frac{1}{2}(\alpha+\beta))\sin(\frac{1}{2}(\alpha-\beta))\Big)\\
=&\,2\sin(\frac{1}{2}(\alpha-\beta))\cos(\frac{1}{2}(\alpha+\beta))
\end{aligned}
$$

- Claim: one can always write linear combinations $a\cos(\theta)+b\sin(\theta)$ in the form $c\sin(\theta+\alpha)$ with some suitable $c,\alpha\in\mathbb{R}$ and with $c\geq 0$
  (let's disregard the trivial case $a=b=0$)

  Proof & construction in three steps:

(i) First note that

$$c\sin(\theta+\alpha) = c\sin(\theta)\cos(\alpha) + c\cos(\theta)\sin(\alpha)$$

Hence we seek c and θ such that

$$a/c = \sin(\alpha) \qquad b/c = \cos(\alpha)$$

Use $\sin^2(\alpha)+\cos^2(\alpha)=1$: $a^2+b^2=c^2$, so $c=\sqrt{a^2+b^2}$

(ii) Next: find α from

$$a/c = \sin(\alpha) \qquad b/c = \cos(\alpha)$$

hence $\tan(\alpha)=a/b$, so we find $\alpha = \arctan(a/b)+n\pi$ with $n\in\mathbb{Z}$
(note: $\arctan(\alpha)$ must be in $(\pi/2,\pi/2)$, but there is no reason why α should be there!)

(iii) Finally: determine n from $a/c=\sin(\alpha)$ and $b/c=\cos(\alpha)$
Just check the quadrant of the solution α in the plane, by inspecting signs.
Note: $\arctan(\alpha)\in(-\pi/2,\pi/2)$ is always in quadrant 1 or 4:

$$\begin{aligned}
b/c=0: &\quad \arctan(\alpha)\ \text{doesn't exist, here}\ \cos(\alpha)=0\ \text{so}\\
&\quad \alpha=\pi/2+2n\pi\quad(n\in\mathbb{Z})\ \text{if}\ a>0\\
&\quad \alpha=3\pi/2+2n\pi\quad(n\in\mathbb{Z})\ \text{if}\ a<0\\
b/c>0: &\quad \text{quadrant 1 or 4,}\quad \text{so}\ \alpha=\arctan(a/b)+2n\pi\quad(n\in\mathbb{Z})\\
b/c<0: &\quad \text{quadrant 2 or 3,}\quad \text{so}\ \alpha=\arctan(a/b)+\pi+2n\pi\quad(n\in\mathbb{Z})
\end{aligned}$$

4.2.4. The $\tan(\theta/2)$ *formulae*

Objective:
write all trigonometric functions in terms of $t=\tan(\frac{1}{2}\theta)$
(useful later in integrals)

$$\tan(\theta) = \frac{2\tan(\frac{1}{2}\theta)}{1-\tan^2(\frac{1}{2}\theta)} = \frac{2t}{1-t^2}$$

$$\begin{aligned}
\cos(\theta) &= \cos^2(\frac{1}{2}\theta)-\sin^2(\frac{1}{2}\theta) = 2\cos^2(\frac{1}{2}\theta)-1 = \frac{2}{\cos^{-2}(\frac{1}{2}\theta)}-1\\
&= \frac{2}{[\sin^2(\frac{1}{2}\theta)+\cos^2(\frac{1}{2}\theta)]\cos^{-2}(\frac{1}{2}\theta)}-1 = \frac{2}{\tan^2(\frac{1}{2}\theta)+1}-1\\
&= \frac{2}{1+t^2}-\frac{1+t^2}{1+t^2} = \frac{1-t^2}{1+t^2}\\
\sin(\theta) &= \cos(\theta)\tan(\theta) = \frac{1-t^2}{1+t^2}\frac{2t}{1-t^2} = \frac{2t}{1+t^2}
\end{aligned}$$

4.3. Definitions of hyperbolic functions

4.3.1. Definition of hyperbolic sine and hyperbolic cosine

Option I. definition via differential equations:

We can define the hyperbolic sine $\sinh(z)$ and hyperbolic cosine $\cosh(z)$ as the solutions of the following equations, with specific initial values:

$$\frac{d}{dz}\sinh(z) = \cosh(z), \qquad \frac{d}{dz}\cosh(z) = \sinh(z), \qquad \cosh(0) = 1, \qquad \sinh(0) = 0$$

(note:
difference with previous eqns defining sine and cosine
is only in a minus sign in the second eqn!)

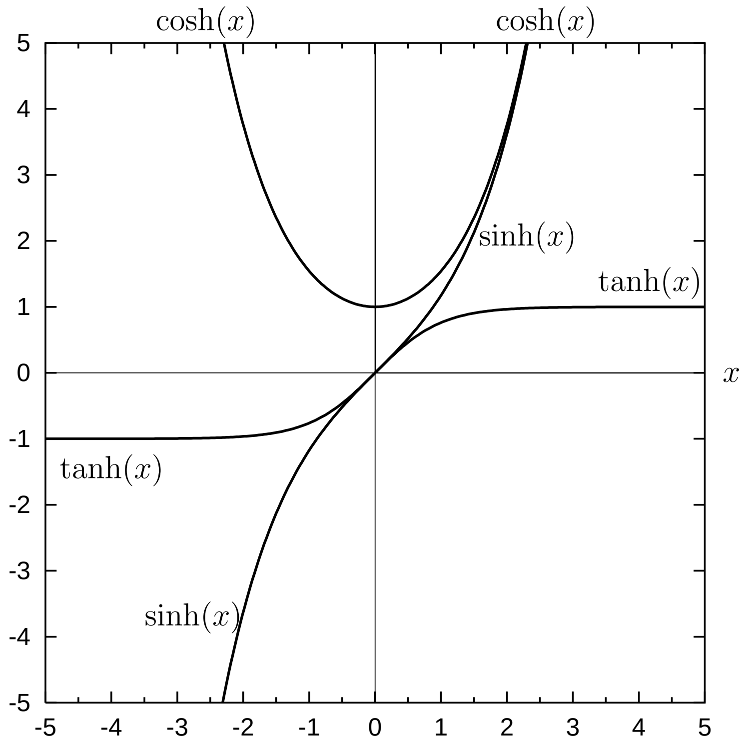
Option II. direct analytic definition

Having already defined e^z
by the series $e^z = \sum_{n=0}^{\infty} z^n/n!$
we define hyperbolic functions via

$$\cosh(z) = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh(z) = \frac{1}{2}(e^z - e^{-z})$$

(this also generalizes
hyperbolic functions
to complex arguments)



Related functions:

hyperbolic tangent : $\tanh(z) = \sinh(z)/\cosh(z)$

hyperbolic cotangent : $\coth(z) = \cosh(z)/\sinh(z)$

hyperbolic secant : $\mathrm{sech}(z) = 1/\cosh(z)$

hyperbolic cosecant : $\mathrm{cosech}(z) = 1/\sinh(z)$

4.3.2. General properties and special values

Properties involving both sinh and cosh:

- One immediately confirms from the direct analytic definition:

$$\frac{d}{dz}\sinh(z) = \cosh(z) \qquad \frac{d}{dz}\cosh(z) = \sinh(z)$$

- For any $z \in \mathbb{C}$: $\cosh^2(z) - \sinh^2(z) = 1$

  proof:

$$\begin{aligned}\cosh^2(z) - \sinh^2(z) &= \Big(\frac{1}{2}(e^z + e^{-z})\Big)^2 - \Big(\frac{1}{2}(e^z - e^{-z})\Big)^2 \\ &= \frac{1}{4}\Big((e^{2z} + 2 + e^{-2z})\Big) - \frac{1}{4}\Big(e^{2z} - 2 + e^{-2z})\Big) \\ &= \frac{1}{4}\Big((e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z})\Big) = 1\end{aligned}$$

  Consequence:
  if for $z \in \mathbb{R}$ the two are regarded as coordinates (X, Y) in a plane, i.e. $X = \cosh(z)$ and $Y = \sinh(z)$, then the possible points (X, Y) define the branches of a hyperbole $X^2 - Y^2 = 1$ (hence the name!)

Properties of sinh:

- $\sinh(-z) = -\sin(z)$

  proof: follows directly from definition,

$$\sinh(-z) = \frac{1}{2}(e^{-z} - e^z) = -\frac{1}{2}(e^z - e^{-z}) = -\sinh(z)$$

- for $z \in \mathbb{R}$: $\sinh(z)$ increases monotonically

  proof: differentiate the analytic definition,
  using $\frac{d}{dz}e^{az} = ae^{az}$,

$$\frac{d}{dz}\sinh(z) = \cosh(z) = \frac{1}{2}(e^z + e^{-z}) > 0$$

- consider $z \in \mathbb{R}$:

  as $z \to -\infty$: $e^z \to 0$ and $e^{-z} = 1/e^z \to \infty$, hence: $\sin(z) = \frac{1}{2}(e^z - e^{-z}) \to -\infty$
  at $z = 0$: $\sinh(0) = \frac{1}{2}(e^0 - e^0) = 0$
  as $z \to \infty$: $e^z \to \infty$ and $e^{-z} = 1/e^z \to 0$, hence: $\sin(z) = \frac{1}{2}(e^z - e^{-z}) \to \infty$

Properties of cosh:

- $\cosh(-z) = \cosh(z)$

  proof: follows directly from definition,
  $$\cosh(-z) = \frac{1}{2}(e^{-z} + e^{z}) = \frac{1}{2}(e^{z} + e^{-z}) = \cosh(z)$$

- for $z \in \mathbb{R}^{+}$: $\cosh(z)$ increases monotonically
  for $z \in \mathbb{R}^{-}$: $\cosh(z)$ decreases monotonically

  proof: differentiate the analytic definition,
  using $\frac{d}{dz}e^{az} = ae^{az}$,
  $$\frac{d}{dz}\cosh(z) = \sinh(z) \quad \begin{cases} > 0 & \text{for } z > 0 \\ = 0 & \text{for } z = 0 \\ < 0 & \text{for } z < 0 \end{cases}$$

- consider $z \in \mathbb{R}$:

  as $z \to -\infty$: $e^{z} \to 0$ and $e^{-z} = 1/e^{z} \to \infty$, hence: $\cosh(z) = \frac{1}{2}(e^{z} + e^{-z}) \to \infty$
  at $z = 0$: $\cosh(0) = \frac{1}{2}(e^{0} + e^{0}) = 0 = 1$
  as $z \to \infty$: $e^{z} \to \infty$ and $e^{-z} = 1/e^{z} \to 0$, hence: $\cosh(z) = \frac{1}{2}(e^{z} + e^{-z}) \to \infty$

Properties of tanh:

- $\tanh(-z) = -\tanh(z)$

  proof: follows directly from definition,
  $$\tanh(-z) = \sinh(-z)/\cosh(-z) = -\sinh(z)/\cosh(z) = -\tanh(z)$$

- for $z \in \mathbb{R}$: $\tanh(z)$ increases monotonically

  proof: differentiate the definition,
  using $\frac{d}{dz}\sinh(z) = \cosh(z)$ and $\frac{d}{dz}\sinh(z) = \cosh(z)$,
  $$\frac{d}{dz}\tanh(z) = \frac{d}{dz}\left(\frac{\sinh(z)}{\cosh(z)}\right) = \frac{\cosh(z)\frac{d}{dz}\sinh(z) - \sinh(z)\frac{d}{dz}\cosh(z)}{\cosh^2(z)}$$
  $$= \frac{\cosh^2(z) - \sinh^2(z)}{\cosh^2(z)} = \frac{1}{\cosh^2(z)} > 0$$

- consider $z \in \mathbb{R}$, rewrite $\tanh(z)$ in two distinct ways:
  $$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} = \begin{cases} (1 - e^{-2z})/(1 + e^{-2z}) & \text{so } \tanh(z) \to 1 \quad \text{if } z \to \infty \\ 0 & \text{for} \quad z = 0 \\ (e^{2z} - 1)/(e^{2z} + 1) & \text{so } \tanh(z) \to -1 \quad \text{if } z \to -\infty \end{cases}$$

4.3.3. Connection with trigonometric functions

More than just similarity between trigonometric and hyperbolic functions,
When defined for complex numbers they can be *expressed in terms of each other!*

Let $x \in \mathbb{R}$:

trigonometric functions are hyperbolic functions of imaginary arguments:

$$\begin{aligned}\sin(x) &= -i\sinh(ix)\\ \cos(x) &= \cosh(ix)\\ \tan(x) &= -i\tanh(ix)\end{aligned}$$

proofs:
just write RHS in terms of exponentials ...

hyperbolic functions are trigonometric functions of imaginary arguments:

$$\begin{aligned}\sinh(x) &= -i\sin(ix)\\ \cosh(x) &= \cos(ix)\\ \tanh(x) &= -i\tan(ix)\end{aligned}$$

proofs:
just write RHS in terms of exponentials ...

4.3.4. Applications of connection with trigonometric functions

All previous identities for trigonometric functions
whose derivation did *not* rely on argument being real (e.g. addition formulae),
translate via the above into identities for hyperbolic functions!

- Addition formulae:

$$\begin{aligned}\sin(\theta+\phi) &= \sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)\\ \cos(\theta+\phi) &= \cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)\\ \tan(\theta+\phi) &= \frac{\tan(\theta)+\tan(\phi)}{1-\tan(\theta)\tan(\phi)}\end{aligned}$$

  give:

$$\begin{aligned}\sinh(\theta+\phi) &= -i\sin(i\theta)\cos(i\phi)-i\cos(i\theta)\sin(i\phi)\\ &= \sinh(\theta)\cosh(\phi)+\cosh(\theta)\sinh(\phi)\end{aligned}$$

$$\cosh(\theta+\phi) = \cos(i\theta)\cos(i\phi)-\sin(i\theta)\sin(i\phi)$$

$$= \cosh(\theta)\cosh(\phi) + \sinh(\theta)\sinh(\phi)$$

$$\tanh(\theta+\phi) = \frac{-i\tan(i\theta) - i\tan(i\phi)}{1-\tan(i\theta)\tan(i\phi)} = \frac{\tanh(\theta)+\tanh(\phi)}{1+\tanh(\theta)\tanh(\phi)}$$

- Formulae for double angles:

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ \tan(2\theta) &= 2\tan(\theta)/[1-\tan^2(\theta)]\end{aligned}$$

give:

$$\begin{aligned}\sinh(2\theta) &= -2i\sin(i\theta)\cos(i\theta) = 2\sinh(\theta)\cosh(\theta)\\ \cosh(2\theta) &= \cos^2(i\theta) - \sin^2(i\theta) = \cosh^2(\theta)+\sinh^2(\theta)\\ \tanh(2\theta) &= -2i\tan(i\theta)/[1-\tan^2(i\theta)] = 2\tanh(\theta)/[1+\tanh^2(\theta)]\end{aligned}$$

4.3.5. Inverse hyperbolic functions

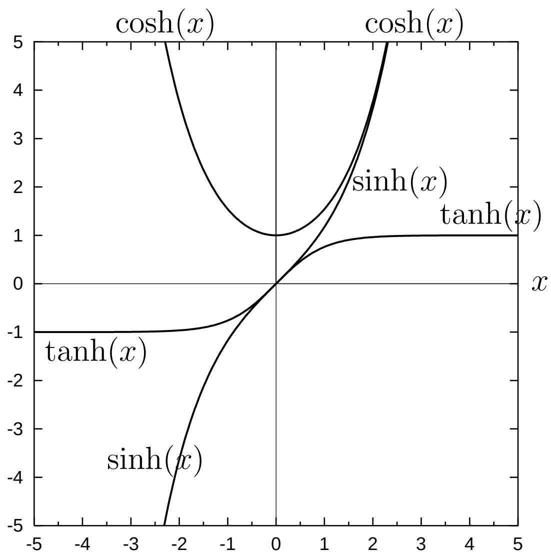
Recall our definitions:

- The inverse f^{-1} of a function $f: D \to R$ is defined by:

$$\begin{aligned}&f^{-1}: R \to D\\ &f^{-1}(f(x)) = x \quad \text{for all } x \in D\\ &f(f^{-1}(x)) = x \quad \text{for all } x \in R\end{aligned}$$

- A function $f: D \to R$ is invertible if and only if:

  $f(x_1) \neq f(x_2)$ for any two $x_1, x_2 \in D$ with $x_1 \neq x_2$



Inspect graphs of hyperbolic functions:

(i) there are generally *two* $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$ such that $\cosh(x_1) = \cosh(x_2)$, namely $x_1 = -x_2$
(in contrast to $\sinh: \mathbb{R} \to \mathbb{R}$ and $\tanh: \mathbb{R} \to (-1,1)$, which are invertible)

(ii) hence: we must limit the domain of $\cosh(x)$ to a set where any two distinct angles will give *different* function values

- <u>definition of the inverse of $\sinh(x)$</u>: $\text{arcsinh}(y)$

$$\begin{aligned} \text{arcsinh} &: \mathbb{R} \to \mathbb{R} \\ \text{arcsinh}(\sinh(x)) &= x \quad \text{for all } x \in \mathbb{R} \\ \sinh(\text{arcsinh}(y)) &= y \quad \text{for all } y \in \mathbb{R} \end{aligned}$$

$\text{arcsinh}(y)$ in words:
gives the value $x \in \mathbb{R}$ such that $\sinh(x) = y$

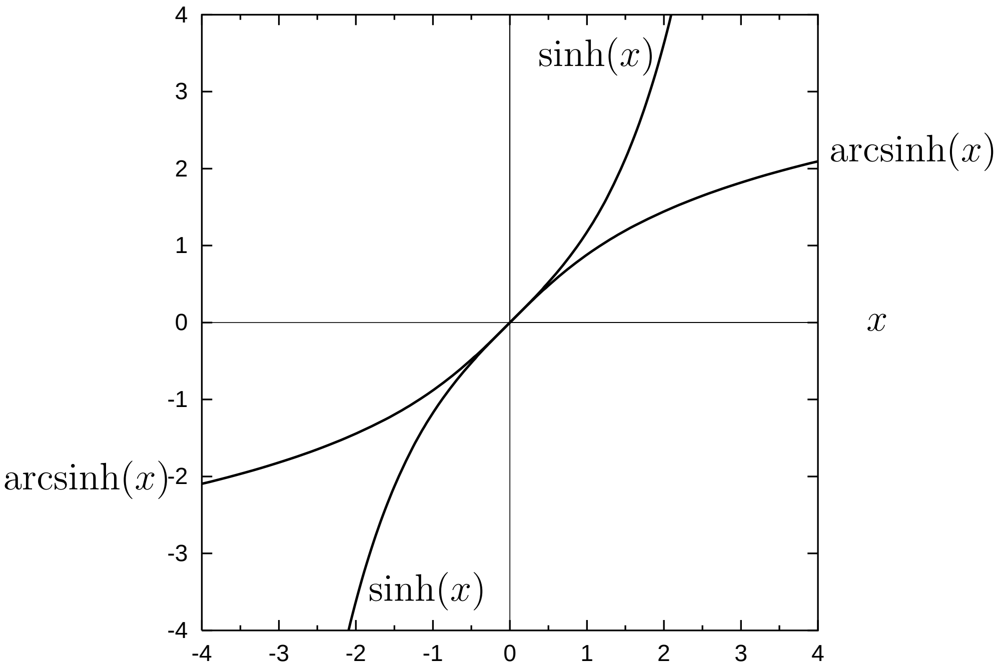
In fact: we can produce a simple formula:

$$y = \sinh(x): \quad y = \frac{1}{2}(e^x - e^{-x}) \quad \Rightarrow \quad e^x - e^{-x} - 2y = 0 \quad \Rightarrow \quad e^{2x} - 2ye^x - 1 = 0$$

$$(e^x)^2 - 2y(e^x) - 1 = 0 \quad \Rightarrow \quad e^x = \frac{1}{2}(2y \pm \sqrt{4y^2+4} = y \pm \sqrt{y^2+1}$$

since $e^x > 0$: $\quad e^x = y + \sqrt{y^2+1}$, so $\quad x = \ln(y + \sqrt{y^2+1})$
hence:

$$\text{arcsinh}(y) = \ln(y + \sqrt{y^2+1}) \quad \text{for all } \; y \in \mathbb{R}$$



- <u>definition of the inverse of $\tanh(x)$</u>: $\text{arctanh}(y)$

$$\text{arctanh} : (-1,1) \to \mathbb{R}$$
$$\text{arctanh}(\tanh(x)) = x \quad \text{for all } x \in \mathbb{R}$$
$$\tanh(\text{arctanh}(y)) = y \quad \text{for all } y \in (-1,1)$$

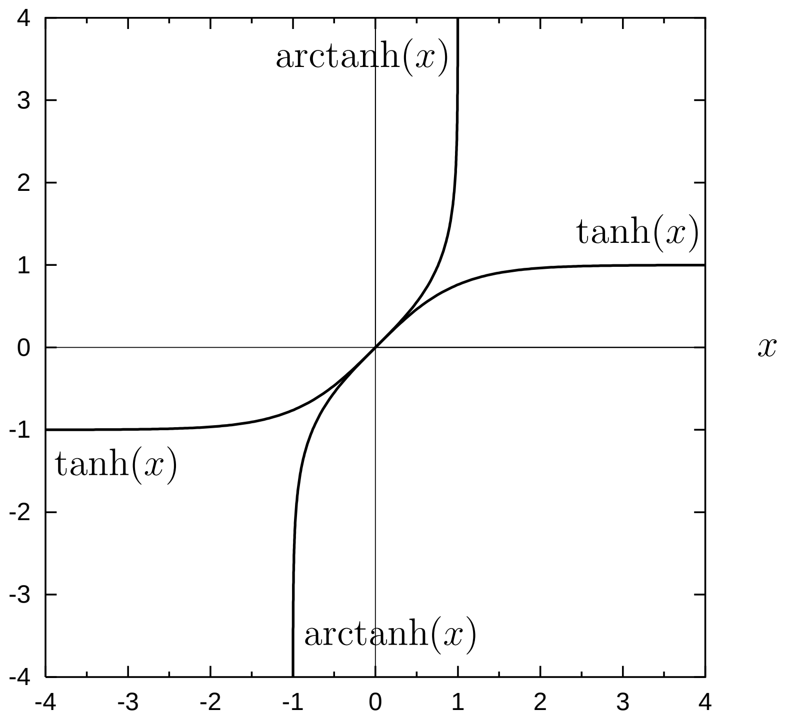
$\text{arctanh}(y)$ in words:
gives the value $x \in \mathbb{R}$ such that $\tanh(x) = y$

In fact: we can produce a simple formula:

$$y = \tanh(x): \quad y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \Rightarrow \quad y(e^x + e^{-x}) = e^x - e^{-x} \quad \Rightarrow \quad y(e^{2x}+1) = e^{2x} - 1$$
$$1 + y = e^{2x}(1-y) \quad \Rightarrow \quad e^{2x} = \frac{1+y}{1-y} \quad \Rightarrow \quad 2x = \ln[(1+y)/(1-y)]$$

hence:

$$\text{arctanh}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \qquad \text{for all } \; y \in (-1,1)$$



- <u>definition of the inverse of $\cosh(x)$</u>: $\quad \text{arccosh}(y)$

need interval D that satisfies
(i) $\cosh(x_1) \neq \cosh(x_2)$ for all $x_1, x_2 \in D$ with $x_1 \neq x_2$
(ii) the range corresponding to D covers all possible values of cosh, $R = [1, \infty)$

Answer: $D = [0, \infty)$, so

$$\text{arccosh} : [1, \infty) \to [0, \infty)$$
$$\text{arccosh}(\cosh(x)) = x \quad \text{for all } x \in [0, \infty)$$
$$\cosh(\text{arccosh}(y)) = y \quad \text{for all } y \in [1, \infty)$$

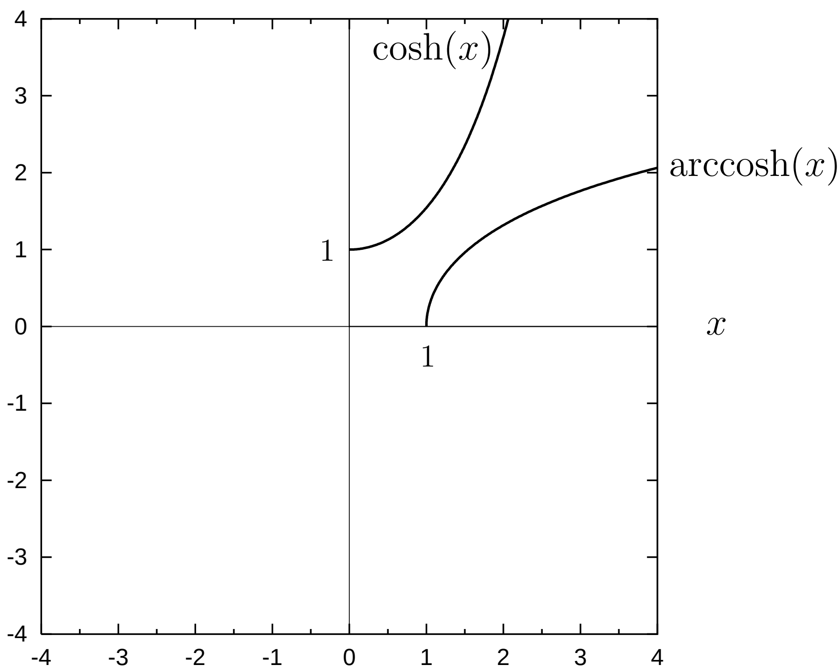
arccosh(y) in words: gives the value $x \in [0, \infty)$ such that $\cosh(x) = y$

In fact: we can produce a simple formula:

$$y = \cosh(x) : \quad y = \frac{1}{2}(e^x + e^{-x}) \quad \Rightarrow \quad e^x + e^{-x} - 2y = 0 \quad \Rightarrow \quad e^{2x} - 2ye^x + 1 = 0$$
$$(e^x)^2 - 2y(e^x) + 1 = 0 \quad \Rightarrow \quad e^x = \frac{1}{2}(2y \pm \sqrt{4y^2 - 4} = y \pm \sqrt{y^2 - 1}$$

since $x \in [0, \infty)$: $\quad e^x \geq 1, \quad$ and hence $\quad e^x = y + \sqrt{y^2 - 1}, \quad$ so $\quad x = \ln(y + \sqrt{y^2 - 1})$
hence:

$$\text{arccosh}(y) = \ln(y + \sqrt{y^2 - 1}) \qquad \text{for all } \; y \in [1, \infty)$$



5. Functions, limits and differentiation

5.1. Introduction

5.1.1. Rate of change, tangent of a curve

'limits' resolved two long-standing problems:
(i) mechanics: how to define and find the *instantaneous* rate of change of a quantity
(ii) geometry: how to define and find the tangent to arbitrary curves in arbitrary points

Consider time-dependent quantity $x(t)$, $t \in \mathbb{R}$ denotes time
(e.g. position of a particle moving along a line)

- rate of change over interval $t \in [t_1, t_2]$: average velocity $\overline{v}$ during the interval

$$\overline{v} = \frac{\text{change in } x}{\text{time taken}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

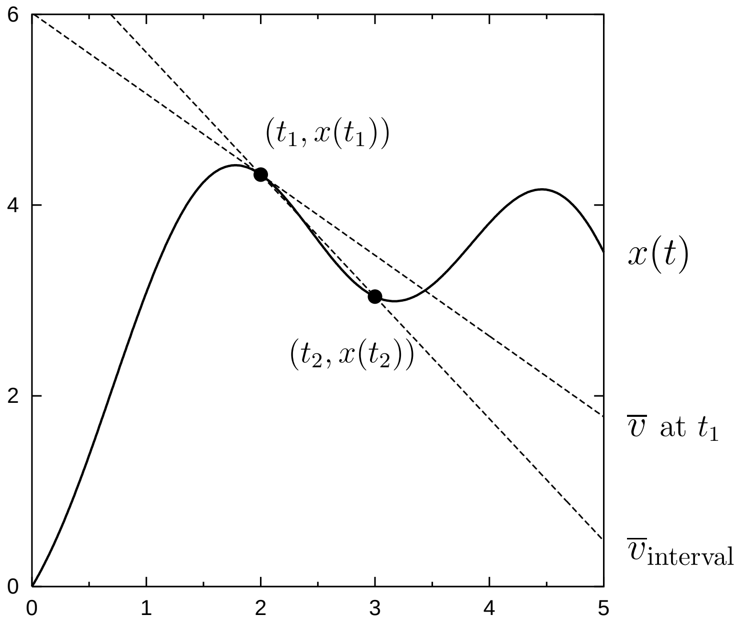
- observation in (t, x) graph:

  $\overline{v}$ is *slope*
  of line through the points
  $(t_1, x(t_1))$ and $(t_2, x(t_2))$

- *instantaneous* speed at time t_1:
  value of $\overline{v}$ when $t_2 \to t_1$

- result: tangent at curve $x(t)$
  at the point $t = t_1$

  problems (i) (mechanics)
  and (ii) (geometry)
  are the same!



So far: only ideas, definitions and pictures ...

<u>Calculus</u>: find *formulas* for the instantaneous velocities (or tangents),
when the curves $x(t)$ are given

note:
not all curves are written as $x(t)$ or $f(x)$ or $y(x)$ ...
(liberate yourself from name conventions!)

5.1.2. Finding tangents and velocities – why we need limits

Calculation would seem obvious, e.g.
Fermat's calculation of instantaneous slope of function $f(x)$ at value x:

- work out formula for average slope during interval $[x, x+h]$
$$slope = \frac{f(x+h)-f(x)}{h}$$
- put $h = 0$ in the result

Often this simple recipe works ...

- $f(x) = ax + b$:
$$slope = \frac{f(x+h)-f(x)}{h} = \frac{[a(x+h)+b]-[ax+b]}{h} = \frac{ah}{h} = a$$
Put $h = 0$: $\quad slope = a$

- $f(x) = ax^2 + bx + c$:
$$\begin{aligned} slope &= \frac{f(x+h)-f(x)}{h} = \frac{[a(x+h)^2+b(x+h)+c]-[ax^2+bx+c]}{h} \\ &= \frac{a(x^2+2xh+h^2)+bx+bh+c-ax^2-bx-c}{h} \\ &= \frac{a(2xh+h^2)+bh}{h} = 2ax+ah+b \end{aligned}$$
Put $h = 0$: $\quad slope = 2ax + b$

- $f(x) = ax^n,\ n \in \mathbb{N}^+$:
use binomial formula,
$$\begin{aligned} slope &= \frac{f(x+h)-f(x)}{h} = \frac{a(x+h)^n - ax^n}{h} = \frac{a}{h}\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k - x^n\right) \\ &= \frac{a}{h}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k = a\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1} = a\sum_{\ell=0}^{n-1}\binom{n}{\ell+1}x^{n-\ell-1}h^{\ell} \end{aligned}$$
Put $h = 0$: $\quad slope = a\binom{n}{1}x^{n-1} = a\ \frac{n!}{1!(n-1)!}\ x^{n-1} = anx^{n-1}$

But equally often it doesn't ...

- $f(x) = a^x$:

$$slope = \frac{f(x+h) - f(x)}{h} = \frac{a^{x+h} - a^x}{h} = a^x \left(\frac{a^h - 1}{h} \right)$$

Putting $h = 0$ gives: $\quad slope = a^x((a^0 - 1)/0) = a^x(0/0)$ ... ??

- $f(x) = \sin(x)$:

$$\begin{aligned} slope &= \frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin(x)}{h} \\ &= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{sin(h)}{h}\right) \end{aligned}$$

Putting $h = 0$ gives: $\quad slope = \sin(x)(0/0) + \cos(x)(0/0)$ ... ??

- $f(x) = \cos(x)$:

$$\begin{aligned} slope &= \frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos(x)}{h} \\ &= \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\ &= \cos(x)\left(\frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right) \end{aligned}$$

Putting $h = 0$ gives: $\quad slope = \cos(x)(0/0) - \sin(x)(0/0)$ ... ??

<u>The problem</u>:

One *cannot* set $h = 0$ in expressions like $(a^h - 1)/h$ or $(\cos(h) - 1)/h$ or $\sin(h)/h$

<u>The solution</u>: (Newton, Leibniz)

The correct thing to do is to take h smaller and smaller, and investigate whether the quantity $[f(x+h) - f(x)]/h$ then *approaches* a well-defined value for $h \to 0$.
If so: that value will be the slope at point x, to be called the 'derivative' of $f(x)$ at x

Notes:

(i) Newton & Leibniz had the concept, the 'intuitive' definition of limit but exact mathematical definition of limit was due to Cauchy, much later ...

(ii) notation for derivative:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We saw:

(i) derivatives of functions that are sums of powers are easy to find
(ii) ergo: another important use of power series!

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{1}{2}x^2 + \dots \\ \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &&= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots \\ \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &&= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots \end{aligned}$$

Remember stumbling blocks on previous page:

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h}(a^h - 1) &= \lim_{h\to 0} \frac{1}{h}\left(e^{\ln(a^h)} - 1\right) = \lim_{h\to 0} \frac{1}{h}\left(e^{h\ln a} - 1\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(1 + h\ln a + \frac{1}{2}(h\ln a)^2 + \dots - 1\right) \\ &= \lim_{h\to 0} \left(\ln a + \frac{1}{2}h(\ln a)^2 + \dots\right) = \ln a \\ \lim_{h\to 0} \frac{1}{h}(\cos(h) - 1) &= \lim_{h\to 0} \frac{1}{h}\left(1 - \frac{1}{2}h^2 + \frac{1}{24}h^4 + \dots - 1\right) \\ &= \lim_{h\to 0} \left(-\frac{1}{2}h + \frac{1}{24}h^3 + \dots\right) = 0 \\ \lim_{h\to 0} \frac{1}{h}\sin(h) &= \lim_{h\to 0} \frac{1}{h}\left(h - \frac{1}{6}h^3 + \frac{1}{120}h^5 + \dots\right) \\ &= \lim_{h\to 0} \left(1 - \frac{1}{6}h^2 + \frac{1}{120}h^4 + \dots\right) = 1 \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dx}a^x &= a^x \lim_{h\to 0} \frac{1}{h}(a^h - 1) = a^x \ln(a) \\ \frac{d}{dx}\sin(x) &= \sin(x) \lim_{h\to 0} \frac{1}{h}(\cos(h) - 1) + \cos(x) \lim_{h\to 0} \frac{1}{h}\sin(h) = \cos(x) \\ \frac{d}{dx}\cos(x) &= \cos(x) \lim_{h\to 0} \frac{1}{h}(\cos(h) - 1) - \sin(x) \lim_{h\to 0} \frac{1}{h}\sin(h) = -\sin(x) \end{aligned}$$

5.2. The limit

Limit in words:
the output value $f(x)$ to which a function tends (if at all)
as x approaches a specific input value x_0

Some simple abbreviations and symbols:

$\exists$: there exists $\quad$ $\forall$: for all $\quad$ $\Leftrightarrow$: if and only if

5.2.1. Left and right limits

The proper $\varepsilon - \delta$ definition ...

- The right limit: approach x_0 from the *right*,
  notation: $\lim_{x\to x_0^+} f(x)$ or $\lim_{x\downarrow x_0} f(x)$

  definition

  $$\lim_{x\downarrow x_0} f(x) = L \quad\Leftrightarrow\quad (\forall\varepsilon>0)(\exists\delta>0): \quad |f(x)-L|<\varepsilon \text{ whenever } x\in(x_0, x_0+\delta)$$

  in words:

  $$\lim_{x\downarrow x_0} f(x) = L \quad\Leftrightarrow\quad \text{for all } \varepsilon>0 \text{ there exists a } \delta>0 \text{ such that } |f(x)-L|<\varepsilon \text{ whenever } x\in(x_0, x_0+\delta)$$

  in sloppy words:
  'one may get $f(x)$ as close as one wishes to the value L
  simply by lowering x sufficiently close to x_0'

- The left limit: approach x_0 from the *left*,
  notation: $\lim_{x\to x_0^-} f(x)$ or $\lim_{x\uparrow x_0} f(x)$

  definition

  $$\lim_{x\uparrow x_0} f(x) = L \quad\Leftrightarrow\quad (\forall\varepsilon>0)(\exists\delta>0): \quad |f(x)-L|<\varepsilon \text{ whenever } x\in(x_0-\delta, x_0)$$

  in words:

  $$\lim_{x\uparrow x_0} f(x) = L \quad\Leftrightarrow\quad \text{for all } \varepsilon>0 \text{ there exists a } \delta>0 \text{ such that } |f(x)-L|<\varepsilon \text{ whenever } x\in(x_0-\delta, x_0)$$

  in sloppy words:
  'one may get $f(x)$ as close as one wishes to the value L
  simply by raising x sufficiently close to x_0'

Notes:

(i) these limits need not always exist
(ii) if they do, then $\lim_{x\downarrow x_0} f(x)$ might be different from $\lim_{x\uparrow x_0} f(x)$
(iii) $\lim_{x\downarrow x_0} f(x)$ and $\lim_{x\uparrow x_0} f(x)$ could exist even if $f(x_0)$ does not exist

Examples (draw pictures!):

$f(x) = 1/x$: neither $\lim_{x\downarrow 0} f(x)$ nor $\lim_{x\uparrow 0} f(x)$ exist, $f(0)$ does not exist

$f(x) = 1/x$: $\lim_{x\downarrow 1} f(x) = \lim_{x\uparrow 1} f(x) = f(1) = 1$

$f(x) = \tanh(1/x)$: $\lim_{x\downarrow 0} f(x) = 1,\ \lim_{x\uparrow 0} f(x) = -1,$ $f(0)$ does not exist

$f(x) = \begin{cases} x/|x| & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$: $\lim_{x\downarrow 0} f(x) = 1,\ \lim_{x\uparrow 0} f(x) = -1,$ $f(0) = 0$

$f(x) = \begin{cases} x & \text{for} \quad x > 0 \\ \pi & \text{for} \quad x = 0 \\ x^{-1} & \text{for} \quad x < 0 \end{cases}$: $\lim_{x\downarrow 0} f(x) = 0,\ \lim_{x\uparrow 0} f(x)$ does not exist, $f(0) = \pi$

5.2.2. Asymptotics - limits involving infinity

- Approach ∞ (always from the left!),
  notation: $\lim_{x\to\infty} f(x)$

  definition

$$\lim_{x\to\infty} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon > 0)(\exists X > 0): \quad |f(x) - L| < \varepsilon \ \text{ whenever } \ x > X$$

  in sloppy words:
  'one may get $f(x)$ as close as one wishes to the value L
  simply by making x larger and larger'

- Approach $-\infty$ (always from the right!),
  notation: $\lim_{x\to -\infty} f(x)$

  definition

  $$\lim_{x\to -\infty} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon > 0)(\exists X < 0): \quad |f(x) - L| < \varepsilon \quad \text{whenever} \quad x < X$$

  in sloppy words:
  'one may get $f(x)$ as close as one wishes to the value L
  simply by making x smaller and smaller'

5.2.3. When left/right limits exists and are identical

- Approach x_0 from *either* side,
  notation: $\lim_{x\to x_0} f(x)$

  definition (version 1)

  $$\lim_{x\to x_0} f(x) = L \quad \Leftrightarrow \quad \lim_{x\downarrow 0} f(x) = \lim_{x\uparrow 0} f(x) = L$$

  definition (version 2)

  $$\lim_{x\to x_0} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon > 0)(\exists \delta > 0): \quad |f(x) - L| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

  in sloppy words:
  'one may get $f(x)$ as close as one wishes to the value L
  simply by taking x sufficiently close to x_0, from either side'

- the concept of continuity of a function

  definition

  A function f is continuous at the point x_0 if
  $\lim_{x\to x_0} f(x)$ exists and $\lim_{x\to x_0} f(x) = f(x_0)$

  continuous functions are those for which the graph can always be drawn
  without lifting one's pen from the paper

Mathematical expressions may involve *multiple* limits,
notation conventions:

$$\lim_{x\to x_0} \lim_{y\to y_0} f(x,y) = \lim_{x\to x_0} \Big(\lim_{y\to y_0} f(x,y) \Big)$$
$$\lim_{y\to y_0} \lim_{x\to x_0} f(x,y) = \lim_{y\to y_0} \Big(\lim_{x\to x_0} f(x,y) \Big)$$

Note: the order in which limits are taken matters! (jargon: limits do not 'commute') e.g.

$$f(x,y) = x^2y - e^{-x-y}: \qquad \lim_{x\to 0}\lim_{y\to 1} f(x,y) = \lim_{x\to 0}(x^2 - e^{-x-1}) = -1/e$$
$$\lim_{y\to 1}\lim_{x\to 0} f(x,y) = \lim_{y\to 1}(-e^{-y}) = -1/e$$

$$f(x,y) = (2x-y)/(x+3y): \qquad \lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}(2x/x) = 2$$
$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}(-y/3y) = -1/3$$

$$f(x,y) = 1 + \tanh(x+y): \qquad \lim_{x\to\infty}\lim_{y\to-\infty} f(x,y) = \lim_{x\to\infty}(1-1) = 0$$
$$\lim_{y\to-\infty}\lim_{x\to\infty} f(x,y) = \lim_{y\to-\infty}(1+1) = 2$$

5.2.4. Rules for limits of composite expressions

(i) If $\lim_{x\to x_0} f(x) = a$, $\lim_{x\to x_0} g(x) = b$:

$$\lim_{x\to x_0}\Big(f(x)+g(x)\Big) = a+b$$

$$\lim_{x\to x_0}\Big(f(x)g(x)\Big) = ab$$

$$\text{if } b\neq 0: \quad \lim_{x\to x_0}\Big(f(x)/g(x)\Big) = a/b$$

$$\text{if } b = 0 \text{ and } a\neq 0: \quad \lim_{x\to x_0}\Big(f(x)/g(x)\Big) \quad \text{does not exist}$$

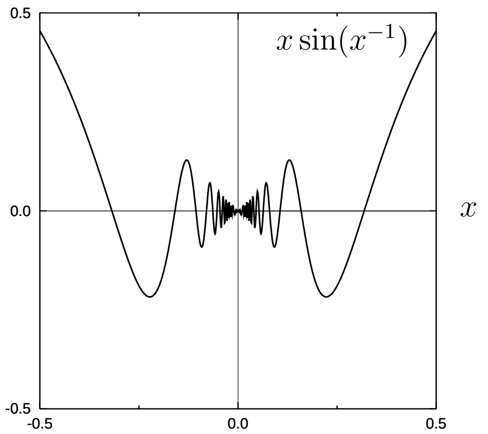
(ii) If $\lim_{x\to x_0} f(x) = a$, $\lim_{y\to a} g(y) = b$, $g(a) = b$:

$$\lim_{x\to x_0} g(f(x)) = b$$

(iii) determine limits via 'pinching' or 'sandwiching':
Let there be a $\delta > 0$ such that

$$\left.\begin{array}{l}(\forall x,\ |x-x_0|<\delta): \ f(x)\leq g(x)\leq h(x)\\ \lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = a\end{array}\right. \quad \Rightarrow \quad \lim_{x\to x_0} g(x) = a$$

5.2.5. Examples



(i) $\lim_{x\to 0} x\sin(x^{-1}) = 0$

proof: via 'sandwiching',
since always $\sin(\ldots) \in [-1, 1]$,

$$(\forall x \in \mathbb{R}): \quad -|x| \leq x\sin(x^{-1}) \leq |x|$$

Clearly: $\lim_{x\to 0} |x| = 0$ and $\lim_{x\to 0}(-|x|) = 0$,
hence $\lim_{x\to 0} x\sin(x^{-1}) = 0$

(ii) $\lim_{x\to 0} x^{-1}\tan(x) = 1$

proof:
use $\lim_{x\to 0} x^{-1}\sin(x) = 1$ and $\lim_{x\to 0}\cos(x) = \cos(0) = 1$

$$x^{-1}\tan(x) = \frac{\sin(x)}{x\cos(x)} = \left(\frac{\sin(x)}{x}\right)\left(\frac{1}{\cos(x)}\right)$$

Hence

$$\lim_{x\to 0} x^{-1}\tan(x) = \left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos(x)}\right) = 1.1 = 1$$

(iii) $\lim_{x\to 0} x^{-1}\ln(1+x) = 1$

proof:
use suitable substitution, e.g. $x = e^y - 1$ with $y \to 0$
as well as $\lim_{x\to 0} x^{-1}(e^x - 1) = 1$

$$x^{-1}\ln(1+x) = (e^y-1)^{-1}\ln(e^y) = \Big((e^y-1)/y\Big)^{-1}$$

Hence

$$\lim_{x\to 0} x^{-1}\ln(1+x) = \Big(\lim_{y\to 0}(e^y-1)/y\Big)^{-1} = 1^{-1} = 1$$

(iv) $\lim_{x\to\infty} x\ln(1+\frac{1}{x}) = 1$

proof:
substitute $x = 1/y$ with $y \to 0$, and convert into limit (iii)

$$x\ln(1+\frac{1}{x}) = y^{-1}\ln(1+y)$$

Hence

$$\lim_{x\to\infty} x\ln(1+\frac{1}{x}) = \lim_{y\to 0} y^{-1}\ln(1+y) = 1$$

(v) $\lim_{x\to 0} x\ln(x) = 0$

proof:
more tricky, substitute $x = e^{-y}$ with $y \to \infty$ and think ...

$$x\ln(x) = \; -ye^{-y}$$

Proof of $\lim_{y\to\infty} ye^{-y} = 0$, in three steps without using power series:

(a) claim: $e^z > z$ for all $z > 0$.
proof: consider $f(z) = e^z - z$ for $z \geq 0$: $f(0) = 1$, $\frac{d}{dz}f(z) = e^z - 1 > 0$ for $z > 0$
so f increases monotonically for $z > 0$, starting at $f(0) = 1$
Thus $e^z > z$ for all $z > 0$

(b) now choose $z = \frac{1}{2}y$: $e^{y/2} > y/2$, so also $e^y > (y/2)^2 = \frac{1}{4}y^2$
equivalently: $ye^{-y} < y/(\frac{1}{4}y^2) = 4/y$

(c) since also $ye^{-y} \geq 0$ for $y > 0$, we can proceed by 'sandwiching':

$$y > 0: \qquad 0 \leq ye^{-y} \leq 4/y$$

Since $\lim_{y\to\infty}(4/y) = 0$, we have proven $\lim_{y\to\infty} ye^{-y} = 0$
and hence also the original statement.

Alternative version of the proof that $\lim_{y\to\infty} ye^{-y} = 0$, using power series:

(a) since $e^y = \sum_{n=0}^{\infty} y^n/n!$, one has $e^y > y^\ell/\ell!$ for any $\ell \in \mathbb{N}$ and any $y > 0$
(b) hence we know for $y > 0$ that $0 \leq ye^{-y} < \ell!\; y^{1-\ell}$
(c) take any $\ell > 1$ and proceed by sandwiching ...

(vi) $\lim_{x\to\infty} x^{-1}\ln(x) = 0$

proof:
substitute $x = 1/y$ with $y \to 0$, and convert into limit (v)

$$\lim_{x\to\infty} x^{-1}\ln(x) = -\lim_{y\to 0} y\ln(y) = 0$$

This completes the proof.

(vii) $\lim_{x\to\infty} x^a e^{-x} = 0$ for any $a \in \mathbb{R}$

proof:
for $a \leq 0$ the claim is trivial, so we concentrate on $a > 0$
we generalize the ideas used in proving (v)

Proof without using power series:

(a) we know that $e^z > z$ for all $z > 0$ (was demonstrated under (iv))

(b) now choose $z = x/2a$: $e^{x/2a} > x/2a$, so also $e^x > (x/2a)^{2a} = (2a)^{-2a}x^{2a}$
equivalently: $x^a e^{-x} < x^a/(2a)^{-2a}x^{2a} = (2a)^{2a}x^{-a}$

(c) we can proceed by 'sandwiching':

$$x > 0: \qquad 0 \leq x^a e^{-x} \leq (2a)^{2a}x^{-a}$$

Since $\lim_{x\to\infty} x^{-a} = 0$, we have proven $\lim_{x\to\infty} x^a e^{-x} = 0$

This completes the proof.

Alternative version using power series:

(a) since $e^x = \sum_{n=0}^{\infty} x^n/n!$, one has $e^x > x^\ell/\ell!$ for any $\ell \in \mathbb{N}$ and any $x > 0$
(b) hence we know for $x > 0$ that $0 \leq x^a e^{-x} < \ell!\, x^{a-\ell}$
(c) take any $\ell > a$ and proceed by sandwiching ...

5.3. *Differentiation*

5.3.1. *Derivatives of functions*

Recall definition of derivative of function f for continuous functions:
(notation: df/dx, or $f'(x)$, or dy/dx with $y = f(x)$)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

calculation of $f'(x)$ from scratch:

- first simplify $(f(x+h) - f(x))/h$ if possible
- if putting $h = 0$ in the formula is allowed (Fermat)
then the result will be $f'(x)$. Done.

If putting $h = 0$ is not allowed, we must determine the limit:

(i) decompose your expression into parts that have known limits,
then use the rules for limits of composite expressions,

or

(ii) substituting suitable power series for the difficult parts,
and simplify further until you *can* put $h = 0$

Note:
the $\varepsilon - \delta$ formulas are used when you *prove* that a limit takes a value L
(they do not help in finding the candidate L in the first place ...)

Elementary derivatives found earlier:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}a^x = a^x \ln(a) \qquad \frac{d}{dx}x^n = nx^{n-1} \quad (n \in \mathbb{Z}^+)$$

Further examples:

- $\frac{d}{dx}\ln(x) = 1/x \quad$ for $x \in \mathbb{R}^+$

  proof:

$$\begin{aligned}\frac{d}{dx}\ln(x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{\ln\Big(x(1+h/x)\Big)-\ln(x)}{h} \\ &= \lim_{h\to 0}\frac{\ln(x)+\ln(1+h/x)-\ln(x)}{h} = \lim_{h\to 0}\frac{\ln(1+h/x)}{h} \quad \text{substitute } y=h/x \\ &= \lim_{y\to 0}\frac{\ln(1+y)}{xy} = \frac{1}{x}\lim_{y\to 0}\frac{\ln(1+y)}{y} = \frac{1}{x} \qquad \text{use limit (iii) of section 4.25}\end{aligned}$$

- $\frac{d}{dx}x^a = ax^{a-1} \quad$ for $a \in \mathbb{R}$, $x \neq 0$
  (so far only proven for $a \in \mathbb{Z}^+$)

  proof:

$$\begin{aligned}\frac{d}{dx}x^a &= \lim_{h\to 0}\frac{(x+h)^a - x^a}{h} = \lim_{h\to 0}\frac{x^a(1+h/x)^a - x^a}{h} \\ &= x^a\lim_{h\to 0}\frac{(1+h/x)^a-1}{h} = x^a\lim_{h\to 0}h^{-1}[e^{a\ln(1+h/x)}-1] \quad \text{substitute } y=h/x \\ &= x^a\lim_{y\to 0}(xy)^{-1}[e^{a\ln(1+y)}-1] = ax^{a-1}\lim_{y\to 0}(ay)^{-1}[e^{ay\left(y^{-1}\ln(1+y)\right)}-1] \\ &= ax^{a-1}\lim_{z\to 0}z^{-1}[e^{z\left(\lim_{y\to 0}y^{-1}\ln(1+y)\right)}-1] \\ &= ax^{a-1}\lim_{z\to 0}z^{-1}[e^z-1] = ax^{a-1}\end{aligned}$$

5.3.2. Rules for derivatives of composite expressions

Objective: efficiency
(we don't want to calculate derivatives always from scratch, via the limit)

(i) The sum rule:

$$y = f(x) + g(x): \qquad \frac{dy}{dx} = f'(x) + g'(x)$$

proof:

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}+\lim_{h\to 0}\left\{\frac{g(x+h)-g(x)}{h}\right\}\\ &= f'(x)+g'(x)\end{aligned}$$

(ii) The product rule:

$$y = f(x)g(x): \qquad \frac{dy}{dx} = f'(x)g(x)+f(x)g'(x)$$

proof:

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x+h)+[g(x+h)-g(x)]f(x)}{h}\\ &= \lim_{h\to 0}\left\{g(x+h)\frac{f(x+h)-f(x)}{h}\right\}+\lim_{h\to 0}\left\{f(x)\frac{g(x+h)-g(x)}{h}\right\}\\ &= g(x)f'(x)+f(x)g'(x)\end{aligned}$$

(iii) The chain rule:

$$y = f(g(x)): \qquad \frac{dy}{dx} = f'(g(x))g'(x)$$

proof:

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\ &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\,\frac{g(x+h)-g(x)}{h}\\ &= \left\{\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right\}\left\{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right\}\\ &= g'(x)\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \quad \text{substitute } \varepsilon = g(x+h)-g(x)\\ &= g'(x)\lim_{\varepsilon\to 0}\frac{f(g(x)+\varepsilon)-f(g(x))}{\varepsilon} = g'(x)f'(g(x))\end{aligned}$$

(some subtleties with this version,
leave that to analysis)

(iv) The quotient rule:

$$y = f(x)/g(x) : \qquad \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

proof:
write $f(x)g^{-1}(x)$, use product rule, chain rule, and $\frac{d}{dx}x^{-1} = -x^{-2}$

$$\begin{aligned}\frac{d}{dx}\left[f(x)g^{-1}(x)\right] &= f'(x)g^{-1}(x) + f(x)\frac{d}{dx}\left[g^{-1}(x)\right] \\ &= f'(x)g^{-1}(x) + f(x)g'(x)\left[-1/g^2(x)\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}\end{aligned}$$

Examples:

$$\begin{aligned}\frac{d}{dx}\left[x^2\sin(x)\right] &= x^2\left[\frac{d}{dx}\sin(x)\right] + \sin(x)\left[\frac{d}{dx}x^2\right] \\ &= x^2\cos(x) + 2x\sin(x)\end{aligned}$$

$$\frac{d}{dx}f(ax) = af'(x)$$

$$\frac{d}{dx}e^{x^n} = e^{x^n}\left(\frac{d}{dx}x^n\right) = nx^{n-1}e^{x^n}$$

$$\frac{d}{dx}\ln(\cosh(x)) = \cosh'(x).\frac{1}{\cosh(x)} = \frac{\sinh(x)}{\cosh(x)} = \tanh(x)$$

$$\begin{aligned}\frac{d}{dx}x^x &= \frac{d}{dx}e^{\ln(x^x)} = \frac{d}{dx}e^{x\ln(x)} \\ &= e^{x\ln(x)}\frac{d}{dx}[x\ln(x)] = e^{x\ln(x)}\left[1.\ln(x) + x\frac{d}{dx}\ln(x)\right] \\ &= e^{x\ln(x)}\left[\ln(x) + \frac{x}{x}\right] = x^x[1 + \ln(x)]\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}f(g(h(x))) &= f'(g(h(x)))\ \frac{d}{dx}g(h(x)) \\ &= f'(g(h(x)))\ g'(h(x))\ \frac{d}{dx}h(x) \\ &= f'(g(h(x)))\ g'(h(x))\ h'(x)\end{aligned}$$

5.3.3. Derivatives of implicit functions

Above methods apply only when we can write an *explicit* formula for the function $f(x)$ we wish to differentiate.
But: some functions are defined by a property, without a formula ...

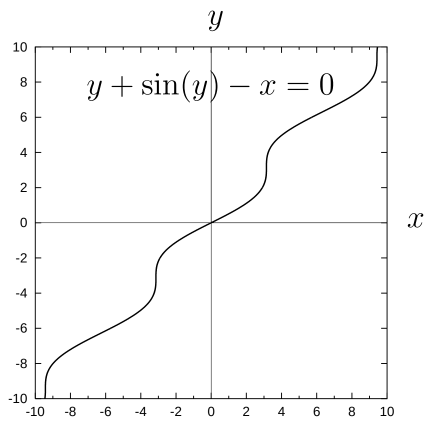
- A. functions defined as solutions of equations for points in the plane

  function $y(x)$: solution of an equation of the type $F(x, y) = 0$
  (for a domain D where each $x \in D$ is associated with only one y)
  How to find $y'(x)$?

  method:

  (i) differentiate the equation, using chain rule
  (ii) then try to solve the result for dy/dx

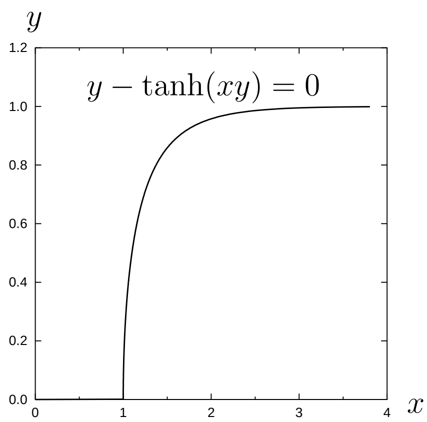
  Example 1:



  $F(x, y) = y + \sin(y) - x$,
  so $y(x)$ is solution of $y + \sin(y) - x = 0$

  (i) differentiate $F(x, y)$:
  $$\frac{dy}{dx} + \cos(y)\frac{dy}{dx} - 1 = 0$$
  (ii) solve for dy/dx:
  $$\frac{dy}{dx}(1 + \cos(y)) = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{1 + \cos(y)}$$

  Example 2:



  $F(x, y) = y - \tanh(xy)$,
  so $y(x)$ is solution of $y - \tanh(xy) = 0$

  (i) differentiate $F(x, y)$:
  $$\frac{dy}{dx} - \tanh'(xy)\frac{d}{dx}(xy) = 0$$
  $$\frac{dy}{dx} - \frac{1}{\cosh^2(xy)}\left(x\frac{dy}{dx} + y\right) = 0$$
  (ii) solve for dy/dx:
  $$\frac{dy}{dx}\left(1 - \frac{x}{\cosh^2(xy)}\right) - \frac{y}{\cosh^2(xy)} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{\cosh^2(xy) - x}$$

- B. functions defined as <u>inverse of another given function</u>

  function $f^{-1}(x)$: solution of the equation $f^{-1}(f(x)) = x$ for all x, with given f
  Suppose we have no formula for $f^{-1}(x)$, e.g. $\arcsin(x)$
  How to find $\frac{d}{dx}f^{-1}(x)$?

  <u>method</u>:

  (i) differentiate the equivalent equation $f(f^{-1}(x)) = x$, using chain rule
  (ii) then solve the result for $\frac{d}{dx}f^{-1}(x)$

  This can be done *generally*:

$$\frac{d}{dx}f(f^{-1}(x)) = 1 \quad \Rightarrow \quad \left(\frac{d}{dx}f^{-1}(x)\right) f'(f^{-1}(x)) = 1$$

so

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Example 1:

let $f(x) = e^x$, so $f^{-1}(x) = \ln(x)$ (with $x > 0$)

$$\frac{d}{dx}\ln(x) = \frac{1}{f'(\ln(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

Example 2:

let $f(x) = \sin(x)$, so $f^{-1}(x) = \arcsin(x)$ (with $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$)

$$\frac{d}{dx}\arcsin(x) = \frac{1}{f'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

Example 3:

let $f(x) = \tan(x)$, so $f^{-1}(x) = \arctan(x)$

$$\frac{d}{dx}\arctan(x) = \frac{1}{f'(\arctan(x))} = \cos^2(\arctan(x))$$

$$= \frac{\cos^2(\arctan(x))}{\sin^2(\arctan(x)) + \cos^2(\arctan(x))} = \frac{1}{\tan^2(\arctan(x)) + 1}$$

$$= \frac{1}{1 + x^2}$$

- C. functions defined <u>parametrically</u>

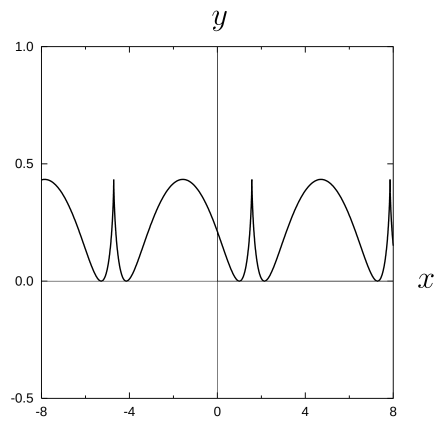
  Given two functions $x(t)$ and $y(t)$, varying t traces out a *curve* in the x-y plane
  This curve defines a function $y(x)$ implicitly
  (for those $t \in \mathbb{R}$ where each x is associated with only one y)
  How to find $y'(x)$?

  <u>method</u>:

  (i) calculate $x'(t) = dx/dt$ and $y'(t) = dy/dt$, then work out $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
  (ii) if possible, use formulas for $x(t)$ and $y(t)$ to eliminate t from your result

Example 1:
for $t \in \mathbb{R}$:

$$x(t) = t + \cos(t)$$
$$y(t) = \ln(\cosh(\sin(t)))$$



(i) differentiate $x(t)$ and $y(t)$:

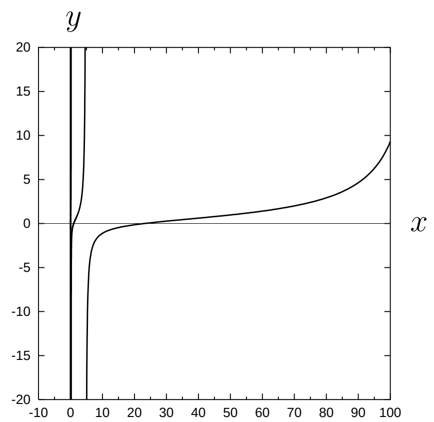
$$x'(t) = 1 - \sin(t)$$
$$y'(t) = \cos(t)\tanh(\sin(t))$$

so
$$\frac{dy}{dx} = \frac{\cos(t)\tanh(\sin(t))}{1 - \sin(t)}$$

(ii) cannot simplify further

Example 2:
for $t \in \mathbb{R}$:

$$x(t) = e^t$$
$$y(t) = \tan(t)$$



(i) differentiate $x(t)$ and $y(t)$:

$$x'(t) = e^t$$
$$y'(t) = 1/\cos^2(t)$$

so
$$\frac{dy}{dx} = \frac{e^{-t}}{\cos^2(t)}$$

(ii) simplify by removing t:

$$\frac{dy}{dx} = \frac{e^{-t}[\sin^2(t) + \cos^2(t)]}{\cos^2(t)} = \frac{\tan^2(t) + 1}{e^t} = \frac{1 + y^2}{x}$$

Finally ...
Are we at all allowed to put $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$?
(even if these derivatives exist)

- Proof 1:

  (i) Assume that y can indeed be written as a function (as yet unknown) of x: $y(t) = f(x(t))$
  We aim to calculate $dy/dx = f'(x)$

  (ii) Apply chain rule to $y(t) = f(x(t))$:

$$y'(t) = f'(x(t))x'(t) \quad \Rightarrow \quad \frac{y'(t)}{x'(t)} = f'(x(t))$$

  Hence $f'(x) = \frac{dy/dt}{dx/dt}$, as claimed.

- Proof 2:

  Via the original definition:

$$\begin{array}{l} x'(t) = \lim_{h\to 0} h^{-1}[x(t+h) - x(t)] \\ y'(t) = \lim_{h\to 0} h^{-1}[y(t+h) - y(t)] \end{array} \quad \Rightarrow \quad \frac{y'(t)}{x'(t)} = \lim_{h\to 0} \frac{y(t+h) - y(t)}{x(t+h) - x(t)}$$

  In the limit we switch from h to the new variable $z = x(t+h) - x(t)$.
  Since $h \to 0$, also $z \to 0$.

  If we write $y(t) = f(x(t))$ for some unknown function f,
  then $y(t+h) = f(x(t+h)) = f(x(t) + z)$:

$$\lim_{h\to 0} \frac{y(t+h) - y(t)}{x(t+h) - x(t)} = \lim_{z\to 0} \frac{f(x(t) + z) - f(x(t))}{z} = f'(x(t))$$

  Hence

$$\frac{y'(t)}{x'(t)} = f'(x(t))$$

5.3.4. Applications of derivative: sketching graphs

standard procedure for sketching the graph of a function $f(x)$
(a reminder – should be secondary school knowledge)

(i) Determine values of x for which f is defined, and those for which it isn't

(ii) Find all stationary points of f (i.e. those x where $f'(x) = 0$, with zero slope)

(iii) Determine the nature of the stationary points (local minimum, local maximum, or neither)

(iv) Find the points where $f(x) = 0$, and calculate $f'(x)$ at these points

(v) State whether f is even, $f(-x) = f(x)$ for all x, or odd, $f(-x) = -f(x)$ for all x, or neither

(vi) Sketch the graph of f over an appropriate range of x

6. Integration

6.1. Introduction

6.1.1. Area under a curve

We define the integral (including notation) in terms of areas:

definition

The integral $\int_a^b f(x)dx$ is the total area in the (x, y) plane between the graph of $y = f(x)$ and the x-axis, counted positively for $f(x) > 0$ (area *above* the x-axis) and negatively for $f(x) < 0$ (area *below* the x-axis).

How to calculate $\int_a^b f(x)dx$?

- The area is easy to calculate for functions that are built of steps only ('staircases'):

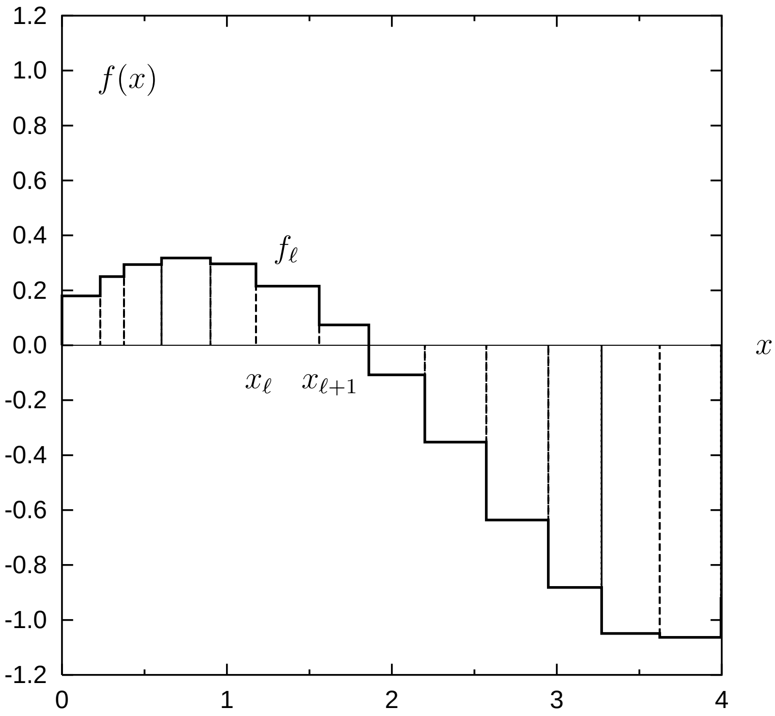
  Denote 'jump' points as x_ℓ, with $\ell \in \mathbb{Z}$ and $x_1 < x_2 < x_3 < \ldots$

  If $x \in [x_\ell, x_{\ell+1})$: $f(x) = f_\ell$

  Contribution to integral from interval $[x_\ell, x_{\ell+1})$:

  area of *rectangle*, $f_\ell(x_{\ell+1} - x_\ell)$ also sign is correct!

  Total integral: add up contributions of all intervals between a and b (let $x_1 = a$, $x_L = b$)



$$\int_a^b f(x)dx = \sum_{\ell=1}^{L-1} f_\ell(x_{\ell+1} - x_\ell)$$

Notes:

(i) previous formula is exact *only* for staircases

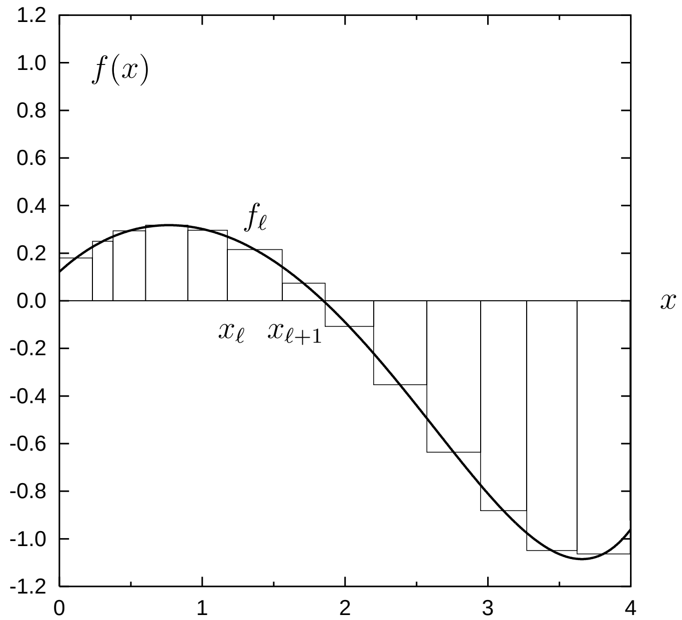
(ii) but we can approximate most functions to arbitrary accuracy using staircases with smaller and smaller steps ... (limits!!)

- What if $f(x)$ is not a staircase?

  Find the integral $\int_a^b f(x)dx$ as the *limit* of the integral for a suitable staircase, where all steps go to zero

  Not as simple as it sounds:

  (i) many staircase possible ...
  (ii) what is a suitable staircase?
  (iii) result ought not to depend on your choice of staircase! (analysis gets involved)



Formal definition of integral (analysis) therefore involves

(i) consider *all* possible staircases where each step touches the curve of $f(x)$

(ii) find area underneath each staircase in the limit width of the *widest* step goes to zero

(iii) check whether all these limits for different staircases are identical

Result in a nutshell:

- If $f(x)$ is a continuous function on $[a, b]$, then the integral $\int_a^b f(x)dx$ exists
- If the integral $\int_a^b f(x)dx$ exists, it will be equal to the limit of *any* staircase approximation, where each step touches the curve of $f(x)$, as the width of the widest step goes to zero

Consequence:
to calculate the integral of a continuous function
we can *choose the most convenient* staircase approximation
and then find its limit

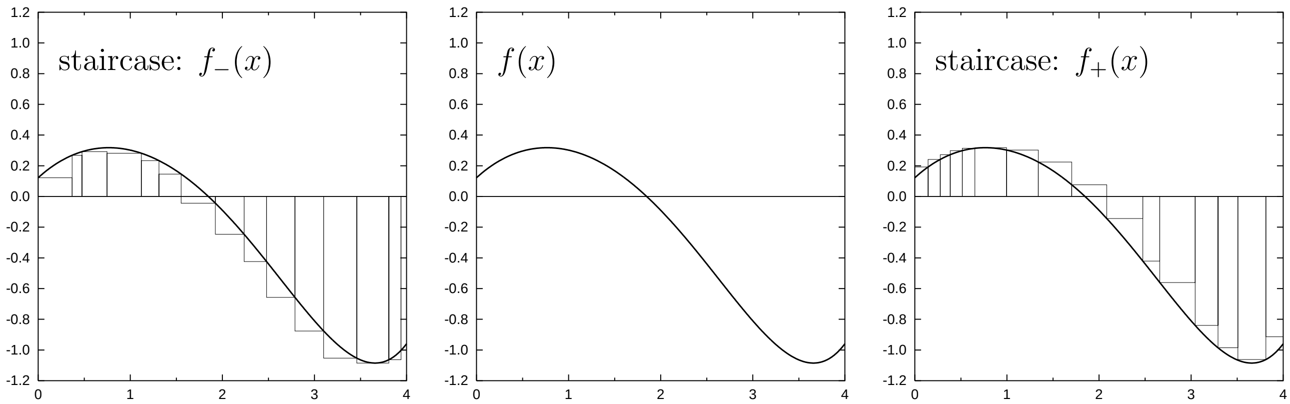
Direct route (nor relying on analysis results): sandwich method using staircases

Step (i):

build staircase functions $f_\pm(x)$ such that
$f_-(x) \leq f(x) \leq f_+(x)$ for all $x \in [a,b]$
e.g.

$$x \in [x_\ell, x_{\ell+1}) : \quad \begin{array}{l} f_+(x) = f_\ell^+ = \max_{x\in[x_\ell,x_{\ell+1})} f(x) \\ f_-(x) = f_\ell^- = \min_{x\in[x_\ell,x_{\ell+1})} f(x) \end{array}$$

(let $x_1 = a$, $x_L = b$)



we are now sure that:

$$\int_a^b f_-(x)dx \quad \leq \quad \int_a^b f(x)dx \quad \leq \quad \int_a^b f_+(x)dx$$

i.e.

$$\sum_{\ell=1}^{L-1} f_\ell^-(x_{\ell+1}-x_\ell) \quad \leq \quad \int_a^b f(x)dx \quad \leq \quad \sum_{\ell=1}^{L-1} f_\ell^+(x_{\ell+1}-x_\ell)$$

Step (ii):

take the limit where $x_\ell - x_{\ell+1} \to 0$ for all ℓ,
in the two bounding areas $A_- = \sum_{\ell=1}^{L-1} f_\ell^-(x_{\ell+1}-x_\ell)$ and $A_+ = \sum_{\ell=1}^{L-1} f_\ell^+(x_{\ell+1}-x_\ell)$

$$\lim_{\text{step widths} \to 0} A_- \;\leq\; \int_a^b f(x)dx \;\leq\; \lim_{\text{step widths} \to 0} A_+$$

Step (iii):

Conclusion:
if $\lim_{\text{step widths} \to 0} A_- = \lim_{\text{step widths} \to 0} A_+ = A$, then: $\quad \int_a^b f(x)dx = A$

6.1.2. Examples of integrals calculated via staircases

- Example 1:

$$A = \int_0^b \cos(x)dx, \qquad \text{with } b \leq \pi$$

(so on $[0, b]$: $\cos(x)$ decreases monotonically, i.e. if $x' > x$ then $\cos(x') < \cos(x)$)

Method: sandwich with staircases
use tutorial exercise 39 (let $m \in \mathbb{Z}$):

$$\theta \neq 2m\pi : \qquad \sum_{k=0}^{n} \cos(k\theta) = \frac{1-\cos(n\theta+\theta)-\cos(\theta)+\cos(n\theta)}{2-2\cos(\theta)}$$

Step (i):
Build staircase functions $f_\pm(x)$ such that
$f_-(x) \leq \cos(x) \leq f_+(x)$ for all $x \in [0, b]$
e.g.

$$x \in [x_\ell, x_{\ell+1}) : \quad \begin{array}{l} f_+(x) = f_\ell^+ = \max_{x\in[x_\ell,x_{\ell+1})} \cos(x) = \cos(x_\ell) \\ f_-(x) = f_\ell^- = \min_{x\in[x_\ell,x_{\ell+1})} \cos(x) = \cos(x_{\ell+1}) \end{array}$$

Choose steps of equal size,
with $x_1 = 0$ and $x_L = b$:

$$x_\ell = (\ell-1)h, \quad \text{with } h = \frac{b}{L-1} : \qquad x_1 = 0,\ x_2 = h,\ x_3 = 2h,\ \ldots\ x_L = (L-1)h = b$$

Upper bound to A:

$$\begin{aligned} A_+ \quad &= \sum_{\ell=1}^{L-1} f_\ell^+(x_{\ell+1} - x_\ell) = h\sum_{\ell=1}^{L-1}\cos(x_\ell) = h\sum_{\ell=1}^{L-1}\cos((\ell-1)h) \\ &= h\sum_{k=0}^{L-2}\cos(kh) = h\frac{1-\cos((L-1)h)-\cos(h)+\cos((L-2)h)}{2-2\cos(h)} \\ \text{eliminate } L : \quad &= h\frac{1-\cos(b)-\cos(h)+\cos(b-h)}{2-2\cos(h)} \\ &= h\frac{1-\cos(b)-\cos(h)+\cos(b)\cos(h)+\sin(b)\sin(h)}{2-2\cos(h)} \\ &= h\frac{[1-\cos(b)][1-\cos(h)]+\sin(b)\sin(h)}{2-2\cos(h)} \\ &= \frac{h}{2}\Big(1-\cos(b)\Big) + \frac{1}{2}\sin(b)\frac{h\sin(h)}{1-\cos(h)} \end{aligned}$$

Lower bound to A:

$$A_- = \sum_{\ell=1}^{L-1} f_\ell^-(x_{\ell+1} - x_\ell) = h\sum_{\ell=1}^{L-1}\cos(x_{\ell+1}) = h\sum_{\ell=1}^{L-1}\cos(\ell h)$$

$$= h\sum_{k=2}^{L}\cos((k-1)h) = h\left[\sum_{k=1}^{L-1}\cos((k-1)h) + \cos((L-1)h) - \cos(0)\right]$$
$$= A_+ + h\cos((L-1)h) - h = A_+ + h[\cos(b) - 1]$$
$$= h[\cos(b) - 1] + \frac{h}{2}\Big(1-\cos(b)\Big) + \frac{1}{2}\sin(b)\frac{h\sin(h)}{1-\cos(h)}$$

We now know that $A_- \leq \int_0^b \cos(x)dx \leq A_+$.

Step (ii):
take the limit $h \to 0$ in the bounds A_+ and A_-

$$\begin{aligned}
\lim_{h\to 0} A_+ &= \lim_{h\to 0}\left\{\frac{h}{2}\Big(1-\cos(b)\Big) + \frac{1}{2}\sin(b)\frac{h\sin(h)}{1-\cos(h)}\right\} \\
&= \lim_{h\to 0}\frac{1}{2}\sin(b)\frac{h\sin(h)}{1-\cos(h)} = \sin(b)\lim_{h\to 0}\left\{\frac{1}{2}\frac{h\sin(h)}{1-\cos^2(h/2)+\sin^2(h/2)}\right\} \\
&= \sin(b)\lim_{h\to 0}\left\{\frac{h\sin(h)}{4\sin^2(h/2)}\right\} = \sin(b)\lim_{h\to 0}\left\{\frac{\sin(h)}{h}\frac{(h/2)^2}{\sin^2(h/2)}\right\} \\
&= = \sin(b)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)\left(\lim_{h\to 0}\frac{h/2}{\sin(h/2)}\right)^2 = \sin(b) \\
\lim_{h\to 0} A_- &= \lim_{h\to 0}\{A_+ + h[\cos(b) - 1]\} = \lim_{h\to 0} A_+ = \sin(b)
\end{aligned}$$

We conclude

$$\lim_{h\to 0} A_- \leq A \leq \lim_{h\to 0} A_+ \quad\Rightarrow\quad \sin(b) \leq A \leq \sin(b) \quad\Rightarrow\quad A = \sin(b)$$

- Example 2:

$$A = \int_0^b \cos(x)dx, \qquad \text{with arbitrary } b > 0$$

($\cos(x)$ need no longer decrease monotonically, so sandwich method becomes messy)

Method: use *most convenient* approximating staircase

(i) $\cos(x)$ is a continuous function on any interval
(ii) both $f_-(x)$ and $f_+(x)$ in example 1 are suitable approximating staircases to $\cos(x)$
(each step in each staircase touches the curve, and all step sizes go to zero)

Hence:

$$A = \lim_{h\to 0} A_- = \lim_{h\to 0} A_+ = \sin(b)$$

- Example 3:

$$A = \int_0^b \sin(x)dx, \qquad \text{with } b \le \pi/2$$

(so on $[0,b]$: $\sin(x)$ increases monotonically, i.e. if $x' > x$ then $\sin(x') > \sin(x)$)

Method: sandwich with staircases
we use tutorial exercise 39 (let $\ell \in \mathbb{Z}$):

$$\theta \neq 2\ell\pi: \qquad \sum_{k=0}^{n} \sin(k\theta) = \frac{-\sin(n\theta{+}\theta){+}\sin(\theta){+}\sin(n\theta)}{2-2\cos(\theta)}$$

Step (i):
Build staircase functions $f_\pm(x)$ such that
$f_-(x) \le \sin(x) \le f_+(x)$ for all $x \in [0,b]$
e.g.

$$x \in [x_\ell, x_{\ell+1}): \qquad \begin{array}{l} f_+(x) = f_\ell^+ = \max_{x\in[x_\ell,x_{\ell+1})} \sin(x) = \sin(x_{\ell+1}) \\ f_-(x) = f_\ell^- = \min_{x\in[x_\ell,x_{\ell+1})} \sin(x) = \sin(x_\ell) \end{array}$$

Choose steps of equal size,
with $x_1 = 0$ and $x_L = b$:

$$x_\ell = (\ell{-}1)h, \quad \text{with } h = \frac{b}{L{-}1}: \qquad x_1 = 0,\ x_2 = h,\ x_3 = 2h,\ \ldots\ x_L = (L{-}1)h = b$$

Upper bound to A:

$$\begin{aligned} A_+ \quad &= \sum_{\ell=1}^{L-1} f_\ell^+(x_{\ell+1} - x_\ell) = h\sum_{\ell=1}^{L-1} \sin(x_{\ell+1}) = h\sum_{\ell=1}^{L-1} \sin(\ell h) \\ &= \ = h\sum_{k=0}^{L-1} \sin(kh) = h\frac{-\sin(Lh){+}\sin(h){+}\sin((L{-}1)h)}{2-2\cos(h)} \\ \text{eliminate } L: \quad &= h\frac{-\sin(b+h){+}\sin(h){+}\sin(b)}{2-2\cos(h)} \\ &= h\frac{-\sin(b)\cos(h){-}\cos(b)\sin(h){+}\sin(h){+}\sin(b)}{2-2\cos(h)} \\ &= h\frac{[1{-}\cos(b)]\sin(h) + [1{-}\cos(h)]\sin(b)}{2-2\cos(h)} \\ &= \frac{h}{2}\sin(b) + \frac{1}{2}[1{-}\cos(b)]\frac{h\sin(h)}{1-\cos(h)} \end{aligned}$$

Lower bound to A:

$$\begin{aligned} A_- &= \sum_{\ell=1}^{L-1} f_\ell^-(x_{\ell+1} - x_\ell) = h\sum_{\ell=1}^{L-1} \sin(x_\ell) = h\sum_{\ell=1}^{L-1} \sin((\ell{-}1)h) \\ &= h\sum_{k=0}^{L-2} \sin(kh) = h\sum_{k=1}^{L-1} \sin(kh) - h\sin((L{-}1)h) = A_+ - h\sin(b) \end{aligned}$$

We now know that $A_- \leq \int_0^b \sin(x)dx \leq A_+$.

Step (ii):
take the limit $h \to 0$ in the bounds A_+ and A_-

$$\begin{aligned}
\lim_{h\to 0} A_+ &= \lim_{h\to 0}\left\{\frac{h}{2}\sin(b)+\frac{1}{2}[1-\cos(b)]\frac{h\sin(h)}{1-\cos(h)}\right\} \\
&= \lim_{h\to 0}\frac{1}{2}[1-\cos(b)]\frac{h\sin(h)}{1-\cos(h)} = [1-\cos(b)]\lim_{h\to 0}\left\{\frac{1}{2}\frac{h\sin(h)}{1-\cos^2(h/2)+\sin^2(h/2)}\right\} \\
&= [1-\cos(b)]\lim_{h\to 0}\left\{\frac{h\sin(h)}{4\sin^2(h/2)}\right\} = [1-\cos(b)]\lim_{h\to 0}\left\{\frac{\sin(h)}{h}\frac{(h/2)^2}{\sin^2(h/2)}\right\} \\
&= = [1-\cos(b)]\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right)\left(\lim_{h\to 0}\frac{h/2}{\sin(h/2)}\right)^2 = 1-\cos(b) \\
\lim_{h\to 0} A_- &= \lim_{h\to 0}\{A_+ - h\sin(b)\} = \lim_{h\to 0} A_+ = 1-\cos(b)
\end{aligned}$$

We conclude

$$\lim_{h\to 0} A_- \leq A \leq \lim_{h\to 0} A_+ \quad\Rightarrow\quad 1-\cos(b) \leq A \leq 1-\cos(b) \quad\Rightarrow\quad A = 1-\cos(b)$$

- Example 4:

$$A = \int_0^b \sin(x)dx, \qquad \text{with arbitrary } b$$

($\sin(x)$ need no longer increase monotonically, so sandwich method becomes more messy)

Method: use *most convenient* approximating staircase

(i) $\sin(x)$ is a continuous function on any interval
(ii) both $f_-(x)$ and $f_+(x)$ in example 3 are suitable approximating staircases to $\sin(x)$
(each step in each staircase touches the curve, and all step sizes go to zero)

Hence:

$$A = \lim_{h\to 0} A_- = \lim_{h\to 0} A_+ = 1-\cos(b)$$

- Example 5:

$$A = \int_a^b x^n dx, \qquad \text{with} \;\; a < b \;\; \text{and} \;\; n \in \mathbb{Z}^+$$

(so we know the function increases monotonically)

Method: sandwich with staircases
we use tutorial exercise 39 (let $\ell \in \mathbb{Z}$):

$$z \neq 1: \qquad \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$$

Step (i):
Build staircase functions $f_\pm(x)$ such that
$f_-(x) \le x^n \le f_+(x)$ for all $x \in [a,b]$
e.g.

$$x \in [x_\ell, x_{\ell+1}): \quad \begin{array}{l} f_+(x) = f_\ell^+ = \max_{x\in[x_\ell,x_{\ell+1})} x^n = x_{\ell+1}^n \\ f_-(x) = f_\ell^- = \min_{x\in[x_\ell,x_{\ell+1})} x^n = x_\ell^n \end{array}$$

Choose steps of non-equal size, in a so-called geometric progression:
with $x_1 = 0$ and $x_L = b$:

$$x_\ell = ah^{\ell-1}, \quad \text{with } b = ah^{L-1}: \qquad x_1 = a,\ x_2 = ah,\ x_3 = ah^2,\ \ldots\ x_L = ah^{L-1} = b$$

Note:
(i) $h > 1$ (since we require $x_{\ell+1} > x_\ell$ always)
(ii) $\ln(b/a) = (L-1)\ln(a)$, so $L-1 = [\ln(b) - \ln(a)]/\ln(a) = \ln(b)/\ln(a) - 1$
(iii) here $x_{\ell+1} - x_\ell = ah^\ell - ah^{\ell-1} = ah^{\ell-1}(h-1)$
(iv) largest step size: $x_L - x_{L-1} = ah^{L-2}(h-1) = ah^{L-1}(1-h^{-1}) = b(1-h^{-1})$
(v) limit of zero step sizes: $h \downarrow 1$

Upper bound to A:

$$\begin{aligned} A_+ &= \sum_{\ell=1}^{L-1} f_\ell^+(x_{\ell+1} - x_\ell) = \sum_{\ell=1}^{L-1} x_{\ell+1}^n ah^{\ell-1}(h-1) = a(h-1)\sum_{\ell=1}^{L-1}[ah^\ell]^n h^{\ell-1} \\ &= a^{n+1}(h-1)\sum_{\ell=1}^{L-1} h^{\ell n+\ell-1} = a^{n+1}(1-h^{-1})\sum_{\ell=1}^{L-1} h^{\ell(n+1)} \\ &= a^{n+1}(1-h^{-1})\sum_{\ell=1}^{L-1}[h^{n+1}]^\ell = a^{n+1}(1-h^{-1})\sum_{\ell=0}^{L-2}[h^{n+1}]^{\ell+1} \\ &= a^{n+1}h^n(h-1)\sum_{k=0}^{L-2}[h^{n+1}]^k = a^{n+1}h^n(h-1)\left(\frac{1-h^{(n+1)(L-1)}}{1-h^{n+1}}\right) \\ \text{eliminate } L: \quad &= a^{n+1}h^n(h-1)\sum_{k=0}^{L-2}[h^{n+1}]^k = a^{n+1}h^n(h-1)\left(\frac{1-(b/a)^{n+1}}{1-h^{n+1}}\right) \end{aligned}$$

Lower bound to A:

$$\begin{aligned} A_- &= \sum_{\ell=1}^{L-1} f_\ell^-(x_{\ell+1} - x_\ell) = \sum_{\ell=1}^{L-1} x_\ell^n ah^{\ell-1}(h-1) = a(h-1)\sum_{\ell=1}^{L-1}[ah^{\ell-1}]^n h^{\ell-1} \\ &= h^{-n}A_+ \end{aligned}$$

We now know that $A_- \le \int_a^b x^n dx \le A_+$.

Step (ii):
take the limit $h \downarrow 1$ in the bounds A_+ and A_-

$$\lim_{h\downarrow 1} A_+ = \lim_{h\downarrow 1}\left\{a^{n+1}h^n(h-1)\left(\frac{1-(b/a)^{n+1}}{1-h^{n+1}}\right)\right\} = \left(a^{n+1} - b^{n+1}\right)\lim_{h\downarrow 1}\frac{h^n(h-1)}{1-h^{n+1}}$$

$$= \left(b^{n+1} - a^{n+1}\right)\left(\lim_{h\downarrow 1} h^n\right)\left(\lim_{h\downarrow 1} \frac{h-1}{h^{n+1}-1}\right) \qquad \text{substitute} \quad h = e^y \text{ with } y \to 0$$

$$= \left(b^{n+1} - a^{n+1}\right)\left(\lim_{y\to 0} \frac{e^y - 1}{e^{(n+1)y}-1}\right)$$

$$= \left(b^{n+1} - a^{n+1}\right)\left(\lim_{y\to 0} \frac{e^y - 1}{y}\frac{(n+1)y}{e^{(n+1)y}-1}\frac{y}{(n+1)y}\right)$$

$$= \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)\left(\lim_{y\to 0} \frac{e^y - 1}{y}\right)\left(\lim_{y\to 0}\frac{(n+1)y}{e^{(n+1)y}-1}\right) = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

$$\lim_{h\downarrow 1} A_- = \lim_{h\downarrow 1}\left\{h^{-n}A_+\right\} = \lim_{h\downarrow 1} A_+ = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

We conclude

$$\lim_{h\downarrow 1} A_- \leq A \leq \lim_{h\downarrow 1} A_+ \quad \Rightarrow \quad \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right) \leq A \leq \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

hence $A = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$.

6.1.3. Fundamental theorems of calculus: integration vs differentiation

Any sane person would like to find a more efficient way to calculate integrals than via the (tedious) construction and analysis of bounding staircases ... Furthermore: we need explicit expressions for the relevant sums to do it ...

help is at hand:

<u>First fundamental theorem of calculus</u>:

If a function $f : [a,b] \to \mathbb{R}$ is continuous on the interval $[a,b]$, and $F : [a,b] \to \mathbb{R}$ is another function such that $F'(x) = f(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

<u>Second fundamental theorem of calculus</u>:

If a function $f : [a,b] \to \mathbb{R}$ is continuous on the interval $[a,b]$, and $F : [a,b] \to \mathbb{R}$ is another function defined by

$$F(x) = \int_a^x f(t)dt \qquad \text{for all } x \in [a,b]$$

then $F'(x) = f(x)$ for all $x \in [a,b]$.

Consequence: a new way to calculate integrals!
Need to find a function $F(x)$ such that $F'(x) = f(x)$

But where do these theorems come from?
Sketch of proofs:

Theorem II:

> If a function $f : [a,b] \to \mathbb{R}$ is continuous on the interval $[a,b]$, and $F : [a,b] \to \mathbb{R}$ is another function defined by
> $$F(x) = \int_a^x f(t)dt \qquad \text{for all } x \in [a,b]$$
> then $F'(x) = f(x)$ for all $x \in [a,b]$.

sketch of proof:
(for differentiable f, for simplicity; proof can be done more generally!)
Let $C = \max_{x\in[a,b]} |f'(t)|$ (maximum slope in absolute sense):

$$\begin{aligned}\frac{dF(x)}{dx} &= \lim_{h\downarrow 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\downarrow 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{h\downarrow 0} \frac{1}{h}\int_x^{x+h} f(t)dt\end{aligned}$$

Easy to bound:
on $t \in [x, x+h]$ one has $f(x) - Ch \le f(t) \le f(x) + Ch$, so

$$\int_x^{x+h} [f(x)-Ch]dt \;\le\; \int_x^{x+h} f(t)dt \;\le\; \int_x^{x+h} [f(x)+Ch]dt$$

$$h[f(x)-Ch] \;\le\; \int_x^{x+h} f(t)dt \;\le\; h[f(x)+Ch]$$

$$\lim_{h\downarrow 0}[f(x)-Ch] \;\le\; \lim_{h\downarrow 0}\frac{1}{h}\int_x^{x+h} f(t)dt \;\le\; \lim_{h\downarrow 0}[f(x)+Ch]$$

Hence

$$F'(x) = \lim_{h\downarrow 0}\frac{1}{h}\int_x^{x+h} f(t)dt = f(x)$$

why was this a sketch rather than formal proof?
(i) restriction to differentiable f can be lifted
(ii) should also do $\lim_{h\uparrow 0}$ in $dF(x)/dx$

Theorem I:

If a function $f : [a, b] \to \mathbb{R}$ is continuous on the interval $[a, b]$, and $F : [a, b] \to \mathbb{R}$ is another function such that $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

proof:
in three steps:

(i) preparation:
Since f is continuous on $[a, b]$ the integral $\int_a^x f(t)dt$ exists for $x \in [a, b]$.
We define $G(x) = \int_a^x f(t)dt$
Properties of G:

$$\begin{aligned} &\text{from Theorem II}: \quad G'(x) = f(x) \quad \text{for all } x \in [a, b] \\ &\text{from definition}: \quad G(a) = 0 \end{aligned}$$

Clearly (using definition of G!): $\int_a^b f(t)dt = G(b)$
We are given another function F with $F'(x) = f(x)$ for all $x \in [a, b]$

(ii) We next show that now $G(x) = F(x) - F(a)$ for all $x \in [a, b]$

How to show this? Call the difference between F and G: $H(x) = F(x) - G(x)$
It follows that $\frac{d}{dx}H(x) = f(x) - f(x) = 0$ for all $x \in [a, b]$
Hence the function $H(x)$ has zero slope on $[a, b]$,
i.e. its graph is horizontal, so $H(x) = H(a)$ for all $x \in [a, b]$
Consequence: $F(x) - G(x) = F(a) - G(a)$ for all $x \in [a, b]$,
Consequence (since $G(a) = 0$): $G(x) = F(x) - F(a)$ for all $x \in [a, b]$

(iii) Combine previous two intermediate results: $\int_a^b f(t)dt = G(b) = F(b) - F(a)$

6.1.4. Indefinite and definite integrals, and other conventions

Some final notation conventions and terminology:

- Definite integral: $A = \int_a^b f(x)dx$
  (the object we worked with so far, always a *number*)
  definition: see earlier

- Indefinite integral: $F(x) = \int f(x)dx$
  (i.e. without any boundary values indicated, always a *function*)

definition: any function F such that $F'(x) = f(x)$
Not uniquely defined: the solutions F differ by a constant, since $\frac{d}{dx}C = 0$

Other names: 'primitive of f', or 'anti-derivative of f'

- Doing an integral via Theorem I requires *finding the primitive* $F(x)$ of $f(x)$. Often we emphasize this intermediate step, show how the integral was done and allow the reader to verify that $F'(x) = f(x)$, by writing

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

- So far we defined $\int_a^b f(x)dx$ for $a \leq b$
Generalization to $a > b$?

Logical choice: via Theorem I
(which will then hold for *any* $a, b \in \mathbb{R}$)

$$a > b: \text{ define } \int_a^b f(x)dx = F(b) - F(a) = -\left(F(a) - F(b)\right) = -\int_b^a f(x)dx$$

6.2. Techniques of integration

- Integration now more or less boils down to this:
when given a function $f(x)$, find a function $F(x)$ (the primitive) such that $F'(x) = f(x)$

- Integration is therefore an art rather than a science:
in contrast to differentiation, where you just follow (carefully) a set of clear rules, integration mostly relies on a mixture of skill, experience, memory and intuition

- The basic strategy: 'divide and conquer'
manipulate (break up, simplify) the integral until it has been reduced to expressions of which you know (i.e. remember) the primitives

- So to be a successful 'integrator' you need to
(i) know and practice the formal manipulation rules and other tricks
(ii) memorize a list of basic primitives
(iii) be creative

We must now
(i) agree on the list of elementary integrals that we will consider known
(ii) describe the tools for manipulation and simplification (most based on rules for differentiation) that we can use to reduce our problem to elementary integrals

6.2.1. List of elementary integrals and general methods for reduction

- Ten elementary integrals

$f(x)$	$F(x)$
$x^a \quad (a \neq -1)$	$\frac{1}{a+1}\, x^{a+1}$
x^{-1}	$\ln\lvert x\rvert$
$\ln(x)$	$x\ln(x) - x$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\frac{1}{\sqrt{x^2-1}}$	$\text{arccosh}(x)$
$\frac{1}{\sqrt{x^2+1}}$	$\text{arcsinh}(x)$

- Four general integration rules
  let $F(x) = \int f(x)dx$, $G(x) = \int g(x)dx$, etc

  (i) Linearity : $\int \Big(cf(x)\Big)dx = cF(x)$

  (ii) Sum rule : $\int \Big(f(x) + g(x)\Big) = F(x) + G(x)$

  (iii) Integration by parts : $\int \Big(f(x)G(x)\Big)dx = F(x)G(x) - \int \Big(g(x)F(x)\Big)dx$

  (iv) Integration by substitution : $\int f(x)dx = \int \Big(f(x(t))\, \frac{dx}{dt}\Big)dt$

  <u>proofs</u>:

  (i) consequence of: $\frac{d}{dx}\Big(cF(x)\Big) = cF'(x)$

  (ii) consequence of: $\frac{d}{dx}\Big(F(x) + G(x)\Big) = F'(x) + G'(x)$

  (iii) consequence of: $\frac{d}{dx}\Big(F(x)G(x)\Big) = F'(x)G(x) + G'(x)F(x)$

  (iv) consequence of: $\frac{d}{dt}F(x(t)) = x'(t)F'(x(t))$

Note:
(iii) and (iv) are usually given as identities for definite integrals,

$$\text{Integration by parts :} \qquad \int_a^b \Big(f(x)G(x)\Big)dx = \Big[F(x)G(x)\Big]_a^b - \int_a^b \Big(g(x)F(x)\Big)dx$$

$$\text{Integration by substitution :} \qquad \int_a^b f(x)dx = \int_{t(a)}^{t(b)} \Big(f(x(t))\ \frac{dx}{dt}\Big)dt$$

Let us inspect the validity of (iv) more carefully:

Since $\int_a^b f(x)dx = F(b) - F(a)$, one knows generally that

$$\frac{d}{db}\int_a^b f(x)dx = F'(b) = f(b)$$

Subtract the left-hand and the right-hand side in the claimed equality,

$$I(b) = \text{LHS} - \text{RHS} = \int_a^b f(x)dx - \int_{t(a)}^{t(b)} \Big(f(x(t))\ \frac{dx}{dt}\Big)dt$$

Then calculate dI/db, via the chain rule:

$$\begin{aligned}\frac{dI}{db} &= \frac{d}{db}\int_a^b f(x)dx - \frac{d}{db}\int_{t(a)}^{t(b)} \Big(f(x(t))x'(t)\Big)dt \\ &= f(b) - t'(b)\frac{d}{dt(b)}\int_{t(a)}^{t(b)} \Big(f(x(t))x'(t)\Big)dt \\ &= f(b) - t'(b)\Big\{f(x(t))x'(t)\Big\}_{t=t(b)} \\ &= f(b) - f(b)\ t'(b)\ x'(t(b)) = f(b) - f(b)\left(\frac{dt}{dx}\frac{dx}{dt}\right)_{x=b} = 0\end{aligned}$$

Thus $I(b)$ is *independent* of b. Hence we may choose $b = a$ to calculate it, i.e.

$$\text{LHS} - \text{RHS} = I(a) = \int_a^a f(x)dx - \int_{t(a)}^{t(a)} \Big(f(x(t))\ \frac{dx}{dt}\Big)dt = 0 - 0 = 0$$

This completes the proof that LHS=RHS for all a and b,
i.e. the claimed identity (iv) is indeed true.

Warning:

changing variables via substitution must involve a one-to-one transformation
(i.e. a unique t for every x, and vice versa)

Example: suppose we are not careful
Clearly $\int_{-1}^1 dx = 2$. Now do the integral via substitution of $y = x^2$, with $dy/dx = 2x$

$$\int_{-1}^1 dx = \int_{y=1}^{y=1} \frac{dy}{2x(y)} = 0\ \ldots$$

6.2.2. Examples: integration by substitution

- $\int \sin^m(x)\cos(x)dx$:

  substitute $t = \sin(x)$, so $dt/dx = \cos(x)$ i.e. $dt = \cos(x)dx$
  $$\int \sin^m(x)\cos(x)dx = \int t^m dt = \frac{1}{m+1}t^{m+1} = \frac{1}{m+1}\sin^{m+1}(x)$$

- $\int e^{x^2} x\ dx$:

  sunstitute $t = x^2$, so $dt/dx = 2x$ i.e. $dt = 2xdx$
  $$\int e^{x^2} x\ dx = \frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \frac{1}{2}e^{x^2}$$

- $\int (1-x^2)^{-1/2}dx$:

  try to use the relation $\cos^2(\theta) + \sin^2(\theta) = 1$
  substitute $x = \sin(\theta)$ with $\theta \in [-\pi/2, \pi/2]$, so $dx/d\theta = \cos(\theta)$ i.e. $dx = \cos(\theta)d\theta$
  $$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = \int \frac{\cos(\theta)d\theta}{\cos(\theta)} = \int d\theta = \theta = \arcsin(x)$$

- $\int (1+x^2)^{-1/2}dx$:

  try to use the relation $\cosh^2(\theta) - \sinh^2(\theta) = 1$
  substitute $x = \sinh(\theta)$, so $dx/d\theta = \cosh(\theta)$ i.e. $dx = \cosh(\theta)d\theta$
  $$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh(\theta)d\theta}{\sqrt{1+\sinh^2(\theta)}} = \int \frac{\cosh(\theta)d\theta}{\cosh(\theta)} = \int d\theta = \theta = \text{arcsinh}(x)$$

- $\int (1+x^2)^{-1}dx$:

  try to use the relation $\cos^2(\theta) + \sin^2(\theta) = 1$
  substitute $x = \tan(\theta)$ with $\theta \in [-\pi/2, \pi/2]$, so $dx/d\theta = \cos^{-2}(\theta)$ i.e. $dx = \cos^{-2}(\theta)d\theta$
  $$\int \frac{dx}{1+x^2} = \int \frac{\cos^{-2}(\theta)d\theta}{1+\tan^2(\theta)} = \int \frac{d\theta}{\cos^2(\theta)+\sin^2(\theta)} = \int d\theta = \theta = \arctan(x)$$

- $\int (1-x^2)^{-1}dx$:

  try to use the relation $\cosh^2(\theta) - \sinh^2(\theta) = 1$
  substitute $x = \tanh(\theta)$, so $dx/d\theta = \cosh^{-2}(\theta)$ i.e. $dx = \cosh^{-2}(\theta)d\theta$
  $$\int \frac{dx}{1-x^2} = \int \frac{\cosh^{-2}(\theta)d\theta}{1-\tanh^2(\theta)} = \int \frac{d\theta}{\cosh^2(\theta)-\sinh^2(\theta)} = \int d\theta = \theta = \text{arctanh}(x)$$

- $\int_2^3 (x^2+2x)^{-1/2}dx$:

  first convert to a form similar to above, using $x^2+2x=(x+1)^2-1$
  $$\int_2^3 \frac{dx}{\sqrt{x^2+2x}} = \int_2^3 \frac{dx}{\sqrt{(x+1)^2-1}}$$
  try to use the relation $\cosh^2(\theta)-\sinh^2(\theta)=1$
  substitute $x+1=\cosh(\theta)$, so $dx/d\theta=\sinh(\theta)$ i.e. $dx=\sinh(\theta)d\theta$
  $$\int_2^3 \frac{dx}{\sqrt{x^2+2x}} = \int_{\text{arccosh}(3)}^{\text{arccosh}(4)} \frac{\sinh(\theta)d\theta}{\sqrt{\cosh^2(\theta)-1}} = \int_{\text{arccosh}(3)}^{\text{arccosh}(4)} \frac{\sinh(\theta)d\theta}{\sinh(\theta)}$$
  $$= \int_{\text{arccosh}(3)}^{\text{arccosh}(4)} d\theta = \Big[\theta\Big]_{\text{arccosh}(3)}^{\text{arccosh}(4)} = \text{arccosh}(4)-\text{arccosh}(3)$$

- $\int_{-3/2}^{-1/2} (-x^2-2x)^{-1/2}dx$:

  first convert to a form similar to above, using $-x^2-2x=1-(x+1)^2$
  $$\int_{-3/2}^{-1/2} \frac{dx}{\sqrt{-x^2-2x}} = \int_{-3/2}^{-1/2} \frac{dx}{\sqrt{1-(x+1)^2}}$$
  try to use the relation $\cos^2(\theta)+\sin^2(\theta)=1$
  substitute $x+1=\sin(\theta)$ with $\theta\in[-\pi/2,\pi/2]$, so $dx/d\theta=\cos(\theta)$ i.e. $dx=\cos(\theta)d\theta$
  $$\int_{-3/2}^{-1/2} \frac{dx}{\sqrt{-x^2-2x}} = \int_{\arcsin(-1/2)}^{\arcsin(1/2)} \frac{\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = \int_{\arcsin(-1/2)}^{\arcsin(1/2)} \frac{\cos(\theta)d\theta}{\cos(\theta)}$$
  $$= \int_{\arcsin(-1/2)}^{\arcsin(1/2)} d\theta = \Big[\theta\Big]_{\arcsin(-1/2)}^{\arcsin(1/2)} = \arcsin(\frac{1}{2})-\arcsin(-\frac{1}{2})$$
  $$= \frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$$

- $\int (6+4x-2x^2)^{-1/2}dx$:

  first convert to a form similar to above, using
  $6+4x-2x^2=-2(x^2-2x-3)=-2((x-1)^2-4)=-8\Big((\frac{1}{2}x-\frac{1}{2})^2-1\Big)$
  $$\int \frac{dx}{\sqrt{6+4x-2x^2}} = \frac{1}{2\sqrt{2}}\int \frac{dx}{\sqrt{1-(\frac{1}{2}x-\frac{1}{2})^2}}$$
  try to use the relation $\cos^2(\theta)+\sin^2(\theta)=1$
  substitute $\frac{1}{2}x-\frac{1}{2}=\sin(\theta)$ with $\theta\in[-\pi/2,\pi/2]$, so $\frac{1}{2}dx/d\theta=\cos(\theta)$ i.e. $dx=2\cos(\theta)d\theta$
  $$\int \frac{dx}{\sqrt{6+4x-2x^2}} = \frac{1}{2\sqrt{2}}\int \frac{2\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} = \frac{1}{\sqrt{2}}\int \frac{\cos(\theta)d\theta}{\cos(\theta)} = \frac{\theta}{\sqrt{2}}$$
  $$= \frac{1}{\sqrt{2}}\arcsin(\frac{1}{2}x-\frac{1}{2})$$

- $\int \tan(x)dx$:

  write $\tan(x) = \sin(x)/\cos(x)$ and substitute $\cos(x) = t$, so $dt/dx = -\sin(x)$ i.e. $dt = -\sin(x)dx$

$$\int \tan(x)dx = \int \frac{\sin(x)dx}{\cos(x)} = -\int \frac{dt}{t} = -\ln|t| = -\ln|\cos(x)|$$

- $\int \sin^{-1}(x)dx$:

  first convert into a form similar to earlier examples, using $\sin(x) = 2\sin(x/2)\cos(x/2)$. Then substitute $\cos(x/2) = t$, so $dt/dx = -\frac{1}{2}\sin(x/2)$ i.e. $dt = -\frac{1}{2}\sin(x/2)dx$:

$$\begin{aligned}\int \frac{dx}{\sin(x)} &= \int \frac{dx}{2\sin(x/2)\cos(x/2)} = -\int \frac{dt}{\sin^2(x/2)t} \\ &= -\int \frac{dt}{t(1-\cos^2(x/2))} = -\int \frac{dt}{t(1-t^2)}\end{aligned}$$

  substitute $u = t^{-2}$, so $du/dt = -2t^{-3}$ i.e. $dt = -\frac{1}{2}t^3du$

$$\begin{aligned}\int \frac{dx}{\sin(x)} &= -\int \frac{dt}{t(1-t^2)} = \frac{1}{2}\int \frac{t^3du}{t(1-u^{-1})} = \frac{1}{2}\int \frac{du}{u(1-u^{-1})} = \frac{1}{2}\int \frac{du}{u-1} \\ &= \frac{1}{2}\ln|u-1| = \ln\left|\frac{1}{t^2} - 1\right|^{\frac{1}{2}} = \ln\left|\frac{1}{\cos^2(x/2)} - 1\right|^{\frac{1}{2}} \\ &= \ln\left|\frac{1-\cos^2(x/2)}{\cos^2(x/2)}\right|^{\frac{1}{2}} = \ln|\tan(x/2)|\end{aligned}$$

6.2.3. Examples: integration by parts

This method works when the function to be integrated is the product of two factors, one which does not get more complicated upon repeated differentiation (e.g. trigonometric or exponential functions), with the other becoming simpler upon repeated differentiation (e.g. powers):

- $\int \arcsin(x)dx$:

  first substitute $x = \sin(\theta)$, so $dx/d\theta = \cos(\theta)$ i.e. $dx = \cos(\theta)d\theta$, then integrate by parts

$$\begin{aligned}\int \arcsin(x)dx &= \int \arcsin(\sin(\theta))\cos(\theta)d\theta = \int \theta\cos(\theta)d\theta \\ &= \theta\sin(\theta) - \int \sin(\theta)\Big(\frac{d}{d\theta}\theta\Big)d\theta = \theta\sin(\theta) - \int \sin(\theta)d\theta \\ &= \theta\sin(\theta) + \cos(\theta) = x\arcsin(x) + \sqrt{1-x^2}\end{aligned}$$

- $\int x^2\cos(2x)dx$:

  integrate by parts twice, to get rid of the annoying factor x^2

$$\begin{aligned}\int x^2\cos(2x)dx &= \frac{1}{2}\sin(2x)x^2 - \int \frac{1}{2}\sin(2x)\Big(\frac{d}{dx}x^2\Big)\\ &= \frac{1}{2}\sin(2x)x^2 - \int x\sin(2x)\\ &= \frac{1}{2}\sin(2x)x^2 - \Big\{-\frac{1}{2}\cos(2x)x + \int \frac{1}{2}\cos(2x)\Big(\frac{d}{dx}x\Big)\Big\}\\ &= \frac{1}{2}\sin(2x)x^2 + \frac{1}{2}\cos(2x)x - \int \frac{1}{2}\cos(2x)\\ &= \frac{1}{2}\sin(2x)x^2 + \frac{1}{2}\cos(2x)x - \frac{1}{4}\sin(2x)\end{aligned}$$

- $\int xe^{-x}dx$:

  integrate by parts to eliminate the factor x:

$$\begin{aligned}\int xe^{-x}dx =&\ -e^{-x}x - \Big\{-\int e^{-x}\Big(\frac{d}{dx}x\Big)\Big\} = -e^{-x}x + \int e^{-x}dx\\ =&\ -(x+1)e^{-x}\end{aligned}$$

Sometimes it is helpful to *create* a form of two factors where initially there was just one, by inserting $1 = (\frac{d}{dx}x)$, e.g.

- $\int \ln(x)dx$:

$$\begin{aligned}\int \ln(x)dx &= \int \ln(x)\Big(\frac{d}{dx}x\Big)dx = x\ln(x) - \int x\Big(\frac{d}{dx}\ln(x)\Big)dx\\ &= x\ln(x) - \int xx^{-1}dx = x\ln(x) - \int dx = x\ln(x) - x\end{aligned}$$

- $\int \arctan(x)dx$:

  first insert $1 = (\frac{d}{dx}x)$, then integrate by parts, then substitute $x^2 = y$, so $dy/dx = 2x$

$$\begin{aligned}\int \arctan(x)dx &= \int \arctan(x)\Big(\frac{d}{dx}x\Big)dx = x\arctan(x) - \int x\Big(\frac{d}{dx}\arctan(x)\Big)dx\\ &= x\arctan(x) - \int \frac{xdx}{1+x^2} = x\arctan(x) - \frac{1}{2}\int \frac{dy}{1+y}\\ &= x\arctan(x) - \frac{1}{2}\ln|1+y| = x\arctan(x) - \frac{1}{2}\ln(1+x^2)\end{aligned}$$

6.2.4. Further tricks: recursion formulae

Certain families of integrals can be calculated by a series of integrations by parts, leading in a natural way to so-called recursion formulae. This is best explained directly via examples, one with a family of definite integrals and one with a family of indefinite ones:

- Example 1:

$$I_n = \int_0^\infty x^n e^{-x} dx \qquad n \in \mathbb{Z}^+$$

Integration by parts (using $\lim_{x\to\infty} x^n e^{-x} = 0$):

$$I_n = -\Big[x^n e^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}\Big(\frac{d}{dx}x^n\Big)dx = n\int_0^\infty e^{-x}x^{n-1}dx = nI_{n-1}$$

Recursion formula: the expression for I_n in terms of I_{n-1}.
Further iteration of the recursion formula: $I_n = n(n-1)I_{n-1} = \ldots = n!I_0$
Thus we need only calculate I_0 to know *all* I_n:

$$I_0 = \int_0^\infty e^{-x}dx = \Big[-e^{-x}\Big]_0^\infty = 0-(-1) = 1 \qquad \text{hence}: \;\; I_n = n!$$

- Example 2:

$$I_{n,m}(x) = \int \sin^n(x)\cos^m(x)dx \qquad n,m \in \mathbb{Z}$$

Just using $\cos^2(x)+\sin^2(x) = 1$ already gives

$$I_{n+2,m}(x) + I_{n,m+2} = \int \sin^n(x)[\sin^2(x)+\cos^2(x)]\cos^m(x)dx = I_{n,m}(x)$$

Integration by parts:

$$\begin{aligned}
I_{n,m}(x) &= \int\left(-\frac{\sin^{n-1}(x)}{m+1}\right)\left(\frac{d}{dx}\cos^{m+1}(x)\right)dx \\
&= -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{m+1} + \frac{1}{m+1}\int\left(\frac{d}{dx}\sin^{n-1}(x)\right)\cos^{m+1}(x)dx \\
&= -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{m+1} + \frac{n-1}{m+1}\int\sin^{n-2}(x)\cos^{m+2}(x)dx \\
&= -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{m+1} + \frac{n-1}{m+1}I_{n-2,m+2}(x) \\
&= -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{m+1} + \frac{n-1}{m+1}\{I_{n-2,m}(x) - I_{n,m}(x)\}
\end{aligned}$$

Solving for $I_{n,m}(x)$ then gives

$$I_{n,m}(x)\left\{1+\frac{n-1}{m+1}\right\} = -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{m+1} + \frac{n-1}{m+1}I_{n-2,m}(x)$$

$$I_{n,m}(x)\frac{m+n}{m+1} = -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{m+1} + \frac{n-1}{m+1}I_{n-2,m}(x)$$

Final result:

$$I_{n,m}(x) = -\frac{\sin^{n-1}(x)\cos^{m+1}(x)}{m+n} + \frac{n-1}{m+n}I_{n-2,m}(x)$$

We need only calculate $I_{1,m}(x)$ and $I_{0,m}(x)$ to know *all* $I_{n,m}(x)$.

The integrals $I_{0,m}(x)$ will be done as a tutorial exercise, the integrals $I_{1,m}(x)$ are:

$$I_{1,m}(x) = \int \sin(x)\cos^m(x)dx \quad \text{put} \quad \cos(x) = y \quad \text{so} \quad dy/dx = -\sin(x)$$

$$= -\int y^m dy = -\frac{1}{m+1}y^{m+1} = -\frac{1}{m+1}\cos^{m+1}(x)$$

- Special case of previous calculation:
  (choose $m = 0$ in above result)

  $$I_n(x) = \int \sin^n(x)dx \qquad n \in \mathbb{Z}$$

  Recursion formula:

  $$I_n(x) = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}I_{n-2}(x)$$

  $$I_0(x) = x$$

Note 1:

The result of example 1 is used to generalize the concept of factorials $n!$ to *real-valued* (and later even complex) numbers n, by using the integral as a definition:

$$\text{for all } z \in \mathbb{R}: \qquad z! = \int_0^\infty x^z e^{-x}dx$$

This prompted the introduction of the so-called 'gamma function' $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$,
Thus $\Gamma(n) = (n-1)!$ for integer n.

Note 2:

As an alternative in the case of integrals with trigonometric functions one could write the latter in complex form, e.g.

$$\int \sin^n(x)dx = (2i)^{-n}\int \left(e^{ix} - e^{-ix}\right)^n dx \quad \text{use Newton's binomial formula}$$

$$= (2i)^{-n}\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}\int e^{ix(2m-n)}dx$$

$$= \begin{cases} \text{if } n \text{ even}: & \frac{1}{(2i)^n}\left\{\sum_{m=0,\ m\neq n/2}^{n}\binom{n}{m}\frac{(-1)^m}{i(2m-n)}e^{ix(2m-n)} + (-1)^{n/2}\binom{n}{n/2}x\right\} \\ \text{if } n \text{ odd}: & \frac{1}{(2i)^n}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^{m+1}}{i(2m-n)}e^{ix(2m-n)} \end{cases}$$

For instance:

$$\int \sin^2(x)dx = \frac{1}{(2i)^2}\left\{\sum_{m=0,\ m\neq 1}^{2}\binom{2}{m}\frac{(-1)^m}{i(2m-n)}e^{ix(2m-2)} - \binom{2}{1}x\right\}$$
$$= -\frac{1}{4}\left\{\binom{2}{0}\frac{1}{-2i}e^{-2ix} + \binom{2}{2}\frac{1}{2i}e^{2ix} - \binom{2}{1}x\right\}$$
$$= -\frac{1}{4}\left\{\frac{1}{-2i}e^{-2ix} + \frac{1}{2i}e^{2ix} - 2x\right\} = -\frac{1}{4}\sin(2x) + \frac{1}{2}x$$

Check against recursion method of example 2:

$$\int \sin^2(x)dx = I_2(x) = -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}I_0(x) = -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x$$
$$= -\frac{1}{4}\sin(2x) + \frac{1}{2}x$$

6.2.5. Further tricks: differentiation with respect to a parameter

Remember: any dirty trick that leads to a proposal for a primitive is allowed, provided one verifies correctness of the proposed answer *a posteriori* via differentiation! Now consider integrals that involve a further parameter $a \in \mathbb{R}$:

$$I(x,a) = \int f(x,a)dx$$

For sufficiently well-behaved functions it is true that

$$\frac{d}{da}\int f(x,a)dx = \int \frac{d}{da}f(x,a)dx$$

But not always ...
(integral is a limit, derivative is a limit, we know that the order of limits matters!)

Our strategy:
(i) *assume* that moving the derivative d/da inside or outside the integral over x is allowed
(ii) use that as a tool to calculate the integral
(iii) check whether the assumption was correct by explicit differentiation of the result

We illustrate the procedure via examples:
(notation: $(\frac{d}{da})^n$ means differentiate n times with respect to a)

- Example 1:

$$I(x,a) = \int x^n e^{ax}dx$$
$$= \int \Big(\frac{d}{da}\Big)^n e^{ax}dx = \Big(\frac{d}{da}\Big)^n \int e^{ax}dx = \Big(\frac{d}{da}\Big)^n \Big(a^{-1}e^{ax}\Big)$$

We get all the integrals we need by differentiating a simple expression.
For instance:

$$\int xe^{ax}dx \ = \frac{d}{da}\left(a^{-1}e^{ax}\right) = (\frac{x}{a} - \frac{1}{a^2})\ e^{ax}$$

$$\int x^2e^{ax}dx = \left(\frac{d}{da}\right)^2\left(a^{-1}e^{ax}\right) = \frac{d}{da}\left\{(\frac{x}{a} - \frac{1}{a^2})e^{ax}\right\} = (\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3})\ e^{ax}$$

Differentiation confirms that these primitives are correct.

- Example 2:

Here we exploit the relations

$$\frac{d}{da}\frac{1}{x^2-a} = \frac{1}{(x^2-a)^2}, \quad \left(\frac{d}{da}\right)^2\frac{1}{x^2-a} = \frac{1.2}{(x^2-a)^3}, \quad \left(\frac{d}{da}\right)^3\frac{1}{x^2-a} = \frac{1.2.3}{(x^2-a)^4}, \quad \text{etc}$$

More generally:

$$\left(\frac{d}{da}\right)^{n-1}\frac{1}{x^2-a} = \frac{(n-1)!}{(x^2-a)^n}$$

Hence, for $a \neq 0$:

$$I(x,a) = \int \frac{dx}{(x^2-a)^n}$$

$$= \int \left\{\frac{1}{(n-1)!}\left(\frac{d}{da}\right)^{n-1}\frac{1}{x^2-a}\right\} dx = \frac{1}{(n-1)!}\left(\frac{d}{da}\right)^{n-1}\int \frac{dx}{x^2-a}$$

Again we obtain all the (complicated) integrals we need
by differentiating some simple basic ones:

$$a > 0: \quad \int \frac{dx}{x^2-a} = \frac{1}{a}\int \frac{dx}{(x/\sqrt{a})^2 - 1} \qquad \text{put} \quad x = y\sqrt{a}$$

$$= \frac{-1}{\sqrt{a}}\int \frac{dy}{1-y^2} = \frac{-1}{\sqrt{a}}\text{arctanh}(y) = -\frac{\text{arctanh}(x/\sqrt{a})}{\sqrt{a}}$$

$$= -\frac{1}{2\sqrt{a}}\ln\left(\frac{1+x/\sqrt{a}}{1-x/\sqrt{\alpha}}\right) = -\frac{1}{2\sqrt{a}}\left\{\ln(\sqrt{\alpha}+x) - \ln(\sqrt{\alpha}-x)\right\}$$

$$a < 0: \quad \int \frac{dx}{x^2-a} = \frac{1}{|a|}\int \frac{dx}{(x/\sqrt{|a|})^2 + 1} \qquad \text{put} \quad x = y\sqrt{|a|}$$

$$= \frac{1}{\sqrt{|a|}}\int \frac{dy}{1+y^2} = \frac{1}{\sqrt{|a|}}\arctan(y) = \frac{\arctan(x/\sqrt{|a|})}{\sqrt{|a|}}$$

For instance:

$$\int \frac{dx}{(x^2+1)^2} = \left\{\frac{d}{da}\int \frac{dx}{x^2-a}\right\}_{a=-1} = \left\{\frac{d}{da}\frac{\arctan(x/\sqrt{-a})}{\sqrt{-a}}\right\}_{a=-1}$$

put $z=(-a)^{-1/2}$ and use the chain rule:

$$\begin{aligned}\int\frac{dx}{(x^2+1)^2} &= \left\{\frac{dz}{da}.\frac{d}{dz}\Big(z\arctan(zx)\Big)\right\}_{a=-1}\\ &= \left\{\Big(\frac{1}{2}(-a)^{-3/2}\Big)\Big(\arctan(zx)+\frac{zx}{1+z^2x^2}\Big)\right\}_{a=-1}\\ &= \frac{1}{2}\arctan(x)+\frac{x}{2(1+x^2)}\end{aligned}$$

Correctness is confirmed by explicit differentiation.

6.2.6. Further tricks: partial fractions

Come into play when integrating

$$\int\frac{p(x)}{q(x)}\,dx \qquad p(x),q(x):\ \ \text{polynomials}$$

Method is based on the following two facts:

- $p(x)$ can always be written as $p(x)=s(x)q(x)+r(x)$ ($r(x)$: the 'remainder')
  $s(x),r(x)$ other polynomials, with $r(x)$ of lower order than $q(x)$
  This gives
  $$\int\frac{p(x)}{q(x)}\,dx=\int s(x)dx+\int\frac{r(x)}{q(x)} \qquad \text{first part : easy!}$$

- If $q(x)$ is of the following form, with $\alpha_i,\beta_j,\gamma_j\in{\rm I\!R}$ (all different),
  $$q(x)=\prod_{i=1}^{n}(x+\alpha_i)^{a_i}\prod_{j=1}^{m}(x^2+\beta_jx+\gamma_j)^{b_j} \qquad \text{with} \qquad a_i,b_i\in\mathbb{Z}^+$$
  (remember: $\prod_{i=1}^{n}k_i=k_1k_2\ldots k_{n-1}k_n$) and the order of $r(x)$ is less than that of $q(x)$, then there always exists constants $A_{ik},B_{j\ell},C_{j\ell}\in{\rm I\!R}$ such that
  $$\frac{r(x)}{q(x)}=\sum_{i=1}^{n}\sum_{k=1}^{a_i}\frac{A_{ik}}{(x+\alpha_i)^k}+\sum_{j=1}^{m}\sum_{\ell=1}^{b_j}\frac{B_{j\ell}x+C_{j\ell}}{(x^2+\beta_jx+\gamma_j)^\ell}$$
  (the latter are called 'partial fractions')

In combination:
our initial integral can always be written as

$$\int\frac{p(x)}{q(x)}\,dx=\int s(x)dx\ +\ \sum_{i=1}^{n}\sum_{k=1}^{a_i}A_{ik}\int\frac{dx}{(x+\alpha_i)^k}\ +\ \sum_{j=1}^{m}\sum_{\ell=1}^{b_j}\int\frac{(B_{j\ell}x+C_{j\ell})dx}{(x^2+\beta_jx+\gamma_j)^\ell}$$

$\int s(x)dx$ with $s(x)$ polynomial: easy

$\int (x+\alpha)^{-k}dx$: easy

$\int (Bx+C)/(x^2+\beta x+\gamma)^\ell$: do-able ...

Our general strategy for integrating a ratio of polynomials:
convert the ratio into the above form

Note 1:

The simplest case of the above is the following

- If $q(x)$ is of the following form, with $\alpha_i \in \mathbb{R}$ (all different),

$$q(x) = \prod_{i=1}^{n}(x+\alpha_i) \qquad \text{with} \;\; \alpha_i \in \mathbb{R}$$

  and the order of $r(x)$ is less than that of $q(x)$, then there are constants $A_i \in \mathbb{R}$ such that

$$\frac{r(x)}{q(x)} = \sum_{i=1}^{n} \frac{A_i}{x+\alpha_i}$$

  and our initial integral can always be written as

$$\int \frac{p(x)}{q(x)}\, dx = \int s(x)dx \;\; + \;\; \sum_{i=1}^{n} A_i \ln|x+\alpha_i|$$

The constants A_j can be found upon multiplying our initial equation by $x+a_j$
followed by setting $x=-a_j$:

$$\frac{r(x)}{\prod_{i=1}^{n}(x+\alpha_i)} = \sum_{i=1}^{n}\frac{A_i}{x+\alpha_i} \quad \Rightarrow \quad \frac{r(x)}{\prod_{i=1,\; i\neq j}^{n}(x+\alpha_i)} = A_j + \sum_{i=1,\; i\neq j}^{n} A_i \frac{x+\alpha_j}{x+\alpha_i}$$

now put $x=-\alpha_j$:

$$A_j = \frac{r(-\alpha_j)}{\prod_{i=1,\; i\neq j}^{n}(\alpha_i-\alpha_j)}$$

Note 2:

Let is inspect the 'not easy but do-able' integrals above in more detail
First: write numerator as derivative of quadratic form in denominator

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^\ell}dx = \frac{B}{2}\int \frac{2x}{(x^2+\beta x+\gamma)^\ell}dx + C\int \frac{dx}{(x^2+\beta x+\gamma)^\ell}$$
$$= \frac{B}{2}\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^\ell}dx + \Big(C-\frac{1}{2}\beta B\Big)\int \frac{dx}{(x^2+\beta x+\gamma)^\ell}$$

$$= \frac{B}{2}\int \frac{d}{dx}\left\{\frac{1}{1-\ell}(x^2+\beta x+\gamma)^{1-\ell}\right\} + \left(C-\frac{1}{2}\beta B\right)\int \frac{dx}{(x^2+\beta x+\gamma)^{\ell}}$$
$$= \frac{B}{2(1-\ell)}(x^2+\beta x+\gamma)^{1-\ell} + \left(C-\frac{1}{2}\beta B\right)\int \frac{dx}{(x^2+\beta x+\gamma)^{\ell}}$$

Second: complete squares in the remaining integral and make an appropriate substitution

$$x^2+\beta x+\gamma = (x+\frac{1}{2}\beta)^2 + (\gamma-\frac{1}{4}\beta^2) \quad \text{put} \quad y = x+\frac{1}{2}, \quad \Delta^2 = |\gamma-\frac{1}{4}\beta^2|$$

$$\int \frac{dx}{(x^2+\beta x+\gamma)^{\ell}} = \int \frac{dy}{(y^2\pm\Delta^2)^{\ell}}$$

The latter integrals are of the form done earlier
(when explaining differentiation with respect to a parameter).

Let us illustrate how this works via examples:

- Example 1:

  $$I = \int \frac{dx}{(x^2-1)(x+3)}$$

  We recognize that

  (i) we integrate a ratio of polynomials → method of partial fractions

  (ii) order of numerator is less than that of denominator,
  i.e. we just have just the 'remainder' $r(x)/q(x)$ (no $s(x)$ needed)

  (iii) in fact the simplest case: $r(x)=1$ and $q(x)=(x+1)(x-1)(x+3)$

  We know that we can always find constants A_1, A_2, A_3 such that

  $$\frac{1}{(x^2-1)(x+3)} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{x+3} \quad \text{for all} \quad x\in\mathbb{R}$$

  Find these constants:

  $$A_1 = \frac{x-1}{(x^2-1)(x+3)} - \frac{A_2(x-1)}{x+1} - \frac{A_3(x-1)}{x+3}$$
  $$= \frac{1}{(x+1)(x+3)} - \frac{A_2(x-1)}{x+1} - \frac{A_3(x-1)}{x+3} \quad \text{put} \quad x=1$$
  $$= \frac{1}{(1+1)(1+3)} = \frac{1}{8}$$

  $$A_2 = \frac{x+1}{(x^2-1)(x+3)} - \frac{A_1(x+1)}{x-1} - \frac{A_3(x+1)}{x+3}$$
  $$= \frac{1}{(x-1)(x+3)} - \frac{A_1(x+1)}{x-1} - \frac{A_3(x+1)}{x+3} \quad \text{put} \quad x=-1$$
  $$= -\frac{1}{4}$$

$$A_3 = \frac{x+3}{(x^2-1)(x+3)} - \frac{A_1(x+3)}{x-1} - \frac{A_2(x+3)}{x+1}$$
$$= \frac{1}{x^2-1} - \frac{A_1(x+3)}{x-1} - \frac{A_2(x+3)}{x+1} \qquad \text{put} \quad x=-3$$
$$= \frac{1}{8}$$

Hence

$$I = \int \left\{ \frac{1}{8(x-1)} - \frac{1}{4(x+1)} + \frac{1}{8(x+3)} \right\} dx$$
$$= \frac{1}{8}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{8}\ln|x+3| = \frac{1}{8}\ln\left|\frac{x^2+2x-3}{x^2+2x+1}\right|$$

- Example 2:

$$I = \int \frac{xdx}{(x^2+1)(x+2)}$$

We recognize that

(i) we integrate a ratio of polynomials → method of partial fractions

(ii) order of numerator is less than that of denominator (no $s(x)$ needed)

(iii) this one is *not* of the simplest form $q(x) = (x+\alpha_1)(x+\alpha_2)(x+\alpha_3)$

We know that we can always find constants A, B, C such that

$$\frac{x}{(x^2+1)(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \quad \text{for all} \quad x \in \mathbb{R}$$

Find first constant:

$$A = \frac{x(x+2)}{(x^2+1)(x+2)} - \frac{(Bx+C)(x+2)}{x^2+1} = \frac{x}{x^2+1} - \frac{(Bx+C)(x+2)}{x^2+1}$$
$$\text{put } x=-2: \quad A = -\frac{2}{5}$$

so we must find constants B, C such that

$$\frac{x}{(x^2+1)(x+2)} = \frac{Bx+C}{x^2+1} - \frac{2}{5(x+2)} \quad \text{for all} \quad x \in \mathbb{R}$$

Insert two convenient values for x and solve the two resulting eqns for B and C:

$$x=-1: \quad B-C = \frac{1}{5} \qquad x=1: \quad B+C = \frac{3}{5}$$

The solution is: $B = 2/5$ and $C = 1/5$. Hence

$$I = \frac{1}{5}\int \left\{ \frac{2x+1}{x^2+1} - \frac{2}{x+2} \right\} dx$$
$$= \frac{1}{5}\int \frac{2xdx}{x^2+1} + \frac{1}{5}\int \frac{dx}{x^2+1} - \frac{2}{5}\int \frac{dx}{x+2}$$
$$= \frac{1}{5}\ln|x^2+1| + \frac{1}{5}\arctan(x) - \frac{2}{5}\ln|x+2|$$

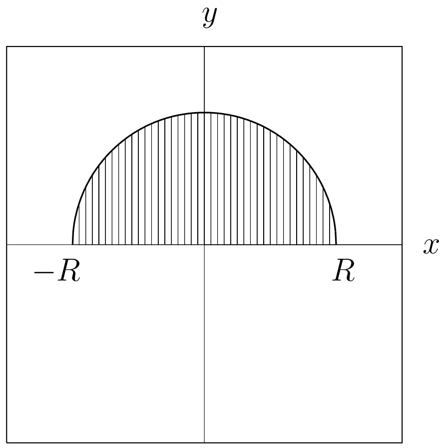
6.3. Some simple applications

6.3.1. Calculation of surface areas

The area A of the surface between the x-axis and a curve $f(x)$,
taken from $x=a$ to $x=b$ (with the accepted sign conventions) is: $A=\int_a^b f(x)dx$

- Area inside a circle:

Equation for circle with radius R,
centred in the origin: $x^2+y^2=R^2$
Upper half of circle: $y=\sqrt{R^2-x^2}$, with $x\in[-R,R]$
Area A between upper half of circle and x-axis:



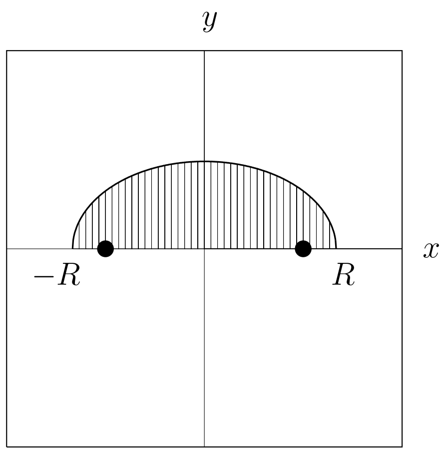
$$A=\int_{-R}^{R}\sqrt{R^2-x^2}\;dx$$
$$\text{put}\quad x=R\cos(\theta)\quad\text{so}\quad dx=-R\sin(\theta)d\theta$$
$$=-R\int_{\pi}^{0}\sqrt{R^2-R^2\cos^2(\theta)}\,\sin(\theta)d\theta$$
$$=R\int_0^{\pi}\sqrt{R^2-R^2\cos^2(\theta)}\,\sin(\theta)d\theta$$
$$=R^2\int_0^{\pi}\sin^2(\theta)d\theta=R^2\int_0^{\pi}\left\{\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right\}d\theta$$
$$=R^2\left[\frac{1}{2}\theta-\frac{1}{4}\sin(2\theta)\right]_0^{\pi}=\frac{1}{2}\pi R^2$$

Since this is exactly half of the surface area inside the circle:

$$A_{\text{circle}}=\pi R^2$$

- Area inside an ellipse:

Equation for an ellipse centred at the origin,
with foci • on x-axis at $x=\pm a$
and with sum of distances to the foci given by $2R$:



$$\sqrt{(x-a)^2+y^2}+\sqrt{(x+a)^2+y^2}=2R$$

Not yet of the required form $y=f(x)$ ...

Rewrite ellipse equation:
move one term to the right, then square both sides

$$\sqrt{(x-a)^2+y^2}=2R-\sqrt{(x+a)^2+y^2}$$
$$(x-a)^2+y^2=4R^2+(x+a)^2+y^2-4R\sqrt{(x+a)^2+y^2}$$

$$
\begin{aligned}
&-2xa = 4R^2+2xa-4R\sqrt{(x+a)^2+y^2}\\
&\sqrt{(x+a)^2+y^2} = R+xa/R\\
&(x+a)^2+y^2 = (R+xa/R)^2
\end{aligned}
$$

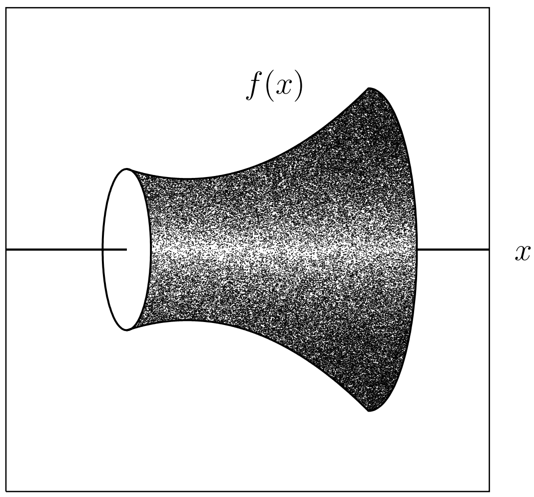
Giving: $y^2 = R^2 - a^2 - x^2(1 - a^2/R^2)$
Upper half of ellipse: $y = \sqrt{R^2-a^2-x^2(1-a^2/R^2)}$, with $x \in [-R, R]$
Hence, area A between upper half of ellipse and x-axis:

$$
\begin{aligned}
A &= \int_{-R}^{R} \sqrt{R^2-a^2-x^2(1-a^2/R^2)}\; dx\\
&= \sqrt{R^2-a^2}\int_{-R}^{R}\sqrt{1-\frac{x^2}{R^2}}\; dx \qquad \text{put} \;\; x = R\cos(\theta) \;\; \text{so} \;\; dx = -R\sin(\theta)d\theta\\
&= \;-R\sqrt{R^2-a^2}\int_{\pi}^{0}\sqrt{1-\cos^2(\theta)}\;\sin(\theta)d\theta\\
&= R\sqrt{R^2-a^2}\int_{0}^{\pi}\sin^2(\theta)d\theta = \frac{1}{2}\pi R\sqrt{R^2-a^2}
\end{aligned}
$$

(θ integral already calculated for the circle)
Since this is exactly half of the surface area inside the ellipse:

$$A_{\text{ellipse}} = \pi R\sqrt{R^2-a^2}$$

6.3.2. Calculation of volumes of revolution



Take the graph of a function $f(x)$
with $f(x) \geq 0$ for all $x \in [a, b]$

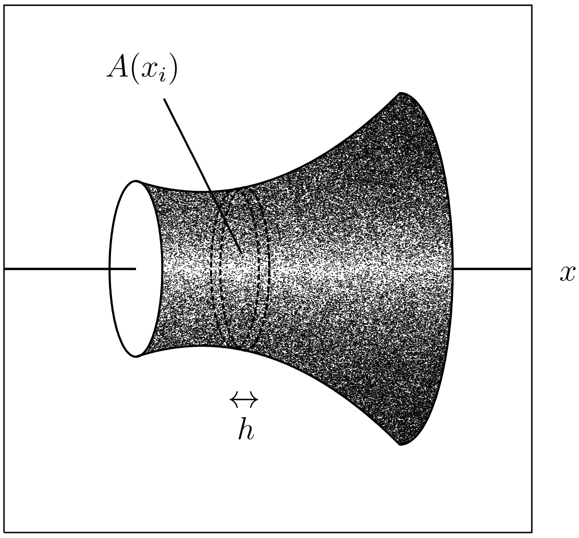
Revolve this *around* the x-axis (see figure)
Result: a *solid* in three dimensions

What is its volume V?

- split the x-axis into small steps of size h
  (i.e. split solid into small slices of width h)
  e.g. $x_i = a + (i-1)h$,
  with $i = 1, \ldots, L$ and $h = (b-a)/L$

- let the cross-section of the solid at point x_i
  have an area equal to $A(x_i)$
  then a 'slice' at position x_i contributes an amount to the volume
  that is approximately $hA(x_i)$ (if h is sufficiently small)

- cross-section of the solid at point x_i is a *circle* with radius $f(x_i)$ hence its area is $A(x_i) = \pi f^2(x_i)$
- Total volume: sum all contributions from the L slices:
$$V \;\sim\; \pi \sum_{i=1}^{L} h f^2(x_i)$$
- Expression becomes *exact* for $h \to 0$:
$$V = \lim_{h\to 0} \pi \sum_{i=1}^{L} h f^2(x_i) = \pi \int_a^b f^2(x)dx$$



<u>Example</u>:

Our solid becomes a *sphere* if what we revolve around the x-axis is a circle:
$$x^2 + y^2 = R^2 \quad \Rightarrow \quad y = \sqrt{R^2 - x^2} \quad \Rightarrow \quad \begin{cases} f(x) = \sqrt{R^2 - x^2} \\ x \in [-R, R] \end{cases}$$
Hence
$$V_{\text{sphere}} = \pi \int_{-R}^{R} f^2(x)dx = \pi \int_{-R}^{R} (R^2 - x^2)dx$$
$$= \pi \left[R^2 x - \frac{1}{3}x^3\right]_{-R}^{R} = 2\pi(R^3 - \frac{1}{3}R^3) = \frac{4}{3}\pi R^3$$

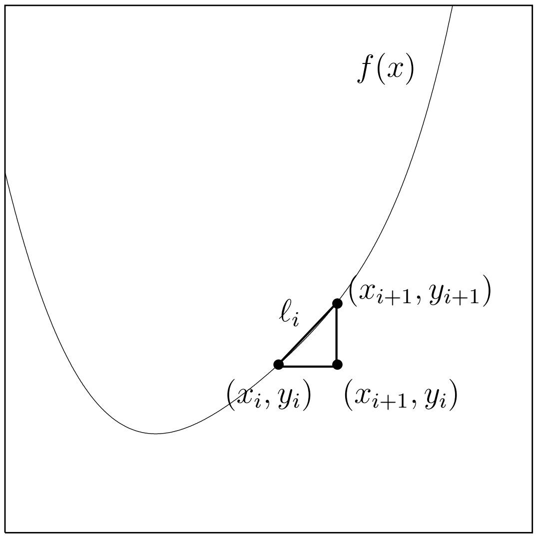
<u>Example</u>:

Our solid becomes a *cigar* if what we revolve around the x-axis is an ellipse:
$$y^2 = R^2 - a^2 - x^2(1 - a^2/R^2) \quad \Rightarrow \quad \begin{cases} f(x) = \sqrt{R^2 - a^2 - x^2(1 - a^2/R^2)} \\ x \in [-R, R] \end{cases}$$
Hence
$$V_{\text{cigar}} = \pi \int_{-R}^{R} f^2(x)dx = \pi \int_{-R}^{R} R^2 - a^2 - x^2(1 - a^2/R^2)dx$$
$$= \pi \left[(R^2 - a^2)x - \frac{1}{3}(1 - a^2/R^2)x^3\right]_{-R}^{R} = 2\pi\Big((R^2 - a^2)R - \frac{1}{3}(R^2 - a^2)R\Big)$$
$$= \frac{4}{3}\pi R(R^2 - a^2)$$

6.3.3. Calculation of the length of curves

Consider an arbitrary curve in the plane, described by a function $y = f(x)$

What is the length L of this curve between, say, $x = a$ and $x = b$?



- split the x-axis into small steps of size h
  e.g. $x_i = a + (i-1)h$,
  with $i = 1, \ldots, K$ and $h = (b-a)/K$
  write $f(x_i) = y_i$

- consider the curve segment between the points (x_i, y_i) and (x_{i+1}, y_{i+1})

  Pythagoras' theorem in triangle:
  the length of this segment is approximately

$$\ell_i \sim \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} \qquad \text{(if } x_{i+1}-x_i \text{ sufficiently small)}$$

- use $x_{i+1} - x_i = h$ and $y_i = f(x_i)$

$$\ell_i \sim h\sqrt{1+\Big(\frac{y_{i+1}-y_i}{h}\Big)^2} = h\sqrt{1+\Big(\frac{f(x_i+h)-f(x_i)}{h}\Big)^2}$$

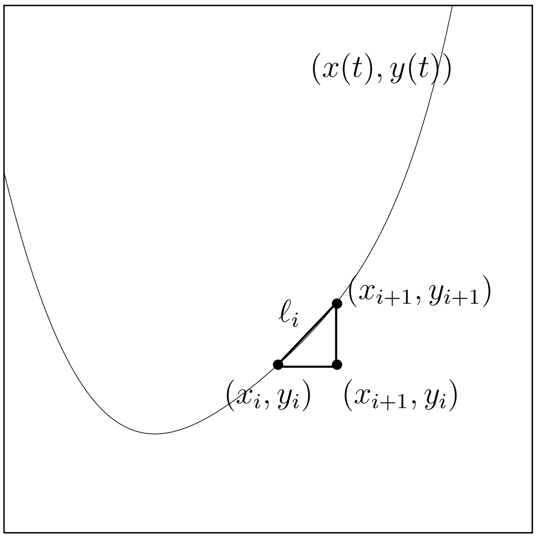
Combine:

$$L = \lim_{h\to 0} \sum_{i=1}^{K} \ell_i = \lim_{h\to 0} h\sum_{i=1}^{K} \sqrt{1+\Big(\frac{f(x_i+h)-f(x_i)}{h}\Big)^2}$$
$$= \int_a^b \sqrt{1+\Big(\frac{df}{dx}\Big)^2}\, dx$$

Example:

Circumference of a circle with radius R?
Twice length of the curve $y = \sqrt{R^2 - x^2}$,
with $x \in [-R, R]$:

$$L_{\text{circle}} = 2\int_{-R}^{R} \sqrt{1+\Big(\frac{d}{dx}(R^2-x^2)^{\frac{1}{2}}\Big)^2}\, dx = 2\int_{-R}^{R} \sqrt{1+\Big(\frac{x}{\sqrt{R^2-x^2}}\Big)^2}\, dx$$
$$= 2\int_{-R}^{R} \sqrt{1+\frac{x^2}{R^2-x^2}}\, dx = 2\int_{-R}^{R} \sqrt{\frac{R^2}{R^2-x^2}}\, dx \qquad \text{put } x = Ry$$
$$= 2R\int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}} = 2R\Big[\arcsin(y)\Big]_{-1}^{1} = 2R\Big(\frac{\pi}{2} - (-\frac{\pi}{2})\Big)$$
$$= 2\pi R$$



Consider an arbitrary curve in the plane, described parametrically by two functions $x(t)$ and $y(t)$

What is the length L of this curve between, say, $t = a$ and $t = b$?

- split the t-axis into small steps of size h
  e.g. $t_i = a + (i-1)h$,
  with $i = 1, \ldots, K$ and $h = (b-a)/K$
  write $x(t_i) = x_i$ and $y(t_i) = y_i$

- consider the curve segment between the points (x_i, y_i) and (x_{i+1}, y_{i+1})

  Pythagoras' theorem in triangle:
  the length of this segment is approximately

$$\ell_i \sim \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2} \qquad \text{(if } t_{i+1}-t_i \text{ sufficiently small)}$$

- use $x_i = x(t_i)$ and $y_i = y(t_i)$

$$\ell_i \sim h\sqrt{\Big(\frac{x_{i+1}-x_i}{h}\Big)^2 + \Big(\frac{y_{i+1}-y_i}{h}\Big)^2}$$
$$= h\sqrt{\Big(\frac{x(t_i+h)-x(t_i)}{h}\Big)^2 + \Big(\frac{y(t_i+h)-y(t_i)}{h}\Big)^2}$$

Combine:

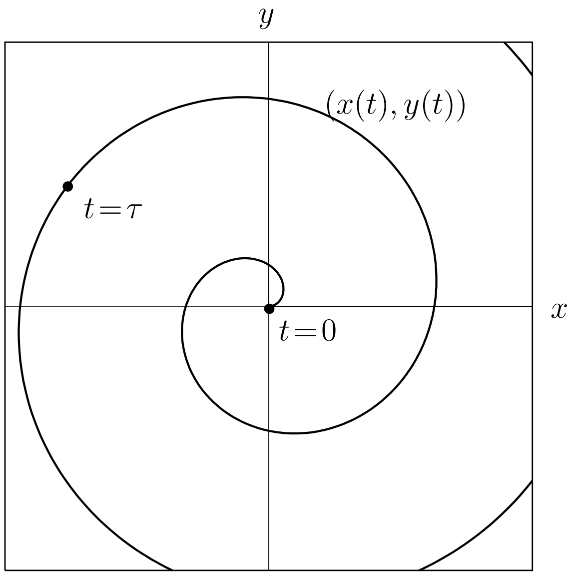
$$L = \lim_{h\to 0} \sum_{i=1}^{K} \ell_i = \lim_{h\to 0} h\sum_{i=1}^{K} \sqrt{\Big(\frac{x(t_i+h)-x(t_i)}{h}\Big)^2 + \Big(\frac{y(t_i+h)-y(t_i)}{h}\Big)^2}$$
$$= \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\; dt$$

Example:

Circumference of a circle with radius R?
parametrize $x(\theta) = R\cos(\theta)$, $y(\theta) = R\sin(\theta)$,
with $\theta \in [0, 2\pi]$:

$$L_{\text{circle}} = \int_0^{2\pi} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\; d\theta = \int_0^{2\pi} \sqrt{R^2\sin^2(\theta) + R^2\cos^2(\theta)}\; d\theta$$
$$= 2\pi R$$

Example:



Length of a spiral with radial velocity v and angular velocity ω?

parametrize
$x(t) = vt\cos(\omega t)$, $y(t) = vt\sin(\omega t)$,
with $t \in [0, \tau]$:

$$\frac{dx}{dt} = v\cos(\omega t) - vt\omega\sin(\omega t)$$
$$\frac{dy}{dt} = v\sin(\omega t) + vt\omega\cos(\omega t)$$

Hence:

$$\begin{aligned}
L_{\text{spiral}} &= \int_0^\tau \sqrt{(dx/dt)^2 + (dy/dt)^2}\; dt \\
&= v\int_0^\tau \sqrt{\Big\{\cos(\omega t) - t\omega\sin(\omega t)\Big\}^2 + \Big\{\sin(\omega t) + t\omega\cos(\omega t)\Big\}}\; dt \\
&= v\int_0^\tau \sqrt{1+\omega^2 t^2}\; dt \qquad \text{put}\ \ \omega t = u\ \ \text{so}\ \ dt = du/\omega \\
&= \frac{v}{\omega}\int_0^{\omega\tau} \sqrt{1+u^2}\; du \qquad \text{put}\ \ u = \sinh(z)\ \ \text{so}\ \ du = \cosh(z)dz \\
&= \frac{v}{\omega}\int_0^{\text{arcsinh}(\omega\tau)} \cosh^2(z)dz = \frac{v}{4\omega}\int_0^{\text{arcsinh}(\omega\tau)} \Big(e^{2z} + 2 + e^{-2z}\Big)\, dz \\
&= \frac{v}{4\omega}\int_0^{\text{arcsinh}(\omega\tau)} \Big[2z + \frac{1}{2}e^{2z} - \frac{1}{2}e^{-2z}\Big]_0^{\omega\tau} \\
&= \frac{v}{4\omega}\Big\{2\ \text{arcsinh}(\omega\tau) + \sinh(2\text{arcsinh}(\omega\tau))\Big\} \\
&= \frac{v}{2\omega}\Big\{\text{arcsinh}(\omega\tau) + \omega\tau\cosh(\text{arcsinh}(\omega\tau))\Big\} \\
&= \frac{v}{2\omega}\ln\Big(\omega\tau+\sqrt{\omega^2\tau^2+1}\Big) + \frac{v\tau}{2}\sqrt{1+\sinh^2(\text{arcsinh}(\omega\tau))} \\
&= \frac{v}{2\omega}\ln\Big(\omega\tau+\sqrt{\omega^2\tau^2+1}\Big) + \frac{v\tau}{2}\sqrt{1+\omega^2\tau^2}
\end{aligned}$$

7. Taylor's theorem and series

7.1. Introduction to series and questions of convergence

7.1.1. Series – notation and elementary properties

definition:

A *series* is an expression of the form $\sum_{n=n_0}^{\infty} a_n$
(usually $n_0 = 0, 1$)

definition:

A *partial sum* of the series is an expression of the form $S_N = \sum_{n=n_0}^{N} a_n$

definition

A series $\sum_{n=n_0}^{\infty} a_n$ is called *convergent* if the limit $S = \lim_{N\to\infty} S_N$ exists. Only then will the series yield a well-defined finite number. If the series does not converge it is called *divergent.*

Notes:

- If a series $\sum_{n=n_0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
  Proof: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = S - S = 0$

- We have already inspected the following series in exercises:

$$\sum_{n=1}^{\infty} 2^{-n}: \quad S_N = \sum_{n=1}^{N} 2^{-n} = 1-2^{-N} \quad \lim_{N\to\infty} S_N = 1, \quad \text{series convergent}$$

$$\sum_{n=1}^{\infty} 1: \quad S_N = \sum_{n=1}^{N} 1 = N \quad \lim_{N\to\infty} S_N \text{ does not exist,} \quad \text{series divergent}$$

However, having $\lim_{n\to\infty} a_n = 0$ is not enough to guarantee that a series converges, the a_n will also have to *go to zero sufficiently fast* as n gets larger. See e.g. the two examples below:

Examples:

- The series $\sum_{n=1}^{\infty} n^{-1}$ is *divergent*

  Proof:
  Consider the partial sums $S_N = \sum_{n=1}^{N} n^{-1}$, they obey

$$S_{2N} - S_N = \frac{1}{N+1} + \frac{1}{N+2} + \ldots + \frac{1}{2N-1} + \frac{1}{2N} \;\geq\; N\frac{1}{2N} = \frac{1}{2}$$

  If we assume that the series converges, then $\lim_{N\to\infty} S_N = S$ exist. However, taking the limit $\lim_{N\to\infty}$ in the previous inequality would then lead us to the contradiction $0 = S - S \geq \frac{1}{2}$. We conclude that the series must be divergent.

- The series $\sum_{n=1}^{\infty} 1/n(n+1)$ is *convergent*

  Proof:
  Since $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, the partial sum S_N can be written as

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \Big(\frac{1}{n} - \frac{1}{n+1}\Big) \\ &= \Big(\frac{1}{1} - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \ldots + \Big(\frac{1}{N} - \frac{1}{N+1}\Big) \\ &= 1 + \Big(-\frac{1}{2} + \frac{1}{2}\Big) + \Big(-\frac{1}{3} + \frac{1}{3}\Big) + \ldots + \Big(-\frac{1}{N} + \frac{1}{N}\Big) - \frac{1}{N+1} = 1 - \frac{1}{N+1} \end{aligned}$$

  It follows that $\lim_{N\to\infty} S_N = 1$, so the series converges: $\sum_{n=1}^{\infty} 1/n(n+1) = 1$.

7.1.2. Series – convergence criteria

If we are only interested in whether a series converges/diverges, the start index n_0 is irrelevant; hence one often speaks simply about the convergence or divergence of a series $\sum_n a_n$.

We now mention a number of useful results on the convergence of series, without proof.
The first applies if our a_n are similar for large n to those of another series that we know.
The others apply if we can calculate either $\lim_{n\to\infty} |a_n|^{1/n}$ or $\lim_{n\to\infty} |a_{n+1}/a_n|$:

$$\begin{array}{llcl} (C1) & \sum_n b_n \text{ is convergent and } \lim_{n\to\infty} |a_n|/b_n \text{ exists} & \Rightarrow & \sum_n a_n \text{ is convergent} \\ (C2) & \lim_{n\to\infty} |a_n|^{1/n} < 1 & \Rightarrow & \sum_n a_n \text{ is convergent} \\ (C3) & \lim_{n\to\infty} |a_n|^{1/n} > 1 & \Rightarrow & \sum_n a_n \text{ is divergent} \\ (C4) & \lim_{n\to\infty} |a_{n+1}/a_n| < 1 & \Rightarrow & \sum_n a_n \text{ is convergent} \\ (C5) & \lim_{n\to\infty} |a_{n+1}/a_n| > 1 & \Rightarrow & \sum_n a_n \text{ is divergent} \end{array}$$

The 'tricky' series are those with
$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |a_{n+1}/a_n| = 1$

Examples:

- The series $\sum_n 1/n^2$ is convergent

  Proof:
  Let $a_n = 1/n^2$, and compare to the convergent series $\sum_n b_n$ with $b_n = 1/n(n+1)$:

$$\lim_{n\to\infty} |a_n|/b_n = \lim_{n\to\infty} \frac{1/n^2}{1/n(n+1)} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} \Big(1+\frac{1}{n}\Big) = 1$$

  It now follows from $(C1)$ above that $\sum_n 1/n^2$ is also convergent.

- The series $\sum_n (-1)^{n+1}/n$ is convergent

  Proof:
  Separate in the partial sum S_N the terms with even n from those with odd n. Write $n = 2m$ for even n, and $n = 2m-1$ for odd n.

$$N \text{ even}: \quad S_N = \sum_{n=1,\ \text{odd}}^{N} \frac{1}{n} - \sum_{n=1,\ \text{even}}^{N} \frac{1}{n} = \sum_{m=1}^{N/2} \frac{1}{2m-1} - \sum_{m=1}^{N/2} \frac{1}{2m} = \sum_{m=1}^{N/2} \frac{1}{2m(2m-1)}$$

$$N \text{ odd}: \quad S_N = \sum_{n=1,\ \text{odd}}^{N} \frac{1}{n} - \sum_{n=1,\ \text{even}}^{N} \frac{1}{n} = \sum_{m=1}^{(N+1)/2} \frac{1}{2m-1} - \sum_{m=1}^{(N-1)/2} \frac{1}{2m}$$

$$= \sum_{m=1}^{(N-1)/2} \frac{1}{2m(2m-1)} + \frac{1}{N}$$

  We see that our present series $\sum_n (-1)^{n+1}/n$ is convergent if and only if the series $\sum_n a_n$ with $a_n = \frac{1}{2n(2n-1)}$ is convergent. We may now use the convergence of $\sum_n b_n$ with $b_n = 1/n^2$ together with statement $(C1)$ to finish the proof:

$$\lim_{n\to\infty} |a_n|/b_n = \lim_{n\to\infty} \frac{1/2n(2n-1)}{1/n^2} = \lim_{n\to\infty} \Big(\frac{1}{4}\frac{1}{1-1/2n}\Big) = \frac{1}{4}$$

  We conclude: $\sum_n a_n$ converges, and therefore also $\sum_n (-1)^{n+1}/n$ converges.

7.1.3. Power series – notation and elementary properties

<u>definition</u>:

A *power series* is an expression of the form $S(x) = \sum_{n=n_0}^{\infty} b_n x^n$
i.e. a series where the terms are $a_n = b_n x^n$

Why our interest?

(i) we saw that some functions can be written as power series $f(x) = \sum_{n=0}^{\infty} b_n x^n$
(ii) each of these functions had its own unique coefficients $\{b_n\}$
(iii) power series representations of functions are extremely useful
(iv) which other functions can perhaps be written as power series $f(x) = \sum_{n=0}^{\infty} b_n x^n$?
(v) how would we find the right coefficients $\{b_n\}$ for any given function?
(vi) how can we know whether the power series would converge?
(vii) how fast does a power series converge to the original function it represents?

Let us apply the convergence criteria $(C2 - C5)$ to power series,
by substitution of $a_n = b_n x^n$
(statements below make sense only if the various limits exist)

$$\text{Let } R_1 = \lim_{n\to\infty} |b_n|^{-1/n}: \qquad \begin{array}{lll} |x| < R_1: & \Rightarrow & \sum_n b_n x^n \text{ converges} \\ |x| > R_2: & \Rightarrow & \sum_n b_n x^n \text{ diverges} \end{array}$$

$$\text{Let } R_2 = \lim_{n\to\infty} |b_n/b_{n+1}|: \qquad \begin{array}{lll} |x| < R_2: & \Rightarrow & \sum_n b_n x^n \text{ converges} \\ |x| > R_2: & \Rightarrow & \sum_n b_n x^n \text{ diverges} \end{array}$$

Clearly, the two limits above must be the same.
This can be shown properly: if the limits exist, then $R_1 = R_2 = R$
We are now led in a natural way to the concept of 'radius of convergence':

<u>definition</u>

The radius of convergence R of a power series $\sum_{n=n_0}^{\infty} b_n x^n$:
a nonnegative number such that the series *converges* for all x with $|x| < R$
and *diverges* for all x with $|x| > R$.

It can be calculated either from $R = \lim_{n\to\infty} |b_n|^{-1/n}$ or from $R = \lim_{n\to\infty} |b_n/b_{n+1}|$
if these latter limits exist

Notes:

(i) Exactly *at* the radius of convergence, i.e. for $|x| = R$,
there is no general rule (just inspect the series at hand in detail)

(ii) If $R = 0$ there is apparently no $x \neq 0$ for which the series converges

(iii) If $R = \infty$ the series converges for *all* $x \in \mathbb{R}$

Examples:

- $e^x = \sum_{n=0}^{\infty} x^n/n!$

  Here $b_n = 1/n!$, so

  $$R = \lim_{n\to\infty} \frac{|b_n|}{|b_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty$$

  Thus the series for e^x converges for *all* $x \in \mathbb{R}$ (as claimed earlier).

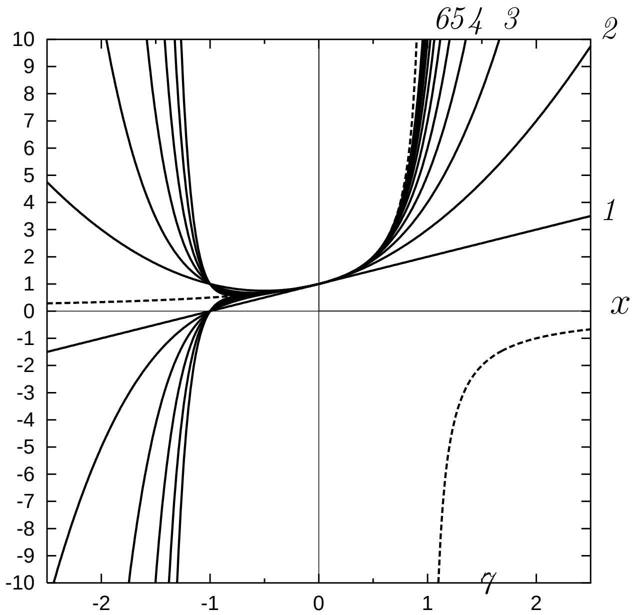
- $f(x) = \sum_{n=0}^{\infty} x^n$

  Here $b_n = 1$, so $R = \lim_{n\to\infty} |b_n|/|b_{n+1}| = 1$.
  Thus the series $f(x) = \sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$.

  Compare this to our earlier result $\sum_{n=0}^{N} x^n = (1 - x^{N+1})/(1 - x)$ for $x \neq 1$.
  We now see that this gives the series expansion for the function $f(x) = 1/(1 - x)$:

  $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Building $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ as a power series,
by taking more and more terms in the summation:



dashed : $\dfrac{1}{1-x}$

solid : $S_N(x) = \sum_{n=0}^{N} x^n$ for different choices of N

– values of N are indicated in italics

– As N increases: $S(x)$ starts to resemble $(1-x)^{-1}$ more and more, but *only on the interval* $(-1, 1)$

– $f(x)$ is only fully *identical* to $(1-x)^{-1}$ for $N \to \infty$, within the radius of convergence, i.e. for $|x| < 1$

- $\sin(x) = \sum_{m=0}^{\infty} (-1)^m x^{2m+1}/(2m+1)!$

(be careful! not yet written in standard form with x^n)
For n odd: $b_n = (-1)^m/(2m+1)!$ (where $n = 2m+1$), i.e. $b_n = (-1)^{(n-1)/2}/n!$
For n even: $b_n = 0$

$$R = \lim_{n\to\infty} |b_n|^{-1/n} = \lim_{n\to\infty} \begin{cases} 0 & \text{if } n \text{ even} \\ (n!)^{1/n} & \text{if } n \text{ odd} \end{cases}$$

It can be shown that $(n!)^{1/n} \to \infty$ as $n \to \infty$, hence our series converges for *all* $x \in \mathbb{R}$.
A quicker way to prove that $R = \infty$ is based on the inequality $|\sum_{n=0}^{\infty} a_n| \leq \sum_{n=0}^{\infty} |a_n|$:

$$|\sum_{m=0}^{\infty}(-1)^m x^{2m+1}/(2m+1)!| \leq \sum_{m=0}^{\infty}|x|^{2m+1}/(2m+1)! \leq \sum_{n=0}^{\infty}|x|^n/n!$$

The last series (the exponential function of $|x|$) converges for all $x \in \mathbb{R}$,
hence also $\sum_{m=0}^{\infty}(-1)^m x^{2m+1}/(2m+1)!$ converges for *all* $x \in \mathbb{R}$ (as claimed earlier).

7.2. Taylor's theorem

We have achieved some understanding of when power series converge,
and next turn to the *construction* of such series for arbitrary functions.
First some further notation: $f^{(n)}(x) = \left(\frac{d}{dx}\right)^n f(x)$
(the 'n-th derivative' of f at the point x)

7.2.1. Expression for the coefficients of power series

We will inspect different routes towards the formula for the coefficients of a power series.
Let us first consider non-pathological functions, where the derivative of the sum equals the sum of the derivatives. Here we can use simple ideas:

Claim:

If a given function $f(x)$ can be written as a power series,
with some convergence radius $R > 0$, i.e.

$$\forall x \in \mathbb{R},\ |x| < R: \qquad f(x) = \sum_{n=0}^{\infty} b_n x^n$$

and if differentiation and summation in the power series for f can be interchanged
(as for finite sums), then $b_n = f^{(n)}(0)/n!$ (if this n-th derivative exists).

Proof:

If the function $f(x)$ is indeed identical to the series $S(x) = \sum_{n=0}^{\infty} b_n x^n$ for $|x| < R$,
then also their derivatives must be identical, i.e. $f^{(m)}(x) = S^{(m)}(x)$ for $|x| < R$. Thus

$$f^{(m)}(x) = \left(\frac{d}{dx}\right)^m \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n \left(\frac{d}{dx}\right)^m x^n$$
$$= \sum_{n=0}^{\infty} b_n n(n-1)(n-2)\ldots(n-m+1)x^{n-m} = \sum_{n=m}^{\infty} b_n \frac{n!}{(n-m)!} x^{n-m}$$

Now choose $x = 0$: all powers of x in the right-hand side with $m > n$ will vanish, giving

$$f^{(m)}(0) = b_m m!$$

Hence $b_m = f^{(m)}(0)/m!$ as claimed. This also shows that under the assumed conditions there can only be *one* power series representation of the function f.

Let us now derive this result in another way, where we don't need to interchange summation and differentiation. It is based on repeated application of the fundamental theorem of calculus:

$$f^{(n)}(x) = f^{(n)}(0) + \int f^{(n+1)}(y)dy$$

Let f be differentiable as many times as we need:

$$\begin{aligned}
f(x) &= f(0) + \int_0^x f^{(1)}(y)dy \qquad \text{put} \;\; y = xz_1 \\
&= f(0) + x\int_0^1 f^{(1)}(xz_1)dz_1 \\
&= f(0) + x\int_0^1 \left\{ f^{(1)}(0) + \int_0^{xz_1} f^{(2)}(y)dy \right\} dz_1 \\
&= f(0) + f^{(1)}(0)x + x\int_0^1 \left\{ \int_0^{xz_1} f^{(2)}(y)dy \right\} dz_1 \qquad \text{put} \;\; y = xz_1z_2 \\
&= f(0) + f^{(1)}(0)x + x^2\int_0^1 z_1 \left\{ \int_0^1 f^{(2)}(xz_1z_2)dz_2 \right\} dz_1 \\
&= f(0) + f^{(1)}(0)x + x^2\int_0^1 z_1 \left\{ \int_0^1 \left\{ f^{(2)}(0) + \int_0^{xz_1z_2} f^{(3)}(y)dy \right\} dz_2 \right\} dz_1 \\
&= f(0) + f^{(1)}(0)x + x^2 f^{(2)}(0)\int_0^1 z_1 \left\{ \int_0^1 dz_2 \right\} dz_1 \\
&\qquad + x^2\int_0^1 z_1 \left\{ \int_0^1 \left\{ \int_0^{xz_1z_2} f^{(3)}(y)dy \right\} dz_2 \right\} dz_1 \qquad \text{put} \;\; y = xz_1z_2z_3 \\
&= f(0) + f^{(1)}(0)x + x^2 f^{(2)}(0)\int_0^1 z_1 dz_1 \\
&\qquad + x^3\int_0^1 z_1^2 \left\{ \int_0^1 \left\{ z_2 \int_0^1 f^{(3)}(xz_1z_2z_3)dz_3 \right\} dz_2 \right\} dz_1 \\
&= f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(0)x^2 + x^3\int_0^1 z_1^2 \left\{ \int_0^1 \left\{ z_2 \int_0^1 f^{(3)}(xz_1z_2z_3)dz_3 \right\} dz_2 \right\} dz_1
\end{aligned}$$

We observe:

(i) we once more generate the coefficients $b_n = f^{(n)}(0)/n!$

(ii) but now we also find explicit formulas for the difference between the true function f and the partial sums $S_N(x) = \sum_{n=0}^{N} f^{(n)}(0)x^n/n!$ of the power series:

$$f(x) \qquad = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}\, x^n + R_{N+1}(x)$$

$$\text{with e.g.} \quad R_1(x) = x\int_0^1 f^{(1)}(xz_1)dz_1$$

$$R_2(x) = x^2\int_0^1 z_1 \left\{ \int_0^1 f^{(2)}(xz_1z_2)dz_2 \right\} dz_1$$

$$R_3(x) = x^3 \int_0^1 z_1^2 \left\{ \int_0^1 \left\{ z_2 \int_0^1 f^{(3)}(xz_1z_2z_3)dz_3 \right\} dz_2 \right\} dz_1$$

7.2.2. Taylor series around $x = 0$

The last result gives us essentially the so-called Taylor series expansion in powers of x of a differentiable function f, the only difference with the above form is that the remainder term is written in a nicer way:

<u>definition</u>

The Taylor expansion to order N around $x = 0$ of a function f is the following (exact) expression, involving an N-th order polynomial and a remainder term $R_{N+1}(x)$:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}\, x^n + R_{N+1}(x) \qquad\qquad R_{N+1}(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(y)(x-y)^N dy$$

<u>Proof</u>:
(of the form claimed to be exact for the remainder term)

This is done by induction. We define as always the difference between left- and right-hand side of the claimed identity,

$$A_N(x) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}\, x^n - \frac{1}{N!}\int_0^x f^{(N+1)}(y)(x-y)^N dy$$

We aim to prove that $A_N(x) = 0$ for all integer $N \geq 0$. First we check the basis, i.e. $N = 0$:

$$\begin{aligned} A_0(x) &= f(x) - \frac{f^{(0)}(0)}{0!}\, x^0 - \frac{1}{0!}\int_0^x f^{(1)}(y)(x-y)^0 dy \\ &= f(x) - f(0) - \int_0^x f'(y)dy = f(x) - f(0) - \Big[f(y)\Big]_0^x = 0 \end{aligned}$$

So the claim is true for $N = 0$. Next we prove the induction step. We assume that $A_N(x) = 0$ (i.e. the Taylor expansion is exact for the value N) and prove from this that also $A_{N+1}(x) = 0$:

$$\begin{aligned} A_{N+1}(x) &= f(x) - \sum_{n=0}^{N+1} \frac{f^{(n)}(0)}{n!}\, x^n - \frac{1}{(N+1)!}\int_0^x f^{(N+2)}(y)(x-y)^{N+1} dy \\ &= f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}\, x^n - \frac{f^{(N+1)}(0)}{(N+1)!}\, x^{N+1} - \frac{1}{(N+1)!}\int_0^x f^{(N+2)}(y)(x-y)^{N+1} dy \\ \text{use } A_N(x) = 0: \quad &= \frac{1}{N!}\int_0^x f^{(N+1)}(y)(x-y)^N dy - \frac{f^{(N+1)}(0)}{(N+1)!}\, x^{N+1} \\ &\qquad - \frac{1}{(N+1)!}\int_0^x f^{(N+2)}(y)(x-y)^{N+1} dy \\ \text{integr by parts}: \quad &= \frac{1}{N!}\int_0^x f^{(N+1)}(y)(x-y)^N dy - \frac{f^{(N+1)}(0)}{(N+1)!}\, x^{N+1} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{N!}\left\{\left[\frac{f^{(N+1)}(y)(x-y)^{N+1}}{N+1}\right]_0^x+\int_0^x f^{(N+1)}(y)(x-y)^N dy\right\}\\
=\;&-\frac{f^{(N+1)}(0)}{(N+1)!}\,x^{N+1}-0+\frac{f^{(N+1)}(0)}{(N+1)!}x^{N+1}=0
\end{aligned}
$$

This completes the proof.

Notes:

- One can regard the Taylor series as an approximation of f for values of x close to zero, with the remainder term $R_{N+1}(x)$ indicating the precise difference between the polynomial approximation up to order N and the true function f
- If the n-th derivative of f is bounded on the interval $[-R,R]$, i.e. $|f^{(n)}(x)|\leq C_n$ for all $x\in[-R,R]$, then $|R_{N+1}(x)|\leq C_n x^{N+1}/(N+1)!$

  Proof:

  $$
  \begin{aligned}
  |R_{N+1}(x)| &\leq \frac{1}{N!}\int_0^x |f^{(N+1)}(y)|(x-y)^N dy \leq C_n\frac{1}{N!}\int_0^x (x-y)^N dy\\
  &= C_n\frac{1}{(N+1)!}\left[-(x-y)^{N+1}\right]_0^x = \frac{C_n}{(N+1)!}x^{N+1}
  \end{aligned}
  $$

  For small $|x|$, the correction is therefore generally much smaller than any of the previous terms in the approximating polynomial.

7.2.3. Taylor series around $x=a$

One can generalize the above Taylor series in order to have an expansion of functions around some other value $x=a$, in terms of powers in the deviation $x-a$.

definition

The Taylor expansion to order N around $x=a$ of a function f is the following (exact) expression,
involving an N-th order polynomial and a remainder term $R_{N+1}(x)$:

$$
f(x)=\sum_{n=0}^{N}\frac{f^{(n)}(a)}{n!}(x-a)^n+R_{N+1}(x) \qquad R_{N+1}(x)=\frac{1}{N!}\int_a^x f^{(N+1)}(y)(x-y)^N dy
$$

Proof:

This can be done by a simple change of variables. Write $x=a+u$, define $g(u)=f(a+u)$, and apply the previous Taylor expansion to $g(u)$:

$$g(u) = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!}\; u^n + R_{N+1}(u) \qquad\qquad R_{N+1}(u) = \frac{1}{N!}\int_0^u g^{(N+1)}(z)(u{-}z)^N dz$$

Use the fact that $g^{(n)}(u) = (\frac{d}{du})^n g(u) = (\frac{d}{du})^n f(a+u) = f^{(n)}(a+u)$. Thus $g^{(n)}(0) = f^{(n)}(a)$. Insertion into the above expansion, together with $g(u) = f(a+u)$ and $u = x-a$, then gives:

$$\begin{aligned} f(a+u) &= \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x{-}a)^n + R_{N+1} \\ R_{N+1} &= \frac{1}{N!}\int_0^{x-a} f^{(N+1)}(a+z)(x{-}a{-}z)^N dz \qquad \text{put} \quad z = y-a \\ &= \frac{1}{N!}\int_a^x f^{(N+1)}(y)(x{-}y)^N dy \end{aligned}$$

Notes:

- One can regard the generalized Taylor series as an approximation of f for values of x close to a, with the remainder term $R_{N+1}(x)$ indicating the precise difference between the polynomial approximation up to order N and the true function f

- If the n-th derivative of f is bounded on the interval $[a-R, a+R]$, i.e. $|f^{(n)}(x)| \leq C_n$ for all $x \in [a-R, a+R]$, then $|R_{N+1}(x)| \leq C_n(x-a)^{N+1}/(N{+}1)!$

  Proof:

  $$\begin{aligned} |R_{N+1}(x)| &\leq \frac{1}{N!}\int_a^x |f^{(N+1)}(y)|(x{-}y)^N dy \leq C_n \frac{1}{N!}\int_a^x (x{-}y)^N dy \\ &= C_n \frac{1}{(N{+}1)!}\Big[-(x{-}y)^{N+1}\Big]_a^x = \frac{C_n}{(N{+}1)!}(x{-}a)^{N+1} \end{aligned}$$

  For small values of $|x{-}a|$, the correction is therefore generally much smaller than any of the previous terms in the approximating polynomial.

7.3. Examples

7.3.1. Series expansions for standard functions

We can now *derive* the power series for arbitrary functions
(including the ones we simply stated in the past):

- $f(x) = e^x$:

  Derivatives: $f^{(n)}(x) = (\frac{d}{dx})^n e^x = e^x$, so $f^{(n)}(0) = e^0 = 1$. Hence

  $$e^x = \sum_{n=0}^{N} \frac{x^n}{n!} + R_{N+1}(x) \qquad\qquad R_{N+1}(x) = \frac{1}{N!}\int_0^x e^y(x{-}y)^N dy$$

So if $\lim_{N\to\infty} R_{N+1}(x) = 0$, which is here true for all $x \in \mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$$

- $f(x) = \sin(x)$:

  Derivatives: $f^{(2m)}(x) = (-1)^m \sin(x)$ and $f^{(2m+1)}(x) = (-1)^m \cos(x)$, so $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$. Hence

$$\begin{aligned}\sin(x) &= \sum_{m=0}^{(N-1)/2} \frac{(-1)^m x^{2m+1}}{(2m+1)!} + R_{N+1}(x)\\ R_{N+1}(x) &= \begin{cases} \frac{1}{N!}(-1)^\ell \int_0^x \cos(y)(x-y)^N dy & \text{if} \quad N = 2\ell \\ \frac{1}{N!}(-1)^{\ell+1} \int_0^x \sin(y)(x-y)^N dy & \text{if} \quad N = 2\ell+1 \end{cases}\end{aligned}$$

  So if $\lim_{N\to\infty} R_{N+1}(x) = 0$, which is here easy to prove for all $x \in \mathbb{R}$, since $\sin(y) \in [-1,1]$ and $\cos(y) \in [-1,1]$:

$$\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{1}{3}x^3 + \frac{1}{120}x^5 + \ldots$$

- $f(x) = \ln(1-x)$:

  Derivatives: $f^{(1)}(x) = -(1-x)^{-1}$, $f^{(2)}(x) = -(1-x)^{-2}$, $f^{(3)}(x) = -2(1-x)^{-3}$, $f^{(4)}(x) = -2.3(1-x)^{-4}$. Further iteration: $f^{(n)}(x) = -(n-1)!(1-x)^{-n}$ for $n \geq 1$ (this one can also prove by induction). So $f^{(n)}(0) = -(n-1)!$ for $n \geq 1$, and $f(0) = 0$. Hence

$$\ln(1-x) = -\sum_{n=1}^{N} \frac{x^n}{n} + R_{N+1}(x) \qquad R_{N+1}(x) = -\int_0^x \frac{(x-y)^N}{(1-y)^{N+1}} dy$$

  So if $\lim_{N\to\infty} R_{N+1}(x) = 0$, which is here true for all $|x| \leq 1$:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

7.3.2. Indirect methods for finding Taylor series

If you only need the first few terms of a Taylor series, combine & manipulate series you know!
(add, multiply, change signs, divide, differentiate, integrate, etc)
but keep track of which powers you keep on board and which you don't, for consistency

Examples:

$$e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots$$

$$
\begin{aligned}
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \\
\frac{1}{1+x} &= \frac{d}{dx}\ln(1{+}x) = \frac{d}{dx}\Big(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\Big) \\
&= 1 - x + x^2 - x^3 + \dots \\
\frac{1}{\cos(x)} &= \frac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots} = \frac{1}{1 + \Big(-\frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots\Big)} \\
&= 1 - \Big(-\frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots\Big) + \Big(-\frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots\Big)^2 + \dots \\
&= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \dots + \frac{1}{4}x^4 + \dots \\
&= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots \\
(1+x)^\alpha &= e^{\alpha\ln(1+x)} = e^{\alpha\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\right)} \\
&= 1 + \alpha\Big(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\Big) + \frac{1}{2}\alpha^2\Big(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\Big)^2 + \dots \\
&= 1 + \alpha x - \frac{1}{2}\alpha x^2 + \dots + \frac{1}{2}\alpha^2 x^2 + \dots \\
&= 1 + \alpha x - \frac{1}{2}\alpha(1{-}\alpha)x^2 + \dots \\
\frac{1}{\sqrt{1+x}} &= (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots \\
\tan(x) &= \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots} \\
&= \Big(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots\Big)\Big(1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots\Big) \\
&= x + \frac{1}{2}x^3 + \frac{5}{24}x^5 - \frac{1}{6}x^3 - \frac{1}{12}x^5 + \frac{1}{120}x^5 + \dots \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots
\end{aligned}
$$

7.4. L'Hopital's rule

We saw earlier how power series can be used to calculate nontrivial limits, for example:

$$\lim_{x\to 0}\frac{\tan(x)}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots}{1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots-1} = \lim_{x\to 0}\frac{1+\frac{1}{3}x^2+\frac{2}{15}x^4+\ldots}{\frac{1}{2}-\frac{1}{8}x+\ldots} = 2$$

$$\lim_{x\to 0}\frac{\frac{1}{3}\tan(3x)-\sin(x)}{2x^3} = \lim_{x\to 0}\frac{\frac{1}{3}\left(3x+\frac{1}{3}(3x)^3+\ldots\right)-\left(x-\frac{1}{6}x^3+\ldots\right)}{2x^3}$$

$$= \lim_{x\to 0}\frac{x+3x^3-x+\frac{1}{6}x^3+\ldots}{2x^3} = \lim_{x\to 0}\frac{\frac{19}{6}x^3+\ldots}{2x^3} = \frac{19}{12}$$

Let us finally use power series to inspect more generally what can be said about limits of fractions:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f(0)+xf'(0)+\frac{1}{2}x^2f''(0)+\ldots}{g(0)+xg'(0)+\frac{1}{2}x^2g''(0)+\ldots}$$

Thus, if $f(0)=g(0)=0$ (the nontrivial limits, where Fermat's simple rule fails):

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{xf'(0)+\frac{1}{2}x^2f''(0)+\ldots}{xg'(0)+\frac{1}{2}x^2g''(0)+\ldots}$$

$$= \lim_{x\to 0}\frac{f'(0)+\frac{1}{2}xf''(0)+\ldots}{g'(0)+\frac{1}{2}xg''(0)+\ldots} = \frac{f'(0)}{g'(0)}$$

This result is called L'Hopital's rule.

Notes:

(i) Never forget that L'Hopital's rule applies only when $f(0)=g(0)=0$

(ii) In contrast to using power series directly,the rule works only if $g'(0)\neq 0$

Examples of the use of L'Hopital's rule:

$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{x\to 0}\frac{\left[\ln(a)e^{x\ln(a)}\right]_{x=0}}{1} = \ln(a)$$

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \frac{\sin'(0)}{1} = \cos(0) = 1$$

$$\lim_{x\to 0}\frac{e^{7x}-1-\sin(\pi x/3)}{\ln(1+x)} = \frac{\left[7e^{7x}-\frac{\pi}{3}\cos(\pi x/3)\right]_{x=0}}{\left[(1+x)^{-1}\right]_{x=0}} = 7-\frac{\pi}{3}$$

8. Exercises

CALCULUS 1 TUTORIAL EXERCISES I

1. Use the induction method to prove the following statements:

$$(\forall n \in \mathbb{N}): \quad \sum_{k=1}^{n} k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)$$

$$(\forall n \in \mathbb{N}): \quad n+1 \leq 2^n \leq (n+1)!$$

$$(\forall n \in \mathbb{N},\ n \geq 4): \quad n^2 \leq 2^n \leq n!$$

2. Express the following three complex numbers in the form $a+ib$, with $a, b \in \mathbb{R}$:

$$(3+i)-(2+6i) \qquad (1+i)(1+2i) \qquad (2-i)^2$$

3. Let z and $\overline{z}$ be a conjugate pair of complex numbers. Prove the following three identities:

$$\mathrm{Re}(z) = \frac{1}{2}(z+\overline{z}) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z-\overline{z}) \qquad \overline{(\overline{z})} = z$$

4. Prove the following two statements:

$$(\forall z, w \in \mathbb{C}): \quad \overline{z+w} = \overline{z} + \overline{w}$$

$$(\forall z, w \in \mathbb{C}): \quad \overline{z.w} = \overline{z}.\overline{w}$$

5. z and $\overline{z}$ be a conjugate pair of complex numbers. Prove the following claims:

$$z\overline{z} \in \mathbb{R} \qquad z\overline{z} \geq 0$$

   Prove that $z.\overline{z} = 0$ if and only if $z = 0$.

6. Let $z, w \in \mathbb{C}$. Prove the following four statements:

$$|\mathrm{Re}(z)| \leq |z| \qquad |\mathrm{Im}(z)| \leq |z| \qquad |z.w| = |z|.|w| \qquad |\overline{z}| = |z|$$

7. Let $z, w \in \mathbb{C}$, with $w \neq 0$. prove the following statements:

$$\overline{z/w} = \overline{z}/\overline{w} \qquad |z/w| = |z|/|w|$$

8. Express the following five complex numbers in the form $a+ib$, with $a, b \in \mathbb{R}$:

$$i^3 \qquad i^4 \qquad i^5 \qquad i^{63} \qquad (i)^{-10}$$

9. Express the following five complex numbers in the form $a+ib$, with $a, b \in \mathbb{R}$:

$$\frac{1}{2-3i} \qquad \frac{2+i}{1+3i} \qquad \frac{1-i}{1+i} \qquad (-1+i\sqrt{3})^3 \qquad (-1-i\sqrt{3})^3$$

10. Find all solutions of the equation $z^3 - 1 = 0$. Hint: write $z^3 - 1$ as the product of $(z-1)$ and another factor.

CALCULUS 1 TUTORIAL EXERCISES II

11. Simplify the four complex numbers $(1+i)^4$, $(1-i)^4$, $(-1+i)^4$, and $(-1-i)^4$. Use your results to find all solutions of the equation $z^4+1=0$.

12. Let $z = 1+2i$. Calculate the following seven complex numbers, and plot them in the complex plane (i.e. on an Argand diagram):
$\overline{z}$, $1/z$, $1/\overline{z}$, $z+\overline{z}$, $z-\overline{z}$, $z\overline{z}$, and $z/\overline{z}$.

13. What are the modulus and the principal values of the arguments of the following complex numbers?

(a) $1+i$	(b) $1-i$	(c) $-2-2i$	(d) $1+\sqrt{3}i$
(e) $1-\sqrt{3}i$	(f) $-1-i$	(g) $e^{2\pi i/3}$	(h) $2e^{-i\pi/4}$
(i) $-e^{3i\pi/4}$	(j) $e^{2+\pi i/2}$		

14. Let $z = re^{i\phi}$, with $r, \theta \in \mathbb{R}$ and $r \geq 0$. Find all values of r and θ such that $z^5 = 1$. Plot the corresponding numbers in the complex plane.

15. Let $z = re^{i\phi}$, with $r, \theta \in \mathbb{R}$ and $r \geq 0$. Find all values of r and θ such that $z^6 = 64$. Plot the corresponding numbers in the complex plane.

16. Find all solutions $z \in \mathbb{C}$ of the equation $(z-1)^5 = 1$. Plot the corresponding numbers in the complex plane.

17. Find all solutions $z \in \mathbb{C}$ of the equation $(z-i)^6 = 64$. Plot the corresponding numbers in the complex plane.

18. Let a, b and u denote real numbers. Find the real and the imaginary parts of the following complex numbers:

(i) e^{a+ib} (ii) $\dfrac{e^{ua}e^{ibu}}{a+ib}$ (iii) $e^{e^{iu}}$

19. Sketch the following sets in the complex plane:

$$A = \{z \in \mathbb{C} \mid |z-i|^2 = 4\}$$
$$B = \{z \in \mathbb{C} \mid (z-i)^2 = 4\}$$
$$C = \{z \in \mathbb{C} \mid 1 < |z+1| \leq 3\}$$

CALCULUS 1 TUTORIAL EXERCISES III

20. Prove by induction that $\sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}$ for all $n \in \mathbb{N}$. Use this result to determine the value of the series $\sum_{k=1}^{\infty} 2^{-k}$. Give an example of a series of the form $\sum_{k=1}^{n} a_k$, with finite values of the sum for any finite n, but such that the summation $\sum_{k=1}^{\infty} a_k$ does *not* exist.

21. Show that for $x \in \mathbb{R}$ the series representation $e^z = \sum_{n=0}^{\infty} z^n/n!$ of the exponential function has the following properties (here and in subsequent exercises you will be allowed to interchange differentiation and summation):

$$\frac{d}{dx}e^{ax} = ae^{ax} \qquad e^0 = 1$$

22. Use the series representations of the trigonometric functions, i.e. $\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$ and $\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$, to derive the following properties for $\theta \in \mathbb{R}$:

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta), \qquad \sin(0) = 0$$
$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta), \qquad \cos(0) = 1$$

23. Give the values of $\cos(\theta)$ and $\sin(\theta)$ for the following choices of the angle: $\theta = 0,\ \frac{\pi}{6},\ \frac{\pi}{4},\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{3\pi}{4},\ \frac{5\pi}{6},\ \pi,\ \frac{5\pi}{4},\ \frac{3\pi}{2},\ \frac{5\pi}{3}$
Give your values as fractions in terms of $\sqrt{2}$, $\sqrt{3}$ etc.
(i.e. not as decimals as given by a calculator)

24. Show that $\cos(\pi/12) = \frac{1}{2}\sqrt{2+\sqrt{3}}$.
(Hint: use the formula for $\cos(2\theta)$ in terms of $\cos(\theta)$)

25. Find all solutions $\theta \in \mathbb{R}$ (if any) of the equation $\cot(\theta) + \tan(\theta) = \alpha$, for the following three cases:

$$\text{(i)}\ \ \alpha = 1 \qquad \text{(ii)}\ \ \alpha = 2 \qquad \text{(iii)}\ \ \alpha = 4$$

26. Let $\alpha, \beta \in \mathbb{R}$. Rewrite each of the following expressions as a constant times a product of trigonometric functions:
(i) $\sin(\alpha - \beta) + \sin(\alpha + \beta)$ (ii) $\cos(\alpha - \beta) - \cos(\alpha + \beta)$

CALCULUS 1 TUTORIAL EXERCISES IV

27. Express the following combinations of functions in the form $c\sin(\theta+\alpha)$, where c and α are real constants which you have to find (assume $|\beta| < \pi/2$):

    (a) $\sin(\theta)+\cos(\theta)$ (b) $\sqrt{3}\sin(\theta)-\cos(\theta)$ (c) $\sin(\theta)-\tan(\beta)\cos(\theta)$

28. Find the values of

    (a) $\arcsin(\sin(5\pi/6))$ (b) $\arcsin(\sin(7\pi/6))$ (c) $\arccos(\cos(7\pi/6))$
    (d) $\arccos(\cos(11\pi/6))$

29. Show that the series representations of sinh and cosh are:

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \qquad \cosh(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

30. Use the above series representations of the hyperbolic functions (rather than the definition in terms of exponentials) to re-derive the following for $x \in \mathbb{R}$:

$$\frac{d}{dx}\sinh(x) = \cosh(x), \qquad \sinh(0) = 0$$
$$\frac{d}{dx}\cosh(x) = \sinh(x), \qquad \cosh(0) = 1$$

31. Show that $\frac{d}{dx}\tanh(x) = 1 - \tanh^2(x)$.

32. Simplify the following complex numbers in which $\alpha, \beta \in \mathbb{R}$ to the standard form $a+ib$, with $a, b \in \mathbb{R}$:

    (a) $\cosh(i\pi)$ (b) $\sinh(i\pi/3)$ (c) $\tanh(i\pi/6)$
    (d) $\sin(\alpha+i\beta)$ (e) $\cos(\alpha-i\beta)$ (f) $\cosh(\pi)$

33. Find formulae that express the hyperbolic functions $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ as ratios of polynomials of $t = \tanh(x/2)$.

34. Calculate the following three derivatives:

    (a) $\frac{d}{dx}\operatorname{arccosh}(x)$ (b) $\frac{d}{dx}\operatorname{arcsinh}(x)$ (c) $\frac{d}{dx}\operatorname{arctanh}(x)$

CALCULUS 1 TUTORIAL EXERCISES V

35. Let $z \in \mathbb{C}$, $z \neq 1$. Prove the following statement by induction:

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$$

Now choose $z = e^{i\theta}$ with $\theta \in \mathbb{R}$, and use the identity that you have just proven to find formulas for the two sums $\sum_{k=0}^{n} \sin(k\theta)$ and $\sum_{k=0}^{n} \cos(k\theta)$. What if θ is a multiple of 2π? Check the correctness of your formulas for the simplest cases $n = 0$ and $n = 1$.

36. Use the explicit formula for the function arcsinh : $\mathbb{R} \to \mathbb{R}$, namely $\text{arcsinh}(y) = \ln(y + \sqrt{y^2+1})$, to show that

$$\text{arcsinh}(\sinh(x)) = x \quad \text{for all } \ x \in \mathbb{R}$$
$$\sinh(\text{arcsinh}(y)) = y \quad \text{for all } \ y \in \mathbb{R}$$

37. Use the explicit formula for the function arccosh : $[1,\infty) \to [0,\infty)$, namely $\text{arccosh}(y) = \ln(y + \sqrt{y^2-1})$, to show that

$$\text{arccosh}(\cosh(x)) = x \quad \text{for all } \ x \in [0,\infty)$$
$$\cosh(\text{arccosh}(y)) = y \quad \text{for all } \ y \in [1,\infty)$$

38. Use the explicit formula for the function arctanh : $(-1,1) \to \mathbb{R}$, namely $\text{arctanh}(y) = \frac{1}{2}\ln[(1+y)/(1-y)]$, to show that

$$\text{arctanh}(\tanh(x)) = x \quad \text{for all } \ x \in \mathbb{R}$$
$$\tanh(\text{arctanh}(y)) = y \quad \text{for all } \ y \in (-1,1)$$

CALCULUS 1 TUTORIAL EXERCISES VI

39. Let $n \in \mathbb{N}$, and calculate the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.
Hint: substitute $n^{-1} = x$ and try to use limits calculated in the lectures.

40. Calculate the following limits, if they exist, without using power series:

(a) $\displaystyle\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$ (b) $\displaystyle\lim_{x\to 0}\frac{\sin(7x)}{\tan(2x)}$

(c) $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{x}$ (d) $\displaystyle\lim_{x\to 0}\cosh(x)$

41. Calculate the following limits, if they exist, using the power series representations of the hyperbolic functions:

(a) $\displaystyle\lim_{x\to 0}\frac{\sinh(x)}{x}$ (b) $\displaystyle\lim_{x\to 0}\frac{1-\cosh(x)}{x}$

(c) $\displaystyle\lim_{x\to 0}\frac{1-\cosh(x)}{x^2}$ (d) $\displaystyle\lim_{x\to 0}\frac{\tanh(x)}{x}$

42. Calculate the following limits by making clever substitutions:

(a) $\displaystyle\lim_{x\to \pi}\frac{\sin(x)}{\pi - x}$ (b) $\displaystyle\lim_{x\to 0}\frac{\arcsin(x)}{x}$

(c) $\displaystyle\lim_{x\to 0}\frac{\arcsin(3x)}{\tan(5x)}$ (d) $\displaystyle\lim_{x\to 0}\frac{\operatorname{arctanh}(x)}{x}$

43. Calculate the following limits, if they exist, using any suitable method:

(a) $\displaystyle\lim_{x\to\infty}\frac{\sin(x)}{x}$ (b) $\displaystyle\lim_{x\to\infty}\frac{\cos(x)}{x^2}$

(c) $\displaystyle\lim_{x\to\infty} x(e^{1/x}-1)$ (d) $\displaystyle\lim_{x\downarrow 0} x^x$

(e) $\displaystyle\lim_{x\to\infty}\frac{\ln(x)}{x^2+1}$ (f) $\displaystyle\lim_{x\downarrow 0}\frac{x^x-1}{x\ln(x)}$

(g) $\displaystyle\lim_{x\downarrow 0} e^{1/x}$ (h) $\displaystyle\lim_{x\downarrow 0} x^{\sin(x)}$

(i) $\displaystyle\lim_{x\to 0}\frac{a^x-b^x}{x}$ $(a>b>0)$ (j) $\displaystyle\lim_{x\downarrow 0}\frac{(x+1)\ln(x)}{\sin(x)}$

(k) $\displaystyle\lim_{x\uparrow 0} e^{1/x}$ (l) $\displaystyle\lim_{x\to\infty} xe^{1/x}$

CALCULUS 1 TUTORIAL EXERCISES VII

44. Let $f_k(x) : \mathbb{R} \to \mathbb{R}$ denote n arbitrary functions, with $k = 1, 2, \ldots, n$, and let $\prod_{k=1}^{n} f_k(x) = f_1(x).f_2(x).\ldots.f_{n-1}(x).f_n(x)$. Prove by induction the following generalized version of the product rule:

$$\text{for all } n \in \mathbb{Z}^+ : \qquad \frac{d}{dx}\left(\prod_{k=1}^{n} f_k(x)\right) = \left(\sum_{\ell=1}^{n} \frac{f'_\ell(x)}{f_\ell(x)}\right)\left(\prod_{k=1}^{n} f_k(x)\right)$$

45. Now prove the previous generalized product rule directly (i.e. without induction), for the special case where $f_k(x) > 0$ for all $x \in \mathbb{R}$ and all k. Hint: write $f(x) = e^{g(x)}$ with $g(x) = \ln(f(x))$, and use the chain rule.

46. calculate the following derivatives:

(a) $\dfrac{d}{dx}\, e^{-x^2}$ (b) $\dfrac{d}{dx}\, \ln(\tan(x))$

(c) $\dfrac{d}{dx}\, (x\ln(x) - x)$ (d) $\dfrac{d}{dx}\, \arcsin(x^2)$

(e) $\dfrac{d}{dx}\, \arctan(e^x)$ (f) $\dfrac{d}{dx}\, e^{\sin(x)}$

(g) $\dfrac{d}{dx}\, xe^{\arctan(x)}$ (h) $\dfrac{d}{dx}\, \ln|\arcsin(x)|$

47. calculate the following derivatives:

(a) $\dfrac{d}{dx}\, \arcsin\left(\dfrac{1-x}{1+x}\right)$ (b) $\dfrac{d}{dx}\, 3^{\sin(x)}$

(c) $\dfrac{d}{dx}\, \ln|x + \sqrt{x^2 - a^2}|$ (d) $\dfrac{d}{dx}\, \dfrac{\sqrt{1+x}}{x}$

(e) $\dfrac{d}{dx}\, \dfrac{1}{x}\arccos\left(\sqrt{1-x^2}\right)$ (f) $\dfrac{d}{dx}\, x^{x\sin(x)}$

(g) $\dfrac{d}{dx}\, \ln(\cosh(x) + \sinh(x))$ (h) $\dfrac{d}{dx}\, \ln\left(\sqrt{(1+x^2)/(1-x^2)}\right)$

CALCULUS 1 TUTORIAL EXERCISES VIII

48. Each of the following is an equation that which determines y as an implicit function of x. Find in all cases an expression for dy/dx.

(a)	$x^2 + y^2 = 1$	(b)	$y^3 + x^3 = 1$
(c)	$\sinh(x) + \cosh(y) = 1$	(d)	$xy - e^{x+y} = 2$
(e)	$\sin(y) + y = x^3$	(f)	$y^2 + x(x-1)(x+1) = 0$

49. Each of the following are a pair of equations which determine x and y in terms of a parameter $t \in \mathbb{R}$, which defines a function $y(x)$ implicitly. Find in all cases an expression for dy/dx.

(a)	$x = \cos(t),\ y = \sin(t)$	(b)	$x = \cosh(t),\ y = \sinh(t)$
(c)	$x = t + \sin(t),\ y = \cos(t)$	(d)	$x = t^3,\ y = t^2$
(e)	$x = e^{2t},\ y = \tanh(t)$	(f)	$x = e^t\cos(t),\ y = e^t\sin(t)$

50. Calculate the integral $A = \int_0^b e^{ax}dx$ (with $a > 0$ and $b > 0$) using the sandwich method, with suitably constructed bounding staircase functions, similar to how this was done in the lectures for integrals such as $\int_0^b \cos(x)dx$ and others. Hint: use staircases with steps of equal size h, and evaluate the summations that show up using the formula in exercise 39, i.e.

$$z \neq 1: \qquad \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

51. Find a 'primitive' for each of the following functions, i.e. a function $F(x)$ such that $F'(x) = f(x)$. Prove your claims.

(a)	$f(x) = 1/(1+x)$	(b)	$f(x) = 1/(1-x)^2$
(c)	$f(x) = 1/(2x+1)^3$	(d)	$f(x) = 1/(1+x^2)$
(e)	$f(x) = 1/(1-x^2)$	(f)	$f(x) = 1/\sqrt{1-x^2}$
(g)	$f(x) = 1/\sqrt{1+x^2}$	(h)	$f(x) = xe^{-\frac{1}{2}x^2}$

CALCULUS 1 TUTORIAL EXERCISES IX

52. Calculate the following indefinite integrals using the method of substitution:

(a) $\int \frac{e^x dx}{1+e^x}$ put $x = \ln(y)$

(b) $\int \sqrt{1-x^2}dx$ put $x = \sin(y)$

(c) $\int \frac{xdx}{\sqrt{1-x^2}}$ put $x^2 = 1-y$

(d) $\int \frac{dx}{\sqrt{a^2+x^2}}$ put $x = at$

(e) $\int \frac{2x^3 dx}{1+x}$ put $x = t-1$

(f) $\int \frac{xdx}{1+\sqrt{x}}$ put $x = t^2$

53. Calculate the following indefinite integrals using the method of substitution:

(a) $\int \cos(\sin(x))\cos(x)dx$ (b) $\int \frac{2xdx}{1+x^4}$

(c) $\int \frac{e^x dx}{e^{2x}-1}$ (d) $\int \frac{dx}{2\sqrt{x}\sqrt{1-x}}$

(e) $\int \frac{xdx}{\sqrt{1-x^2}}$ (f) $\int \frac{x^2 dx}{\sqrt{1-x^2}}$

(g) $\int \frac{dx}{(1-x^2)^{3/2}}$ (h) $\int \frac{xdx}{(1-x^2)^{3/2}}$

(i) $\int \frac{x^2 dx}{(1-x^2)^{3/2}}$ (j) $\int \frac{dx}{\cos(x)}$

54. Calculate the following indefinite integrals using integration by parts:

(a) $\int x\sinh(x)dx$ (b) $\int (1+x^2)e^x dx$

(c) $\int \arcsin(x)dx$ (d) $\int x^3\sin(x)dx$

(e) $\int x^2\Big(\ln(x)\Big)^2 dx$ (f) $\int x\arctan(x)dx$

CALCULUS 1 TUTORIAL EXERCISES X

55. Define the indefinite integrals $I_n(x)$ for $n \in \mathbb{Z}^+$ as follows:
$$I_n(x) = \int \frac{dx}{(1+x^2)^n}$$
Use integration by parts to derive the following recursion formula:
$$I_n(x) = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2n-2}\, I_{n-1}(x)$$
Use the recursion formula to find the indefinite integral $\int (1+x^2)^{-2} dx$.

56. Define the indefinite integrals $I_n(x)$ for $n \in \mathbb{Z}$ as $I_n(x) = \int \cos^n(x) dx$. Use integration by parts to derive the following recursion formula:
$$I_n(x) = \frac{1}{n} \sin(x) \cos^{n-1}(x) + \frac{n-1}{n}\, I_{n-2}(x)$$
Use the recursion formula to find the indefinite integral $\int \cos^4(x) dx$.

57. (i) Define the definite integrals $I_n(a)$ for $n \in \mathbb{Z}^+$ and $a \in \mathbb{R}$ as $I_n(a) = \int_{-\infty}^{\infty} x^{2n} e^{-ax^2}$. Assume that differentiation with respect to a can be moved from outside to inside the integral. Use repeated differentiation with respect to a to find a formula that expresses $I_n(a)$ in terms of $I_0(a)$.
    (ii) Show that $I_0(a) = C/\sqrt{a}$, where $C = \int_{-\infty}^{\infty} e^{-y^2} dy$. From now on you may use (without proof) that $C = \sqrt{\pi}$.
    (iii) Use the previous results to show that $\int_{-\infty}^{\infty} x^2 e^{-x^2/2} = \sqrt{2\pi}$ and that $\int_{-\infty}^{\infty} x^4 e^{-x^2/2} = 3\sqrt{2\pi}$.

58. Use the method of partial fractions to calculate the following integrals:
(a) $\displaystyle\int \frac{dx}{x^2+x-6}$ (b) $\displaystyle\int \frac{xdx}{(1-x)^2(1+x)}$
(c) $\displaystyle\int \frac{dx}{2x^2+x-1}$ (d) $\displaystyle\int \frac{xdx}{12x^2-7x-12}$

59. Calculate the length L of the curve in the plane described by the function $y = \ln(\cos(x))$, between the points $x = -\pi/4$ and $x = \pi/4$. Hint: use the result of an earlier exercise to deal with the integral.

CALCULUS 1 TUTORIAL EXERCISES XI

60. Find out for the following series whether they are convergent or divergent, using the various results stated and/or derived in the lectures or otherwise:

    (a) $\sum_n n^{-3}$ (b) $\sum_n (-1)^n n^{-2}$

    (c) $\sum_n 1/\sqrt{n}$ (d) $\sum_n n^\alpha e^{-n} \quad (\alpha > 0)$

61. Find the radii of convergence for the following powers series:

    (a) $\sum_n x^n/n^3$ (b) $\sum_n (-1)^n x^n/n^2$

    (c) $\sum_n x^n/\sqrt{n}$ (d) $\sum_n x^n n^\alpha e^{-n} \quad (\alpha > 0)$

62. Derive the Taylor expansions for the following functions, up to order $N = 3$ for the first three functions and up to order N for the last one, and give in each case an exact expression for the remainder term:

    (a) $f(x) = \tan(x)$ (b) $f(x) = \tanh(x)$

    (c) $f(x) = x^3$ (d) $f(x) = \sqrt{1+x}\, e^{\sin(x)}$

63. Write the following two functions as power series of the form $\sum_{n=0}^{\infty} b_n x^n$, and determine for each the associated radius of convergence:

    (a) $f(x) = 1/(1-x)^2$ (b) $f(x) = \arctan(x)$

    Substitute $x = 1$ into your result under (b) (why is it not obvious that this is allowed?) and derive an expression for the number π as a series (the so-called Leibniz series).

64. Find the first three terms (i.e. up to order x^3) of the Taylor expansions for the following two functions, by combining and/or manipulation the Taylor expansions of other functions that you know:

    (a) $f(x) = \tanh(x)$ (b) $f(x) = \ln\left(\frac{1+2x}{1-2x}\right)$