Computing with Calabi–Yau manifolds: Lecture I

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19 October 2015

Why Calabi-Yau?

CY manifolds and algebraic geometry

Projective space

The quintic threefold

Affine varieties

Why Calabi–Yau?

- Consider a compactification of string theory or M-theory which describes an effective supersymmetric theory of lower dimension on Minkowski space. What is needed to construct such a compactification?

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- For the simplest solutions (without any background fluxes), one needs a compact Riemannian manifold X, together with whatever geometric structure is required to specify expectation values in directions parallel to X of gauge fields or k-form fields from the associated supergravity theory. (For example, one may ask for a G-bundle with connection if the gauge fields are $\mathfrak{g}$-valued.)

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- For the simplest solutions (without any background fluxes), one needs a compact Riemannian manifold X, together with whatever geometric structure is required to specify expectation values in directions parallel to X of gauge fields or k-form fields from the associated supergravity theory. (For example, one may ask for a G-bundle with connection if the gauge fields are $\mathfrak{g}$-valued.)
- The key condition we focus on today is the requirement of supersymmetry in the compactified theory. For this, X must be a spin manifold with a covariantly constant spinor. Moreover, the Einstein equations of motion imply that the metric on X must be Ricci flat in order to obtain an effective theory in Minkowski space.

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- Every Riemannian metric g_{ij} has a holonomy group $\mathrm{Hol}(g_{ij})$, which is the group of symmetries of the tangent space $T_{X,x}$ at a point $x \in X$ obtained by parallel transport along loops based at x. We use the Levi–Civita connection of g_{ij} to do parallel transport; other connections can also have holonomy groups.

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- (Technicality #2: there is also a restricted holonomy group obtained by using contractible loops only. If X is not simply connected, this group may be different.)
- For a generic Riemannian metric on an oriented ℓ-manifold, the holonomy group is isomorphic to $SO(\ell)$. However, the existence of a covariantly constant spinor or covariantly constant k-form reduces the holonomy.

Why Calabi–Yau?

- The possible holonomy groups with a covariantly constant spinor are known (due to Berger). All of the corresponding metrics are Ricci flat. For X of dimension ℓ, the possibilites with irreducible holonomy are:
  - A flat metric, with trivial holonomy.
  - If $\ell = 4n$, a hyperKähler metric, with holonomy $Sp(n) \subset SO(4n)$. These metrics have three covariantly constant 2-forms, any nonzero linear combination of which will serve as the (rescaled) Kähler form on X for an appropriate choice of complex structure on X.
  - If $\ell = 2m \geq 6$, a "strict" Calabi–Yau metric, with holonomy $SU(m) \subset SO(2m)$. These metrics have a covariantly constant 2-form ω, a complex structure which makes ω into the Kähler form, and a covariantly constant holomorphic m-form Ω (i.e. an $(m, 0)$-form).
  - If $\ell = 7$, a G_2 metric, with holonomy $G_2 \subset SO(7)$. These metrics feature a covariantly constant 3-form.
  - If $\ell = 8$, a Spin(7) metric, with holonomy $\mathrm{Spin}(7) \subset SO(8)$. These metrics feature a covariantly constant self-dual 4-form.

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- More generally, if there is a covariantly constant spinor then X is a finite quotient of a product of manifolds of the types in Berger's classification.
- Here is a brief review of Kähler manifolds. If $\ell = 2m$, X has a complex structure if it can be constructed by gluing open subsets of $\mathbb{C}^m$ together with coordinate change maps which are holomorphic. Any Hermitian metric $g_{\alpha\bar{\beta}}$ on a complex manifold has an associated 2-form $\omega := (\sqrt{-1}/2) \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}$ called the Kähler form of the metric; the metric is Kähler if the Kähler form is closed, i.e., $d\omega = 0$.

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- On any complex manifold, the exterior derivative d acting on complex-valued differential forms splits into two operators ∂ and $\overline{\partial}$ indicating the holomorphic and anti-holomorphic parts of differentiation. The complex valued k-forms $\mathcal{A}^k$ can be decomposed as a sum of complex valued (p,q)-forms (with p dz's and q $d\bar{z}$'s).

Why Calabi–Yau?

- Kähler review, con. On a compact Kähler manifold, many wonderful things happen, the first of which is that the Laplace operator $\Delta_d = dd^* + d^*d$ coincides with one-half of the Laplace operator $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and also with one-half of the Laplace operator $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. Each element in de Rham cohomology

$$H^k_{dR}(X) = \ker(d : \mathcal{A}^k \to \mathcal{A}^{k+1}) / \operatorname{im}(d : \mathcal{A}^{k-1} \to \mathcal{A}^k)$$

has a unique harmonic representative, and each element in Dolbeault cohomology

$$H^{p,q}_{\overline{\partial}}(X) = \ker(\overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}) / \operatorname{im}(\overline{\partial} : \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q})$$

also has a unique harmonic representative. This leads to the Hodge decomposition:

Why Calabi–Yau?

- Kähler review, con.

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

As a consequence, the k^{th} Betti number $b_k := \dim H^k(X)$ is a sum of Hodge numbers $h^{p,q} := \dim H^{p,q}(X)$.

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- HyperKähler manifolds are Riemannian manifolds which are Kähler in more than one way, that is, there is more than one complex structure for which the metric is Kähler. HyperKähler manifolds are usually studied with respect to some fixed choice of complex structure, but always bear in mind that this choice can be varied.

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- On any Kähler manifold X, the Ricci tensor of the metric can similarly be made into a 2-form, and this 2-form gives a de Rham representative of the first Chern class $c_1(X)$. In particular, if the metric is Ricci flat then the first Chern class must vanish.

Why Calabi–Yau?

- In the 1950's, Eugenio Calabi proved that if X is a compact complex manifold of complex dimension m with $c_1(X) = 0$ and if ω_1 and ω_2 are the Kähler forms of two Ricci-flat Kähler metrics on X which have the same de Rham cohomology class, i.e., $\omega_1 - \omega_2 = d\eta$, then the metrics are the same, i.e., $\omega_1 = \omega_2$. This gives a uniqueness result for Ricci-flat Kähler metrics in terms of the de Rham cohomology class of their Kähler form, and Calabi went on to conjecture that a Ricci-flat metric exists for any Kähler class of X. Note that any such metric has holonomy contained in $SU(m)$.

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- The existence was a very challenging problem, finally settled by Shing-Tung Yau in 1978. The name "Calabi–Yau manifold" was conferred by Candelas, Horowitz, Strominger, and Witten in the first application to string compactification (1985).

Why Calabi–Yau?

- At this level of generality, a Calabi–Yau manifold is a compact complex manifold equipped with a Ricci-flat Kähler metric. Complex tori, compact hyperKähler manifolds with a chosen complex structure, and "strict" compact Calabi–Yau manifolds all provide examples. Additional examples arise by taking products and finite quotients. The key criterion is: Kähler and $c_1(X) = 0$.

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- We remark that there is a large (stringy) literature about “non-compact Calabi–Yau manifolds,” which have a slightly different definition. Not relevant here.

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- We remark that there is a large (stringy) literature about "non-compact Calabi–Yau manifolds," which have a slightly different definition. Not relevant here.
- Many Calabi–Yau manifolds, including all "strict" compact Calabi–Yau manifolds as well as some complex tori and some hyperKähler manifolds, can be embedded as complex submanifolds of a complex projective space (although they do not inherit their Riemannian metric from the ambient space).

Calabi–Yau manifolds and algebraic geometry

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- A theorem of Chow says that any compact complex submanifold $X \subset \mathbb{CP}^N$ is the set of common zeros of a finite collection of homogeneous polynomials (taken from the homogeneous coordinate ring of $\mathbb{CP}^N$). This applies to Calabi–Yau manifolds, and shows that our task belongs to the mathematical field of algebraic geometry.

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- The basic objects of study in algebraic geometry are sets of commons zeros of collections of polynomials, not necessarily homogeneous ones. Such a set U is called an affine variety, and more general algebraic varieties are built up as $X = \bigcup_\alpha U_\alpha$, with the affine varieties U_α attached by maps which are ratios of polynomials whose denominators do not vanish on the intersections.

Calabi–Yau manifolds and algebraic geometry

- To study the Kähler classes on X, we first observe that they form a real cone, because if $\omega_1, \ldots, \omega_r$ are Kähler forms, then $\sum_{j=1}^{r} a_j\omega_j$ is a Kähler form for any positive real numbers a_j. We will determine a set of Kähler classes of the form $[c_1(L_j)]$ for "ample line bundles" L_j on X (to be described later), and the real cone spanned by these classes will be the full Kähler cone of X whenever $h^{2,0}(X) = 0$. (This includes the case of "strict" compact Calabi–Yau manifolds.) In general, this span will only generate what we call the algebraically spanned subcone of the Kähler cone.

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- The Calabi–Yau condition is checked in a practical way as follows: we will seek a holomorphic m-form on X; if it has no zeros, then $c_1(X) = 0$.

Projective space as an algebraic variety

- Our first example is the construction of $\mathbb{CP}^N$ as an algebraic variety. A familiar construction to many of you is

$$\mathbb{CP}^N = (\mathbb{C}^{N+1} - \{\vec{0}\})/(z_0, \ldots, z_N) \sim (\lambda z_0, \ldots, \lambda z_N)$$

for $\lambda \neq 0$, $\lambda \in \mathbb{C}$. The ring of polynomials $\mathbb{C}[z_0, \ldots, z_N]$ is known as the homogenous coordinate ring of the projective space, and a homogeneous polynomial $f(z_0, \ldots, z_N)$ of degree k will scale as λ^k with λ. Thus, f is not a well-defined function on $\mathbb{CP}^N$ but rather (as we will explain soon) a section of a line bundle.

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- To express $\mathbb{CP}^N = \bigcup_\alpha U_\alpha$ is conceptually clear but notationally awkward. We want U_α to be the subset where $z_\alpha \neq 0$. On that subset, we can take $\lambda = 1/z_\alpha$ so that the z_j/z_α for $j \neq \alpha$ serve as coordinates on U_α. The set of polynomials of which we take the common zeros is empty in this case, and we use the entire $\mathbb{C}^N$ described by those coordinates as U_α.

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- As an example of the gluing, note that $x_i := z_i/z_0$, $i = 1, \ldots, N$ give coordinates on U_0 and $y_j := z_j/z_N$, $j = 0, \ldots, N-1$ give coordinates on U_N. The coordinate change is $y_0 = 1/x_N$ and $y_j = x_j/x_N$ for $j = 1, \ldots, N-1$.

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- What is the fate of the homogeneous polynomial f? On U_0 it takes the form $f(1, x_1, \ldots, x_N)$. On $U_0 \cap U_N$, since f is homogeneous of degree k this transforms to

$$x_N^k f(\frac{1}{x_N}, \frac{x_1}{x_N}, \ldots, 1) = y_0^{-k} f(y_0, \ldots, y_{N-1}, 1).$$

Projective space as an algebraic variety

- We define the line bundle $\mathcal{O}_{\mathbb{CP}^N}(k)$ as $\bigcup_\alpha (U_\alpha \times \mathbb{C})$, where a coordinate s_α on $\mathbb{C}$ is added to the coordinates of U_α. For any line bundle, the coordinate change will take the form $s_\beta = \psi_{\beta\alpha} s_\alpha$ for a non-vanishing function $\psi_{\beta\alpha}$ on $U_\alpha \cap U_\beta$. In this particular case, we wish to choose the "transition functions" $\psi_{\beta\alpha}$ so that $f(z_0, \ldots, z_N)$ can be interpreted as a section of the line bundle.

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- That is, if we define "local sections" $s_0 = f(1, x_1, \dots, x_N)$ and $s_N = f(y_0, \dots, y_{N-1}, 1)$ (and similarly for the other coordinate charts) then $s_0 = y_0^{-k} s_N$ so that $\psi_{0N} := y_0^{-k} = x_N^k$ defines the transition function and hence the line bundle.

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- The set $\{f(z_0, \ldots, z_N) = 0\}$ of zeros of f defines an algebraic subvariety of $\mathbb{CP}^N$ known as the divisor of f, or $\mathrm{div}(f)$. Note that while f itself is not invariant under scaling by λ, $\mathrm{div}(f)$ is invariant.

Projective space as an algebraic variety

- To write an N-form on $\mathbb{CP}^N$, we start with the symmetric expression

$$dz_0 \wedge dz_1 \wedge \cdots \wedge dz_N.$$

This has too high a degree, but can be corrected by contraction with the "Euler vector field" to obtain

$$\Omega := dz_0 \wedge dz_1 \wedge \cdots \wedge dz_N \lrcorner \sum z_j \frac{\partial}{\partial z_j}$$
$$= z_0 dz_1 \wedge dz_2 \cdots \wedge dz_N - z_1 dz_0 \wedge dz_2 \cdots \wedge dz_N + \cdots$$

This is not invariant under scaling, but if we divide by a homogeneous polynomial of degree $N+1$, it becomes invariant:

Projective space as an algebraic variety

- $$\frac{\Omega}{f_{N+1}(z_0,\ldots,z_N)}$$

  is a meromorphic differential form on $\mathbb{CP}^N$. Since it has poles, $c_1(\mathbb{CP}^N) \neq 0$. That is, complex projective space is not itself a Calabi–Yau manifold.

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- (This is not a surprise!)

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- As a first example of a Calabi–Yau manifold, we choose a (general) homogeneous polynomial of degree 5 in 5 variables. To be concrete, we may take

$$f_5(z_0, \ldots, z_5) := z_0^5 + \cdots + z_4^5 + 5\psi z_0 z_1 z_2 z_3 z_4.$$

The zeros of this polynomial define a subvariety $X \subset \mathbb{CP}^4$.

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The zeros of this polynomial define a subvariety $X \subset \mathbb{CP}^4$.
- The description of projective space in terms of coordinate charts extends immediately to this case. We have
$$U_\alpha = \{z_\alpha = 1, f(z_0,\ldots,z_4) = 0\}$$
with the same coordinate change maps as before. What changes is the functions which we use: since f restricts to 0 on X, the functions on U_0 are described as
$$R_0 := \mathbb{C}[x_1,\ldots,x_4]/I_0$$
where I_0 is the ideal in $\mathbb{C}[x_1,\ldots,x_4]$ consisting of all multiples of $f_5(1,x_1,\ldots,x_4)$.

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- Similarly, the functions on U_4 are described as

$$R_4 := \mathbb{C}[y_0, \ldots, y_3]/I_4$$

with I_4 being all multiples of $f_5(y_0, \ldots, y_3, 1)$.

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- We would like to interpret the change of coordinates $y_0 = 1/x_4$, $y_j = x_j/x_4$ as a homomorphism of rings $R_4 \to R_0$, i.e.,

$$\mathbb{C}[y_0, \ldots, y_3]/I_4 \to \mathbb{C}[x_1, \ldots, x_4]/I_0$$

  obtained by substituting the coordinate change into polynomials, but due to the denominators in the coordinate change formulas, we need to do this carefully.

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- Similarly, the functions on U_4 are described as

$$R_4 := \mathbb{C}[y_0, \ldots, y_3]/I_4$$

  with I_4 being all multiples of $f_5(y_0, \ldots, y_3, 1)$.
- We would like to interpret the change of coordinates $y_0 = 1/x_4$, $y_j = x_j/x_4$ as a homomorphism of rings $R_4 \to R_0$, i.e.,

$$\mathbb{C}[y_0, \ldots, y_3]/I_4 \to \mathbb{C}[x_1, \ldots, x_4]/I_0$$

  obtained by substituting the coordinate change into polynomials, but due to the denominators in the coordinate change formulas, we need to do this carefully.
- The complement of $U_0 \cap U_4$ inside U_4 is defined by $y_0 = 0$, so functions on $U_0 \cap U_4$ should be allowed to have powers of y_0 in the denominator. This ring of functions is known as a localization of the original ring R_4, and is denoted by $(R_4)_{y_0}$. To repeat:

The quintic threefold

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- Note that the line bundle $\mathcal{O}_X(k)$ is defined in exactly the same way as on the ambient projective space.

The quintic threefold

- To compute a holomorphic 3-form on X, we use the Poincaré residue formula. This is a generalization of the familiar Cauchy residue formula in one complex variable:

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{dz}{z} = g(0),$$

where γ circles the origin once in a counterclockwise direction.

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- The Poincaré residue applies to a complex manifold of dimension $m+1$, and associates to a meromorphic differential form of degree $m+1$ whose poles are on $f=0$, a holomorphic differential form on the divisor $\{f=0\}$, by specifying the integrals of that holomorphic form over arbitrary m-cycles γ on $\{f=0\}$, as follows:

$$\int_\gamma \mathrm{Res}(\Omega/f) = \frac{1}{2\pi i}\int_\Gamma \Omega/f,$$

where Γ is formed from γ by taking counterclockwise circles within the the normal directions to $\{f=0\}$.

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- The conclusion for a divisor X of degree $N+1$ in $\mathbb{CP}^N$ is that it has a holomorphic $(N-1)$-form with neither zeros nor poles, in other words, that $c_1(X) = 0$. In particular, this applies to the quintic threefold, and gives the first example of a Calabi–Yau manifold of complex dimension three.

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- The possible complex structures in this case are determined by the coefficients in $f_5(z_0, \ldots, z_4)$, which in general has many more terms than in our example. In fact, there are 101 independent complex parameters here.
- On the other hand, the set of Kähler classes is only one-dimensional, and the only effective Kähler parameter in this case is the volume.

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- Given a finite collection of polynomials $f_1, \dots, f_r$, in the ring $\mathbb{C}[x_1, \dots, x_n]$, we define their vanishing locus to be

$$V(f_1, \dots, f_r) = \{(x_1, \dots, x_n) \mid f_j(x_1, \dots, x_n) = 0 \text{ for all } j\}.$$

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- In fact, the vanishing locus only depends on the ideal I generated by $f_1, \ldots, f_r$, and we can denote it by $V(I)$.
- On the other hand, given a subset $V \subset \mathbb{C}^n$, we can ask for the ideal $I(V)$ of polynomials vanishing at all points of V. Hilbert's "Nullstellensatz" asserts that

$$I(V(I)) = \sqrt{I} := \{g \mid g^k \in I \text{ for some } k > 0\}.$$

Affine varieties

- An affine algebraic variety is a set of the form $V(I) \subset \mathbb{C}^n$ for a "radical" ideal, i.e., one which satisfies $\sqrt{I} = I$. Such a variety has a coordinate ring $R := \mathbb{C}[x_1, \ldots, x_n]/I$ of functions defined on it. Moreover, it has natural open sets (forming the so-called Zariski topology) generated by open sets of the form $\{g \neq 0\}$ for some nonzero $g \in R$.

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- The functions on the open set $\{g \neq 0\}$ belong to the localization of R at g, denoted by R_g. Formally, the way to allow denominators along g is to introduce an auxiliary variable:

$$R_g = R[y]/(yg - 1).$$