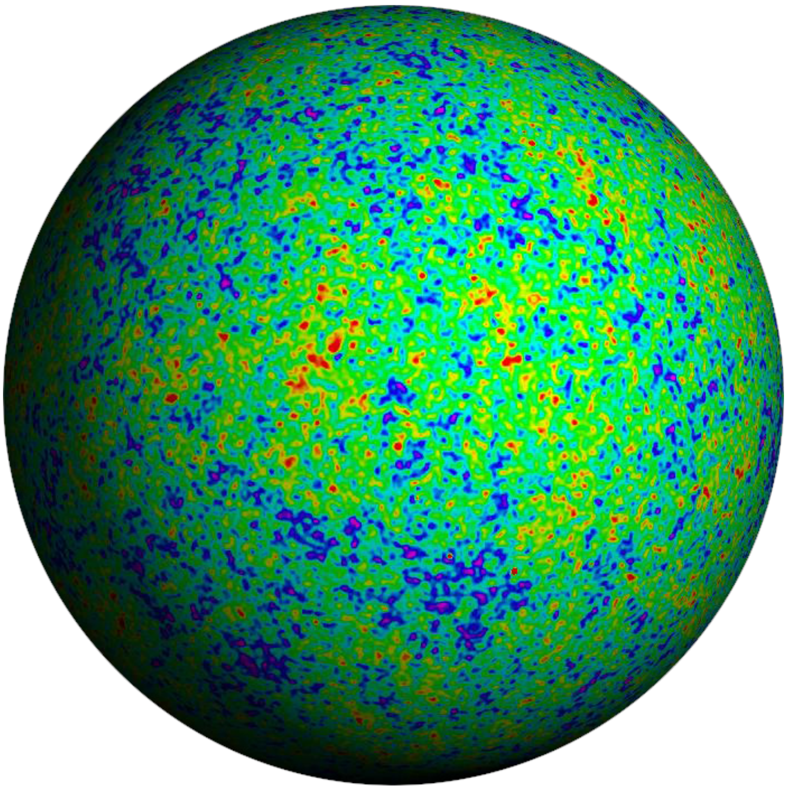
Physics of the Cosmic Microwave Background

Daniel Baumann

Department of Applied Mathematics and Theoretical Physics,
Cambridge University, Cambridge, CB3 0WA, UK

Institute of Theoretical Physics, University of Amsterdam,
Science Park, 1090 GL Amsterdam, The Netherlands



Contents

1 Motivation

Why should you learn about the physics of the cosmic microwave background (CMB)? First and foremost, because it is beautiful physics. Second, because it is important physics: the CMB is a fundamental feature of our universe and so you should know about it. Moreover, the physics of the CMB is perfectly understood; it is one of the big triumphs of modern cosmology. In this course I look forward to telling you about this.

The CMB was created 380,000 years after the Big Bang, when the universe had cooled enough for neutral atoms to become stable. Photons stopped scattering off the electrons in the primordial plasma and started streaming freely through the universe. About 13.8 billion years later, this radiation is captured by our telescopes. The frequency spectrum of the CMB is that of a nearly perfect blackbody with an effective temperature of about 2.7 degrees Kelvin. This temperature isn't completely uniform, but varies across the sky (see Fig. 1). These fluctuations are a directly consequence of density perturbations in the primordial plasma and therefore teach us a lot about the early universe. The physics of the formation and evolution of the CMB fluctuations will be the main focus of these lectures.

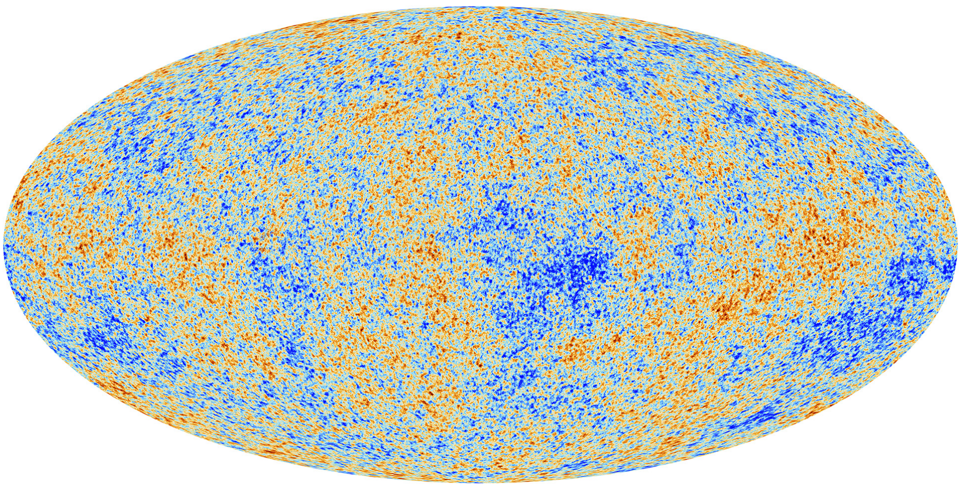


Figure 1. Map of the CMB fluctuations as measured by the Planck satellite.

References

This course is a follow-up on my Part III course on *Cosmology* [1], to which I refer you for background material. These notes draw heavily on Anthony Challinor's excellent Part III course on *Advanced Cosmology* [2]. In preparing these lectures, I also found Julien Lesgourgues' TASI lectures [3] and Wayne Hu's CMB lectures [4] very helpful. For more details, I recommend the textbooks by Dodelson [5] and by Uzan and Peter [6]. Finally, the PhD thesis' of Matias Zaldarriaga [7] and Wayne Hu [8] are rather remarkable documents and contain some nice pedagogical material.

Software

These lectures will be complemented by a series of example classes (taught by Dr. Enrico Pajer). A key element of these classes will be an introduction to the standard Boltzmann codes used by the CMB community. The goal is to provide you with the tools to perform your own computations.

We will give a tutorial on the Boltzmann code CLASS. This software can be downloaded **here**. You should install it on your laptop before coming to the classes. Some documentation on the installation is available on the same page. A thorough discussion of the most common issues can be found in the **installation wiki**. A **forum for support** is contained in the same wiki. The code is successfully installed when the command "./class explanatory.in" returns something like "Running CLASS version v2.4.3 ...", without any errors. Notice that to run just CLASS, it is sufficient to perform the installation with "make class" rather than "make", which compiles a bunch of other things. A series of lectures on CLASS can be found **here**.

Notation and Conventions

Throughout, we will use natural units, in which the speed of light and Planck's constant are set equal to one, $c = \hbar \equiv 1$. Our metric signature is $(+ - --)$. Greek indices $\mu, \nu, \cdots$ stand for spacetime indices, e.g. (x^μ, p^μ), and run from 0 to 3. We will use the Einstein summation convention where repeated indices are summed over. Latin indices $i, j, k, \cdots$ will stand for spatial indices, e.g. (x^i, p^i). Bold font will denote spatial three-vectors, e.g. $(\mathbf{x}, \mathbf{p})$. We will use η for conformal time and overdots will stand for derivatives with respect to η.

2 Preliminaries

We will begin with a lightning review of some elementary concepts in cosmology. I will assume that you have seen most, if not all, of this material before, so I will cite many results without detailed derivations. Further details can be found in my *Cosmology* course [1] or any of the standard textbooks (e.g. [5, 6]).

2.1 Homogeneous Cosmology

The FRW metric of a homogenous and isotropic spacetime is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\gamma_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\,, \tag{2.1}$$

where γ_{ij} denotes the metric of a maximally symmetric 3-space. Throughout these lectures, we will restrict to the special case of flat spatial slices, i.e. $\gamma_{ij} = \delta_{ij}$, and define $\mathrm{d}\mathbf{x}^2 \equiv \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. We will also introduce conformal time, $\mathrm{d}\eta = \mathrm{d}t/a(t)$, so that the metric becomes

$$\mathrm{d}s^2 = a^2(\eta)\left(\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right). \tag{2.2}$$

We will first discuss the kinematics of particles in an FRW spacetime for an arbitrary scalar factor $a(\eta)$. Then we will show how the Einstein equations determine $a(\eta)$ in terms of the stress-energy content of the universe.

2.1.1 Kinematics

The kinematics of particles in the FRW spacetime follows from the geodesic equation

$$p^\nu\nabla_\nu p^\mu = p^\nu\left(\frac{\partial p^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}p^\rho\right) = 0\,, \tag{2.3}$$

where $p^\mu \equiv dx^\mu/d\lambda$ is the 4-momentum of the particle. We will find it convenient (especially when we study perturbations) to parameterize the 4-momentum in terms of the components relative to an orthonormal tetrad $(E_0)^\mu$ and $(E_i)^\mu$ with the property

$$g_{\mu\nu}(E_\alpha)^\mu(E_\beta)^\nu = \eta_{\alpha\beta}\,. \tag{2.4}$$

We take $(E_0)^\mu$ to be the four-velocity of an observer at rest with respect to the coordinate system. This implies, $(E_0)^\mu = a^{-1}\delta^\mu_0$ and $(E_i)^\mu = a^{-1}\delta^\mu_i$. Let the energy and 3-momentum of a particle in the orthonormal frame be $p^{\hat{0}} \equiv E$ and $p^{\hat{\imath}}$. (We will use hat's on the momentum indices to denote components defined in the orthonormal frame.) The 4-momentum seen by a general observer then is

$$p^\mu = E(E_0)^\mu + p^{\hat{\imath}}(E_i)^\mu = a^{-1}[E,\mathbf{p}]\,. \tag{2.5}$$

For massless particles, we have the constraint $g_{\mu\nu}p^\mu p^\nu = E^2 - |\mathbf{p}|^2 = 0$, so we can write $\mathbf{p} = E\mathbf{e}$, where $\mathbf{e}$ is a unit vector in the direction of propagation.

Exercise.—Show that the non-zero Christoffel symbols associated with the metric (2.2) are

$$\Gamma^0_{00} = \mathcal{H}\,, \quad \Gamma^0_{ij} = \mathcal{H}\delta_{ij}\,, \quad \Gamma^i_{j0} = \mathcal{H}\delta^i_j\,. \tag{2.6}$$

The $\mu = 0$ component of the geodesic equation (2.3) becomes

$$p^0 \frac{dp^0}{d\eta} = -\Gamma^0_{\alpha\beta} p^\alpha p^\alpha \,. \tag{2.7}$$

Substituting (2.5) and (2.6), we get

$$(a^{-1}E)\frac{d}{d\eta}(a^{-1}E) = -\mathcal{H}a^{-2}E^2 - a^{-2}\mathcal{H}|\mathbf{p}|^2 \,, \tag{2.8}$$

which simplifies to

$$\frac{1}{E}\frac{dE}{d\eta} = -\frac{1}{a}\frac{da}{d\eta} \quad \to \quad E \propto a^{-1} \,. \tag{2.9}$$

This describes the redshifting of the photon energy in an expanding spacetime.

2.1.2 Dynamics

The evolution of the scale factor is determined by the Friedmann equations

$$3\mathcal{H}^2 = 8\pi G a^2 \bar{\rho} \,, \tag{2.10}$$

$$2\dot{\mathcal{H}} + \mathcal{H}^2 = -8\pi G a^2 \bar{P} \,, \tag{2.11}$$

where $\mathcal{H} \equiv \partial_\eta \ln a$ is the conformal Hubble parameter, and $\bar{\rho}$ and $\bar{P}$ are the background density and pressure, respectively.

Exercise.—By substituting (2.6) into

$$R_{\mu\nu} \equiv \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \,, \tag{2.12}$$

show that

$$R_{00} = -3\dot{\mathcal{H}} \,, \quad R_{ij} = (\dot{\mathcal{H}} + 2\mathcal{H}^2)\delta_{ij} \quad \Rightarrow \quad a^2 R = -6(\dot{\mathcal{H}} + \mathcal{H}^2) \,. \tag{2.13}$$

Hence, show that the non-zero components of the Einstein tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$, are

$$G_{00} = 3\mathcal{H}^2 \,, \quad G_{ij} = -(2\dot{H} + \mathcal{H}^2)\delta_{ij} \,. \tag{2.14}$$

Use this to confirm that the 00-Einstein equation, $G_{00} = 8\pi G T_{00}$, implies (2.10) and the ij-Einstein equation, $G_{ij} = 8\pi G T_{ij}$, leads to (2.11).

Combining (2.10) and (2.11), we can write an evolution equation for the density

$$\dot{\bar{\rho}} = -3\mathcal{H}(\bar{\rho} + \bar{P}) \,. \tag{2.15}$$

For pressureless matter ($\bar{P}_m \approx 0$) this implies $\bar{\rho}_m \propto a^{-3}$, while for radiation ($\bar{P}_r = \frac{1}{3}\bar{\rho}_r$) we have $\bar{\rho}_r \propto a^{-4}$.

Exercise.—Derive the continuity equation (2.15) from $\nabla^\mu T_{\mu\nu} = 0$.

2.1.3 The Cosmic Inventory

At early times, particle interactions were efficient enough to keep the different species of the primordial plasma (photons (γ), electrons (e), neutrinos (ν), cold dark matter (c), etc.) in local equilibrium. The different species then shared a common temperature T. The number density, energy density and pressure of each species a is then given by

$$n_a = g_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, f_a(\mathbf{x}, \mathbf{p})\,, \tag{2.16}$$

$$\rho_a = g_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, f_a(\mathbf{x}, \mathbf{p})\, E(p)\,, \tag{2.17}$$

$$P_a = g_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, f_a(\mathbf{x}, \mathbf{p})\, \frac{p^2}{3E(p)}\,, \tag{2.18}$$

where $f_a(\mathbf{x}, \mathbf{p})$ is the (phase space) distribution function of the species a. In the unperturbed universe, the distribution functions should not depend in the position and the direction of the momentum, i.e. $f_a(\mathbf{x}, \mathbf{p}) \to \bar{f}_a(E(p))$. The equilibrium distribution functions are

$$\bar{f}_a(E) = \frac{1}{e^{(E-\mu_a)/T} \pm 1}\,, \tag{2.19}$$

with $+$ for bosons and $-$ for fermions. The chemical potential μ vanishes for photons and is (likely) small for all other species. We will henceforth set it to zero. When the temperature drops below the mass of the particles, $T \ll m$, they become non-relativistic and their distribution function receives an exponential suppression, $f \to e^{-m/T}$. This means that relativistic particles ('radiation') dominate the density and pressure of the primordial plasma. By performing the integrals (2.17) and (2.18) in the limit $E \to p$, one finds

$$\bar{\rho}_a = \frac{\pi^2}{30}\, g_a T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases} \qquad \text{and} \qquad \bar{P}_a = \frac{1}{3}\bar{\rho}_a\,. \tag{2.20}$$

The total radiation density is

$$\bar{\rho}_r = \frac{\pi^2}{30}\, g_\star T^4\,, \quad \text{where} \quad g_\star \equiv \sum_{a=b} g_a + \frac{7}{8} \sum_{a=f} g_a\,. \tag{2.21}$$

If equilibrium had persisted until today, all species with masses greater than 10^{-3} eV would be exponentially suppressed. Not a very interesting world. Fortunately, many massive particle species (dark matter, baryons) have sufficiently weak interactions, so that they decoupled from the primordial plasma. The freeze-out abundance of the dark matter has shaped the large-scale structure of the universe.

2.1.4 A Brief Thermal History

The key to understanding the thermal history of the universe is understanding the competition between the interaction rate of particles, Γ, and the expansion rate, H. Particles maintain equilibrium as long as $\Gamma \gg H$ and freeze out when $\Gamma \lesssim H$ (see Fig. 2).

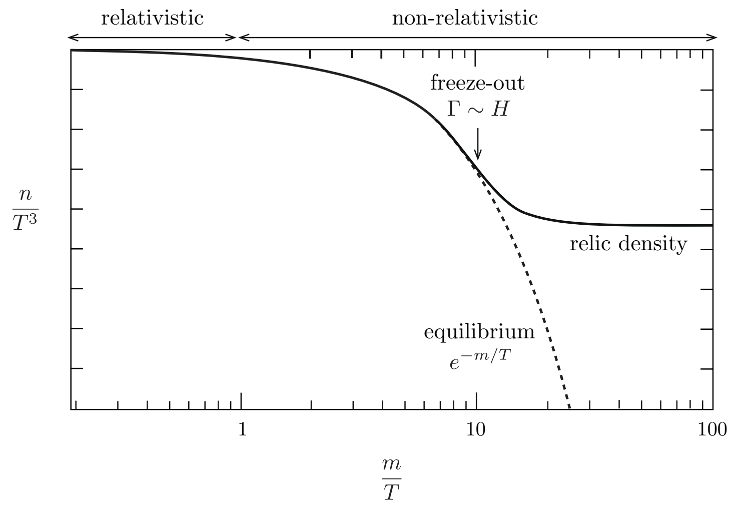


Figure 2. A schematic illustration of particle freeze-out. At high temperatures, $T \gg m$, the particle abundance tracks its equilibrium value. At low temperatures, $T \ll m$, the particles freeze out and maintain a density that is much larger than the Boltzmann-suppressed equilibrium abundance.

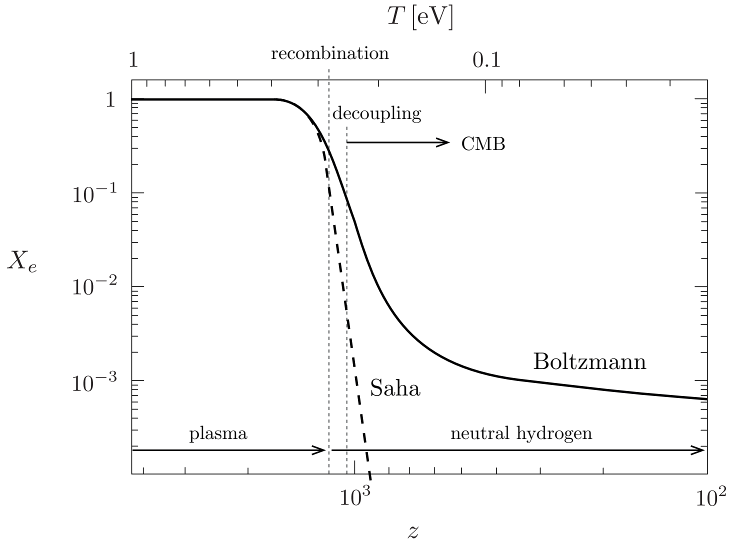


Figure 3. Fraction of free electrons as a function of redshift (or temperature).

Neutrinos are the most weakly interacting particles of the Standard Model and therefore decoupled first (around 0.8 MeV or 1 sec after the Big Bang). Shortly after neutrino decoupling, electrons and positrons annihilated. The energies of the electrons and positrons got transferred to the photons, but not the neutrinos. The temperature of the photons is therefore slightly bigger today than that of the neutrinos. At around the same time, neutron-proton interactions became inefficient, leading to a relic abundance of neutrons. These neutrons were essential for the formation of the light elements during Big Bang nucleosynthesis (BBN), which occurred around

3 minutes after the Big Bang.

Below about 1 eV, or 380,000 years after the Big Bang, the temperature had become low enough for neutral hydrogen to form through the reaction $e^- + p^+ \to \mathrm{H} + \gamma$. This is the moment of *recombination*. At this point the density of free electrons dropped dramatically (see Fig. 3). Before recombination the strongest coupling between the photons and the rest of the plasma was through Thomson scattering, $e^- + \gamma \to e^- + \gamma$. The sharp drop in the free electron density after recombination means that this process became inefficient and the photons decoupled. After *decoupling* the photons streamed freely through the universe and are today observed as the cosmic microwave background.

2.2 Cosmological Perturbation Theory

In this course, we will be concerned with the fact that the CMB isn't perfectly uniform, but varies in intensity across the sky. These anisotropies of the CMB are the imprints of density fluctuations in the primordial plasma. In this section, we will remind ourselves how we study these fluctuations in cosmological perturbation theory. The Einstein equation's couple the matter perturbations to the metric, so the two need to be studied simultaneously. We write the small perturbations of the metric and the matter fields as

$$g_{\mu\nu}(\eta, \mathbf{x}) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \mathbf{x})\,, \tag{2.22}$$

$$T_{\mu\nu}(\eta, \mathbf{x}) = \bar{T}_{\mu\nu}(\eta) + \delta T_{\mu\nu}(\eta, \mathbf{x})\,. \tag{2.23}$$

To avoid clutter we will often drop the argument $(\eta, \mathbf{x})$ on the perturbations.

2.2.1 Metric Perturbations

We will work in *Newtonian gauge* and write the perturbed metric as

$$\mathrm{d}s^2 = a^2(\eta)\left\{(1+2\Psi)\,\mathrm{d}\eta^2 - \left[(1-2\Phi)\,\delta_{ij} + h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}, \tag{2.24}$$

where Ψ is the gravitational potential, Φ is a local perturbation of the average scale factor and h_{ij} is a transverse and traceless tensor. For now, we will only study scalar fluctuations and set $h_{ij} = 0$. The perturbed orthonormal frame of vectors is

$$(E_0)^\mu = a^{-1}(1-\Psi)\delta^\mu_0\,, \quad (E_i)^\mu = a^{-1}(1+\Phi)\delta^\mu_i\,. \tag{2.25}$$

You should confirm that this satisfies the defining property $g_{\mu\nu}(E_\alpha)^\mu(E_\beta)^\nu = \eta_{\alpha\beta}$.

Exercise.—Show that the Christoffel symbols associated with the metric (2.24) are

$$\Gamma^0_{00} = \mathcal{H} + \dot{\Psi}\,, \tag{2.26}$$

$$\Gamma^0_{i0} = \partial_i \Psi\,, \tag{2.27}$$

$$\Gamma^i_{00} = \delta^{ij}\partial_j \Psi\,, \tag{2.28}$$

$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left[\dot{\Phi} + 2\mathcal{H}(\Phi + \Psi)\right]\delta_{ij}\,, \tag{2.29}$$

$$\Gamma^i_{j0} = \left[\mathcal{H} - \dot{\Phi}\right]\delta^i_j\,, \tag{2.30}$$

$$\Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l \Phi\,. \tag{2.31}$$

2.2.2 Matter Perturbations

The tetrad components of the stress-energy tensor are

$$T^{\hat{0}\hat{0}} = \bar{\rho}(\eta) + \delta\rho \,, \tag{2.32}$$

$$T^{\hat{0}\hat{\imath}} = [\bar{\rho}(\eta) + \bar{P}(\eta)]v^{\hat{\imath}} \,, \tag{2.33}$$

$$T^{\hat{\imath}\hat{\jmath}} = [\bar{P}(\eta) + \delta P]\delta^{ij} - \Pi^{\hat{\imath}\hat{\jmath}} \,. \tag{2.34}$$

The components in a general frame, $T^{\mu\nu} = T^{\hat{\alpha}\hat{\beta}}(E_\alpha)^\mu(E_\beta)^\nu$, contain a mixture of matter and metric fluctuations. We have written the momentum density $T^{\hat{0}\hat{\imath}}$ in terms of the bulk velocity $v^{\hat{\imath}}$. For scalar fluctuations, we can write the bulk velocity as $v^{\hat{\imath}} = \partial^i v$ and the anisotropic stress as $\Pi^{\hat{\imath}\hat{\jmath}} = (\bar{\rho}+\bar{P})(\partial^i\partial^j - \frac{1}{3}\delta^{ij}\nabla^2)\sigma$. Sometimes the scalar part of the velocity is written in terms of the velocity divergence $\theta \equiv \partial_i v^{\hat{\imath}}$. It also convenient to write the density perturbations in terms of the dimensionless density contrast $\delta \equiv \delta\rho/\rho$. Four perturbations, $(\delta, \delta P, v, \sigma)$, therefore characterize the scalar perturbations of the total matter. Similarly, we can write the stress tensors for the distinct species $a = \gamma, \nu, c, b, \cdots$ and study their perturbations $(\delta_a, \delta P_a, v_a, \sigma_a)$.

2.2.3 Equations of Motion

The equations of motion for the matter perturbations follow from conservation of the stress tensor, $\nabla^\mu T_{\mu\nu} = 0$. If there is no energy and momentum transfer between the different components, then the species are separately conserved and we also have $\nabla^\mu T^{(a)}_{\mu\nu} = 0$. The covariant conservation of the stress tensor contains two scalar equations: the relativistic continuity equation,

$$\dot{\delta}_a + \left(1 + \frac{\bar{P}_a}{\bar{\rho}_a}\right)(\partial_i v_a^{\hat{\imath}} - 3\dot{\Phi}) + 3\mathcal{H}\left(\frac{\delta P_a}{\bar{\rho}_a} - \frac{\bar{P}_a}{\bar{\rho}_a}\delta_a\right) = 0 \,, \tag{2.35}$$

and the relativistic Euler equation,

$$\dot{v}_a^{\hat{\imath}} + \left(\mathcal{H} + \frac{\dot{\bar{P}}_a}{\bar{\rho}_a + \bar{P}_a}\right)v_a^{\hat{\imath}} + \frac{1}{\bar{\rho}_a + \bar{P}_a}\left(\partial^i \delta P_a - \partial_j \Pi_a^{\hat{\imath}\hat{\jmath}}\right) + \partial^i \Psi = 0 \,. \tag{2.36}$$

Exercise.—Derive eqs. (2.35) and (2.36) from $\nabla^\mu T^{(a)}_{\mu\nu} = 0$. Given a physical interpretation for each term in the equations.

Equations (2.35) and (2.36) aren't sufficient to completely describe the evolution of the four perturbations $(\delta_a, \delta P_a, v_a, \sigma_a)$. To make progress, we either must make further simplifying assumptions or find additional evolution equations. We will do both.

- A perfect fluid is characterized by strong interactions which keep the pressure isotropic, $\sigma_a = 0$. In addition, pressure perturbations satisfy $\delta P_a = c_{s,a}^2 \delta\rho_a$, where $c_{s,a}$ is the adiabatic sound speed of the fluid. The perturbations of a perfect fluid are therefore described by only two independent variables, say δ_a and v_a, and the continuity and Euler equations are sufficient for closing the system.

- Decoupled or weakly interacting species (e.g. neutrinos) cannot be described by a perfect fluid and the above simplifications for the anisotropic stress and the pressure perturbation do not apply. In that case, we can't avoid solving the Boltzmann equation for the evolution of the perturbed distribution function f_a. We will have fun with this in Section 3.
- Decoupled cold dark matter is a peculiar case. It is collisionless and has a negligible velocity dispersion. It therefore behaves like a pressureless perfect fluid although it has no interactions and therefore really isn't a fluid.

The different matter components are gravitationally coupled through the metric fluctuations in the continuity and Euler equations. The 00-Einstein equation leads to the relativistic generalization of the Poisson equation

$$\nabla^2\Phi + 3\mathcal{H}(\dot{\Phi} + \mathcal{H}\Psi) = 4\pi G a^2 \delta\rho \,, \tag{2.37}$$

where $\delta\rho \equiv \sum_a \delta\rho_a$ is the *total* density perturbation. The transverse part of the ij-Einstein equation implies

$$\Phi - \Psi = -16\pi G a^2 (\bar{\rho} + \bar{P})\sigma \,, \tag{2.38}$$

where $(\bar{\rho} + \bar{P})\sigma \equiv \sum_a (\bar{\rho}_a + \bar{P}_a)\sigma_a$. Equations (2.37) and (2.38) are non-dynamical constraint equations. The evolution of the metric potentials is determined by the remaining Einstein equations. If the total anisotropic stress is negligible, we have $\Psi \approx \Phi$. The Einstein equation for the evolution of Φ is then given by

$$\ddot{\Phi} + 3\mathcal{H}\dot{\Phi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P \,. \tag{2.39}$$

If we don't ignore the anisotropic stress, eq. (2.39) will get an additional source term.

2.2.4 Initial Conditions

At sufficiently early times, all scales of interest to current observations were outside the Hubble radius. On super-Hubble scales, the evolution of perturbations becomes very simple and completely trivial for adiabatic initial conditions.

Adiabatic perturbations

Adiabatic perturbations have the property that the local state of matter (determined, for example, by the energy density ρ and the pressure P) at some spacetime point $(\eta, \mathbf{x})$ of the perturbed universe is the same as in the *background* universe at some slightly different time $\eta + \delta\eta(\mathbf{x})$. (Notice that the time shift varies with location $\mathbf{x}$.) We can thus view adiabatic perturbations as some parts of the universe being "ahead" and others "behind" in the evolution. If the universe is filled with multiple fluids, adiabatic perturbations correspond to perturbations induced by a *common, local shift in time* of all background quantities; e.g. adiabatic density perturbations are defined as

$$\delta\rho_a(\eta, \mathbf{x}) \equiv \bar{\rho}_a(\eta + \delta\eta(\mathbf{x})) - \bar{\rho}_a(\eta) = \dot{\bar{\rho}}_a \, \delta\eta(\mathbf{x}) \,, \tag{2.40}$$

where $\delta\eta$ is the same for all species a. This implies

$$\delta\eta = \frac{\delta\rho_a}{\dot{\bar{\rho}}_a} = \frac{\delta\rho_b}{\dot{\bar{\rho}}_b} \qquad \text{for all species } a \text{ and } b \,. \tag{2.41}$$

Using $\dot{\bar{\rho}}_a = -3\mathcal{H}(1+w_a)\bar{\rho}_a$, we can write this as

$$\frac{\delta_a}{1+w_a} = \frac{\delta_b}{1+w_b} \qquad \text{for all species } a \text{ and } b\ . \tag{2.42}$$

Thus, for adiabatic perturbations, all matter components ($w_m \approx 0$) have the same fractional perturbations, while all radiation perturbations ($w_r = \frac{1}{3}$) obey

$$\delta_r = \frac{4}{3}\delta_m\ . \tag{2.43}$$

It follows that, for adiabatic fluctuations, the total density perturbation, $\delta\rho \equiv \sum_a \bar{\rho}_a\delta_a$, is dominated by the species that is dominant in the background since all the δ_a's are comparable. At early times, the universe is radiation dominated, so it natural to set the initial conditions for all super-Hubble Fourier modes then. Equation (2.39) implies that $\Phi = const.$ on super-Hubble scales, while equation (2.37) leads to

$$\delta \approx \delta_r = -2\Phi = const. \tag{2.44}$$

Equations (2.44) and (2.43) show that, for adiabatic initial conditions, all matter perturbations are given in terms of the super-Hubble value of the potential Φ. In these lectures, we will be concerned with the evolution of photons, baryons and cold dark matter (CDM). Their fractional density perturbations will satisfy the relation (2.43) on super-Hubble scales, but will start to evolve in distinct ways inside of the horizon (cf. Fig. 4).

Comoving curvature perturbations

The metric potentials are only constant on super-Hubble scales if the equation of state of the background is constant. Whenever the equation of state evolves (e.g. in the transitions from inflation to radiation domination and from radiation to matter domination), so will the metric potentials. It will be convenient to identify an alternative perturbation variable that stays constant on large scales even in these more general situations. Such a variable is the *comoving curvature perturbation* $\mathcal{R}$. Defining the initial conditions in terms of $\mathcal{R}$ allows us to match predictions made by inflation to fluctuations in the primordial plasma most easily (see Fig. 5).

To define $\mathcal{R}$, let us first consider the most general perturbation to the background metric

$$\mathrm{d}s^2 = a^2(\eta)\left\{(1+2\Psi)\mathrm{d}\eta^2 - 2B_i\mathrm{d}x^i\mathrm{d}\eta - \left[(1-2\Phi)\delta_{ij} + 2E_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}. \tag{2.45}$$

The induced metric on surfaces of constant η is

$$\gamma_{ij} = a^2\left[(1-2\Phi)\delta_{ij} + 2E_{ij}\right]. \tag{2.46}$$

The intrinsic curvature scalar (i.e. the Ricci scalar computed from γ_{ij}) of these surfaces is

$$a^2R_{(3)} = 4\nabla^2\Phi + 2\partial^i\partial^j E_{ij}\ . \tag{2.47}$$

This vanishes for vector and tensor perturbations (as do all perturbed scalars), and for scalar perturbations it becomes

$$a^2R_{(3)} = 4\nabla^2\left(\Phi + \frac{1}{3}\nabla^2 E\right) \equiv -4\nabla^2\mathcal{R}\,, \tag{2.48}$$

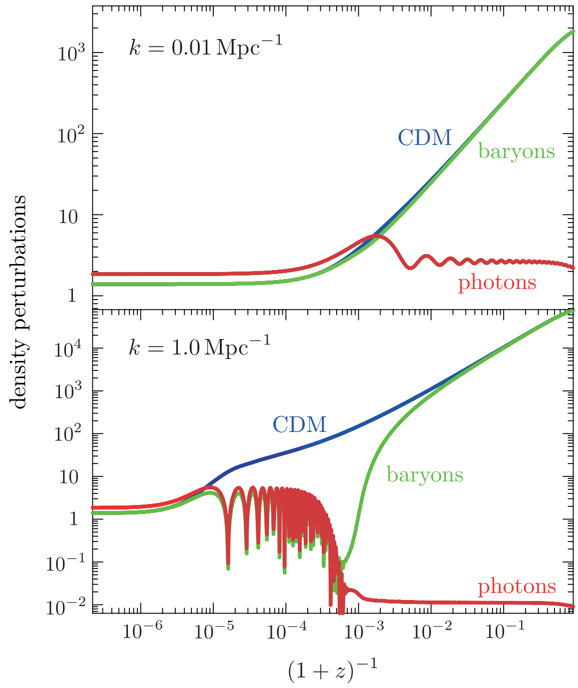


Figure 4. Evolution of photons, baryons and dark matter. On super-Hubble scales (early times), the fluctuations in each component remain adiabatic and constant. On sub-Hubble scales, CDM first only grows slowly during the radiation era and then as a power law during the matter era. Before recombination, photons and baryons are tightly coupled and oscillate on sub-Hubble scales. After recombination baryons fall into CDM potential wells.

where the minus sign in the definition of $\mathcal{R}$ is conventional. We see that the comoving curvature perturbation $\mathcal{R}$ parameterizes the curvature perturbation in comoving gauge. Who would have guessed. A gauge-invariant definition of the comoving curvature perturbation is

$$\mathcal{R} = -\Phi - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v)\,. \tag{2.49}$$

> *Exercise.*—Show that (2.49) is indeed gauge-invariant.

In Newtonian gauge ($B = E = 0$), this becomes

$$\mathcal{R} = -\Phi + \mathcal{H}v = -\Phi - \frac{\mathcal{H}(\dot{\Phi} + \mathcal{H}\Phi)}{4\pi G a^2(\bar{\rho} + \bar{P})}\,, \tag{2.50}$$

where in the last equality we have used an Einstein equation to write v in terms of Φ.

> *Exercise.*—By differentiating (2.50) and using the Einstein equations, show that
>
> $$-\,4\pi G a^2(\bar{\rho}+\bar{P})\,\frac{\dot{\mathcal{R}}}{\mathcal{H}} \;=\; 4\pi G a^2\,\delta P_{en} + \frac{\dot{\bar{P}}}{\dot{\bar{\rho}}}\nabla^2\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi)\,, \tag{2.51}$$

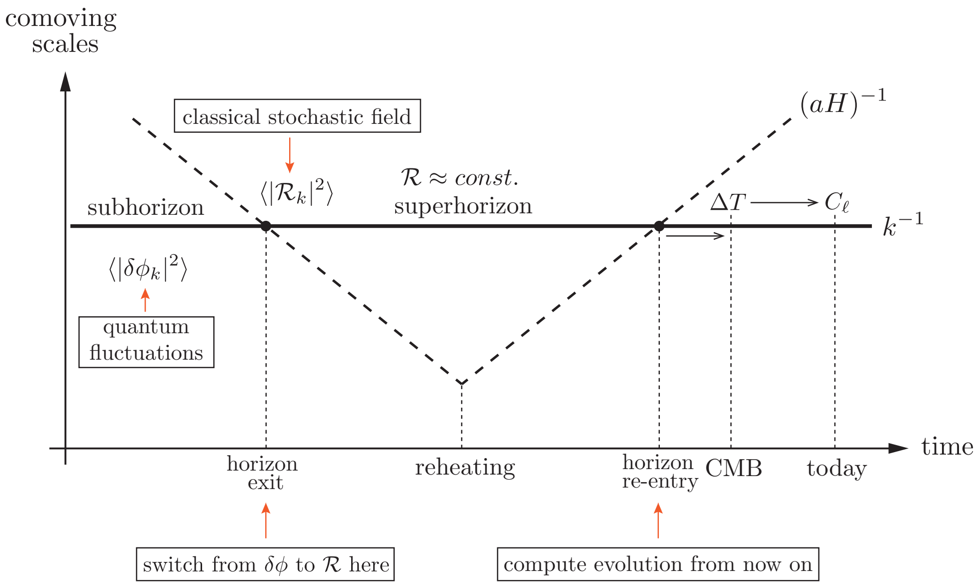


Figure 5. Curvature perturbations during and after inflation: The comoving horizon $(aH)^{-1}$ shrinks during inflation and grows in the subsequent FRW evolution. This implies that comoving scales k^{-1} exit the horizon at early times and re-enter the horizon at late times. While the curvature perturbations $\mathcal{R}$ are outside of the horizon they don't evolve, so our computation for the correlation function $\langle|\mathcal{R}_k|^2\rangle$ at horizon exit during inflation can be related directly to observables at late times.

where $\delta P_{en} \equiv \delta P - (\dot{P}/\dot{\rho})\,\delta\rho$ is the non-adiabatic pressure perturbation.

The last exercise proves that the curvature perturbation is conserved on super-Hubble scales ($k \ll aH$) and for adiabatic perturbations ($\delta P_{en} = 0$), independent of the equation of the state of the background.

In the special case where the equation of state of the background is a constant, the super-Hubble limit of the curvature perturbation is

$$\mathcal{R} \xrightarrow{k \ll aH} -\frac{5+3w}{3+3w}\,\Phi = \begin{cases} -\dfrac{3}{2}\Phi & \text{RD} \\ -\dfrac{5}{3}\Phi & \text{MD} \end{cases} . \tag{2.52}$$

We see that during the radiation era $\mathcal{R}$ and Φ are simply proportional. However, $\mathcal{R}$ has the advantage over Φ that is stays constant even during reheating, so we can relate it more easily to the fluctuations created by inflation (see §2.3).

Statistics

Quantum mechanics during inflation only predicts the statistics of the initial conditions, i.e. it predicts the correlation between the CMB fluctuations in different directions, rather than the specific value of the temperature fluctuation in a specific direction. For Gaussian initial conditions, these correlations are completely specified by the two-point correlation function

$$\langle \mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{x}')\rangle \equiv \xi_{\mathcal{R}}(\mathbf{x},\mathbf{x}') = \xi_{\mathcal{R}}(|\mathbf{x}'-\mathbf{x}|)\,, \tag{2.53}$$

where the last equality holds as a consequence of statistical homogeneity and isotropy. The Fourier transform of $\mathcal{R}$ then satisfies

$$\langle \mathcal{R}(\mathbf{k})\mathcal{R}^*(\mathbf{k}')\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}(k)\,\delta_D(\mathbf{k}-\mathbf{k}')\,, \tag{2.54}$$

where $\mathcal{P}_{\mathcal{R}}(k)$ is the (dimensionless) *power spectrum.*

Exercise.—Show that

$$\xi_{\mathcal{R}}(\mathbf{x},\mathbf{x}') = \int \frac{\mathrm{d}k}{k}\,\mathcal{P}_{\mathcal{R}}(k)\,\mathrm{sinc}(k|\mathbf{x}-\mathbf{x}'|)\,. \tag{2.55}$$

2.3 Initial Conditions from Inflation

The most remarkable feature of inflation is that it provides at natural mechanism for producing the initial conditions (see Fig. 6). The reason why inflation inevitably produces fluctuations is simple: the evolution of the inflaton field $\phi(t)$ governs the energy density of the early universe $\rho(t)$ and hence controls the end of inflation. Essentially, the field ϕ plays the role of a local "clock" reading off the amount of inflationary expansion still to occur. By the uncertainty principle, arbitrarily precise timing is not possible in quantum mechanics. Instead, quantum-mechanical clocks necessarily have some variance, so the inflaton will have spatially varying fluctuations $\delta\phi(t,\mathbf{x})$. There will hence be local differences in the time when inflation ends, $\delta t(\mathbf{x})$, so that different regions of space inflate by different amounts. These differences in the local expansion histories lead to differences in the local densities after inflation, $\delta\rho(t,\mathbf{x})$, and the curvature perturbations in comoving gauge, $\mathcal{R}(\mathbf{x})$. It is worth remarking that the theory wasn't engineered to produce these fluctuations, but their origin is instead a natural consequence of treating inflation quantum mechanically.

2.3.1 Scalar Fluctuations

The details of the computation of the inflationary quantum fluctuations can be found in my *Cosmology* lectures [1]. Here, I will just cite the main results. The power spectrum of inflaton fluctuations $\delta\phi$ in zero-curvature gauge ($\phi + \frac{1}{3}\nabla^2 E = 0$) is found to be

$$\mathcal{P}_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2\bigg|_{k=aH}\,. \tag{2.56}$$

This is the universal result for any nearly massless scalar field in a quasi-de Sitter background. The right-hand side in (2.56) is evaluated at horizon crossing to avoid making an error due to the superhorizon evolution of $\delta\phi$. This power spectrum is then matched to the power spectrum for $\mathcal{R}$ which is conserved outside the horizon. The curvature perturbation in the zero-curvature gauge is

$$\mathcal{R} = -H\,\frac{\delta\phi}{\dot{\phi}}\,. \tag{2.57}$$

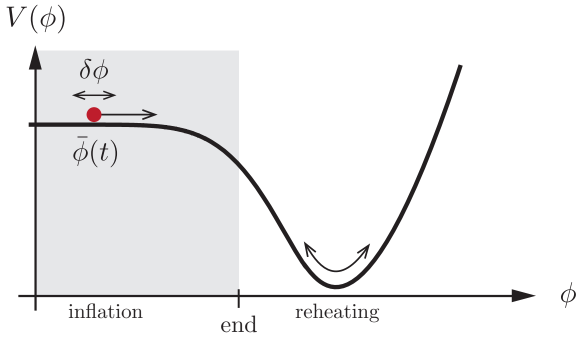


Figure 6. Quantum fluctuations $\delta\phi(t, \mathbf{x})$ around the classical background evolution $\bar{\phi}(t)$. Regions acquiring a negative fluctuations $\delta\phi$ remain potential-dominated longer than regions with positive $\delta\phi$. Different parts of the universe therefore undergo slightly different evolutions. After inflation, this induces density fluctuations $\delta\rho(t, \mathbf{x})$.

Exercise.—Show that (2.49) implies (2.57) for a scalar field dominated universe.

The power spectrum of $\mathcal{R}$ at horizon exit therefore is

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\dot{\bar{\phi}}}\right)^2\Bigg|_{k=aH} . \tag{2.58}$$

From now on, we will drop the $k = aH$ label to avoid clutter. The result in (2.58) may also be written as

$$\boxed{\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{8\pi^2\varepsilon}\frac{H^2}{M_{\rm pl}^2}} , \quad \varepsilon \equiv \frac{\frac{1}{2}\dot{\bar{\phi}}^2}{M_{\rm pl}^2 H^2} , \tag{2.59}$$

where ε is the inflationary slow-roll parameter. The time dependence of H and ε leads to a small scale dependence of $\mathcal{P}_{\mathcal{R}}(k)$. The form of the spectrum is approximately a power law, $\mathcal{P}_{\mathcal{R}}(k) = A_s(k/k_*)^{n_s-1}$, with the following spectral index

$$\boxed{n_s - 1 = -2\varepsilon - \eta} , \tag{2.60}$$

where $\eta \equiv \dot{\varepsilon}/(H\varepsilon)$ is the second slow-roll parameter. The observational constraint on the scalar spectral index is $n_s = 0.9603 \pm 0.0073$.

2.3.2 Tensor Fluctuations

Inflation also produces fluctuations in the transverse, traceless part of the metric, h_{ij}. These tensor modes of the metric correspond to gravitational waves. Their power spectrum is found to be

$$\boxed{\mathcal{P}_h(k) = \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2}} . \tag{2.61}$$

Notice that this power spectrum only depends on the expansion rate during inflation, while the scalar power spectrum has an additional dependence on the slow-roll parameter ε (coming from the conversion of $\delta\phi$ to $\mathcal{R}$). The form of the tensor power spectrum is also a power law, $\mathcal{P}_h(k) = A_t(k/k_*)^{n_t}$, with the following spectral index

$$n_t = -2\varepsilon\,. \tag{2.62}$$

Observationally, a small value for n_t is hard to distinguish from zero. The tensor amplitude is often normalized with respect to the measured scalar amplitude, $A_s = (2.196 \pm 0.060) \times 10^{-9}$ (at $k_* = 0.05$ Mpc^{-1}). The *tensor-to-scalar ratio* is

$$\boxed{r \equiv \frac{A_t}{A_s} = 16\varepsilon}\,. \tag{2.63}$$

Inflationary models make predictions for (n_s, r). The latest observational constraints on these parameters are shown in Fig. 7.

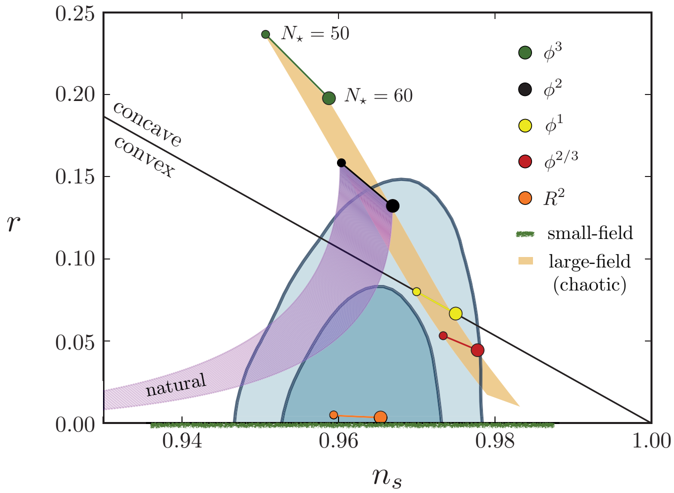


Figure 7. Planck+WMAP+BAO constraints on n_s and r, together with predictions from a few representative inflationary models.

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