

# Differential Equations and Asymptotic Solutions for Arithmetic Asian Options: “Black-Scholes formulae” for Asian-Rate Calls

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## Abstract

In this article we present a simplified means of pricing Asian options using partial differential equations. We first provide a concise derivation of the well-known similarity reduction and exact Laplace transform solution. We then analyse the problem afresh as a power series in the volatility-scaled contract duration, with a view to obtaining an asymptotic solution for the low-volatility limit, a limit which presents difficulties in the context of the general Laplace transform solution. The problem is approached anew from the point of view of asymptotic expansions and the results are compared with direct, high precision, inversion of the Laplace transform and with numerical results obtained by V. Linetsky and J. Vecer. Our asymptotic formulae are little more complicated than the standard Black-Scholes formulae and, working to third order in the volatility-scaled expiry, are accurate to at least four significant figures for standard test problems. In the case of zero risk-neutral drift we have the solution to fifth order and, for practical purposes, the results are effectively exact.

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# 1 Introduction

The subject of arithmetically-averaged Asian options has been a source of interest in mathematical finance for several decades. The essential difficulty in dealing with them is the awkward nature of the terminal distribution of a path-wise arithmetic-average of an asset price which is itself log-normally distributed. A detailed probabilistic analysis for the case of an Asian-rate call by Geman and Yor [1] gave, for the first time, a closed-form mathematical representation of the answer, albeit in terms of the Laplace transform of the solution. This Laplace transform involves special functions of a somewhat awkward character. The extraction of numerical results from this formula using high-level mathematical programming systems to do a direct inversion of the transform is straightforward (see e.g. Shaw, 1998 [2]), but there are still computational difficulties for *low* volatilities. These were highlighted by Fu *et al* [3], who also gave a useful family of test cases that we, in common with several other authors, shall use. The analysis in [3] did not do full justice to the analysis given in [2] and overstated the use of Monte Carlo methods as benchmarking tools, a matter discussed in [4], but the family of test problems and general methodology of [3] has proved useful for the purposes of comparison. While the case of low volatility remains easy in Monte Carlo approaches, it continues to present difficulties in context of the analytical model of [1].

A completely different, spectral approach by Linetsky [5] gives another way of extracting accurate numerical results, and the methods of [5, 2], when correctly deployed, are consistent with each other and the numerical methods pioneered by Vecer [6, 7]. It should be pointed out that the results cited in [6] do resolve some difficulties in [3], notably that [3] did not correctly use the exact inversion method of [2], and perhaps placed undue emphasis on certain Monte-Carlo benchmarking results that have turned out to be less accurate than the methods of [7, 6, 2, 5].

The approach of Geman and Yor, however, leaves some questions unanswered. Is it possible to give a more straightforward derivation of the solution? Can one treat the case of low volatility more effectively? Is there a continuity correction to it that allows discrete sampling of the average to be treated? We shall show that the answer to the first two of these questions is yes, and will give several options for the low volatility situation. The issue of continuity corrections will be discussed elsewhere. Note that a simplified approach to the Geman-Yor model, based on hitting times, has also been given by Lewis [8], who also explains the use of a quadrant based method, which is also discussed here in what we call the reduced domain approach.

Another thread of work is the reduced-form partial differential equation approach originally pioneered by Ingersoll [9]. This was developed by Rogers and Shi [10] and used by Zhang to develop a semi-analytical approximation [11]. The asymptotic expansion we present here has, as its first term, the analytical contribution to the solution developed by Zhang, who computed the remaining contributions by the numerical solution of a partial differential equation for the residual terms. By a more detailed formalization of the problem in the asymptotic setting, however, we shall see how to obtain more accuracy within a purely analytical setting.

The plan of this paper is as follows. In Section Two we derive the partial differential equations and terminal conditions that are fundamental to our approach. In Section Three we explain how to derive the Geman-Yor solution using methods that are comparatively straightforward, and also relate our approach to Vecer's partial differential equation. In particular, we explain how Vecer's method may be further optimized using the reduced domain method employed to give an efficient derivation of the Geman-Yor model. In Section Four we present an analysis based on asymptotic expansions for the average rate-call. In Section Five we obtain an asymptotic solution for the rate-call based on a formal asymptotic solution of the reduced underlying risk-neutral stochastic differential equation. In Section Six we present comparisons of results from our various approaches. The test problems we employ are a natural extension of the Fu-Madan-Wang test problems introduced in [3]. Here we define analogues of their test problems to treat the cases  $r = q$ ,  $q > r$  in addition to problems where  $r > q$  discussed in [3], and also consider variants with  $\sigma\sqrt{T} \rightarrow 0$ , where  $r$  is the risk-free rate,  $q$  the continuous dividend yield,  $\sigma$  the volatility and  $T$  the duration of the contract.

## 2 Derivation of the fundamental PDE

We begin by assuming that the spot price,  $S_t$ , of the asset underlying our Asian option satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where  $W_t$  is a standard Brownian motion and  $\mu$  and  $\sigma$  are constants. We let

$$I_t = \int_0^t w(u) S_u du$$

denote the running  $w$ -weighted integral of the asset price  $S_t$ .

If we let  $r$  denote the constant risk-free rate,  $q$  the constant continuous dividend yield, then using standard Black-Scholes dynamic hedging arguments as shown in [12, 13], the value,  $V(S, I, t)$ , of a European option which depends on  $S$ ,  $I$  and  $t$ , and has the payoff

$$V(S, I, T) = P_o(S, I)$$

at expiry,  $T$ , satisfies the partial differential equation final-value problem

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} + w(t)S \frac{\partial V}{\partial I} - rV &= 0, \\ V(S, I, T) &= P_o(S, I). \end{aligned} \tag{1}$$

The solution of this problem may be found and, via Feynman-Kac formula, can be shown to be the discounted expected value of the payoff  $P_o(S_t, I_t)$ , under the risk neutral process

$$\begin{aligned} dS_t &= (r - q)S_t dt + \sigma S_t dW_t, \\ dI_t &= w(t)S_t dt. \end{aligned}$$

We now carry out a series of elementary steps to simplify problem (1). One would normally expect to begin this process by removing the discounting term,  $-rV$ , but it turns out that this would be premature and would only have to be undone later. So we concentrate first on the essential *symmetries* of the problem.

### 2.1 Autonomy

We now wish to exploit the fact that (1) is *autonomous* in  $I$ . The details of how we shall use this depend on the nature of the payoff function  $P_o$ . We distinguish two cases of importance:

1. *Average-rate* options, where the payoff has the form

$$P_o(S, I) = P_r(I/T - K)$$

for some function  $P_r(\cdot)$ . For an average-rate call option with value  $C_r(S, I, t)$  and payoff

$$C_r(S, I, T) = \max\left(\frac{I}{T} - K, 0\right)$$

we have  $P_{rc}(x) = \max(x, 0)$  and for an Asian-rate put option with value  $P_r(S, I, t)$  and payoff

$$P_r(S, I, T) = \max\left(K - \frac{I}{T}, 0\right)$$

we have  $P_{rp}(x) = \max(-x, 0)$ ;

2. *Average-strike* options, where the payoff has the form

$$P_o(S, I) = P_s(S - I/T)$$

for some function  $P_s(\cdot)$ . For average-strike calls,  $C_s(S, I, t)$ , the payoff is  $P_{sc}(x) = \max(x, 0)$  and for puts,  $P_s(S, I, t)$ , the payoff is  $P_{sp}(x) = \max(-x, 0)$ .

In the first case, an average-rate option, we set  $I' = I - KT$ , and in the second case we leave  $I$  alone and set  $I' = I$ . Regarding an option's value,  $V$ , as a function  $V(S, I', t)$ , the partial differential equation remains unaltered,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} + w(t)S \frac{\partial V}{\partial I'} - rV = 0, \quad (2)$$

but the terminal conditions are now:

1. average-rate call:  $C_r(S, I', T) = 1/T \max(I', 0)$ ;
2. average-rate put:  $P_r(S, I', T) = 1/T \max(-I', 0)$ ;
3. average-strike call:  $C_s(S, I', T) = \max(S - I'/T, 0)$ ;
4. average-strike put:  $P_s(S, I', T) = \max(I'/T - S, 0)$ .

## 2.2 Scaling, dividend-discounting and time reversal

The use of reduction techniques for Asian options has been considered by many people. The first that we are aware of is the discussion of the Asian-strike case by Ingersoll [9].

Note that  $V$ ,  $S$  and  $I'/T$  are all prices, so (2) and the associated terminal conditions must be, and indeed are, invariant under the scaling:

$$S \rightarrow \lambda S, \quad V \rightarrow \lambda V, \quad I' \rightarrow \lambda I', \quad \forall \lambda > 0.$$

Two non-dimensional invariants associated with the PDE *and the payoffs given above* are:

$$\phi = \frac{V}{S}, \quad \eta = \frac{I'}{ST}.$$

Since the PDE and its terminal conditions inherit this scaling symmetry, we can reduce the problems above to ones involving only these reduced variables and time. This is equivalent to noting that, since cash does not explicitly occur in the payoffs for the calls and puts above, we can employ a change of numeraire and measure prices relative to  $S$  rather than cash.

It is an elementary exercise to apply the chain rule and show that  $\phi$  satisfies the partial differential equation (PDE)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 \eta^2 \frac{\partial^2 \phi}{\partial \eta^2} + \left( \frac{w(t)}{T} - (r - q)\eta \right) \frac{\partial \phi}{\partial \eta} - q\phi = 0.$$

We now perform a further ‘‘dividend-discounting’’, by setting

$$\psi = e^{-q(T-t)} \phi,$$

and we also make the usual time reversal, setting

$$\tau_1 = T - t \quad \text{and} \quad W(\tau_1) = w(t),$$

to obtain our ‘‘master’’ PDE,

$$\frac{\partial \psi}{\partial \tau_1} + \frac{1}{2}\sigma^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \left( \frac{W(\tau_1)}{T} - (r - q)\eta \right) \frac{\partial \psi}{\partial \eta} = 0. \quad (3)$$

The original terminal conditions become initial conditions in terms of  $\tau_1$ ,

1. average-rate call:  $\psi(\eta, 0) = \max(\eta, 0)$ ;
2. average-rate put:  $\psi(\eta, 0) = \max(-\eta, 0)$ ;
3. average-strike call:  $\psi(\eta, 0) = \max(1 - \eta, 0)$ ;
4. average-strike put:  $\psi(\eta, 0) = \max(\eta - 1, 0)$ .

For average-rate options

$$V(S, I, t) = S e^{-q(T-t)} \psi \left( \frac{I - K T}{S T}, T - t \right)$$

and for average-strike options

$$V(S, I, t) = S e^{-q(T-t)} \psi \left( \frac{I}{S T}, T - t \right).$$

It should be noted that our master PDE is a simple generalization of the equation (derived by stochastic methods) of Rogers and Shi [10] to the case of non-zero (and not necessarily continuously paid) dividend yield. Links between the different types of Asian options are discussed by Henderson *et al* [14].

### 2.3 Choices of weight function

Different types of averaging may be achieved through various choices of the weight function  $w$ . We list those of most obvious interest:

1.  $w(t) = 1$ , which corresponds to continuous arithmetic averaging throughout the life of the instrument;
2.  $w(t) = 0$  for  $t < T_1 < T$  and  $w(t) = 1$  for  $T_1 < t < T$ , which corresponds to a continuously sampled, arithmetic-average Asian-tail;
3.  $w(t) = \frac{T}{N} \sum_{k=1}^N \delta(t-t_k)$ , corresponding to discrete arithmetic averaging of the asset value with sampling at times  $t_k$ ,  $k = 1, \dots, N$ .

### 2.4 The associated SDE

Having reduced the partial differential equation to (3), involving just one independent price-like variable, we can now appeal to the Feynman-Kac theorem to see that the solution of the PDE with the appropriate payoff is equivalent to the expected value of that payoff, where the expectation is taken with respect to the process

$$d\eta_t = \left( \frac{w(t)}{T} - (r - q)\eta_t \right) dt + \sigma \eta_t dW_t, \quad (4)$$

where  $W_t$  is a standard Brownian motion. This is perhaps the Asian problem, for payoffs with the appropriate symmetries, in its most basic and pure form. It is easy to make this the basis of a European-style Monte Carlo simulation. In general there is no analytic formula for the terminal distribution of this process, so one must numerically integrate process paths in order to effect a Monte-Carlo simulation.

So far as we are aware, stochastic differential equations of this mixed type, which may be considered as models for a situation in which an “asset” ( $\eta$ ) pays both proportional dividend-yields  $((r - q)\eta)$  and constant dividends  $(w(t)/T)$  either continuously or discretely, were first considered by E. Wong in 1953, [15], who obtained some solutions for the transition probability density for certain signs of the parameters and with  $w(t)$  constant.

### 2.5 Linear solutions and parity

We now reduce the number of cases that need to be considered by establishing the relevant parity relationships between calls and puts. As with the standard Black-Scholes problem,<sup>1</sup> this can be treated by finding the solutions of (3) that are *linear* in  $\eta$ . Thus, we assume a solution linear in  $\eta$ ,

$$\psi(\eta, \tau_1) = a(\tau_1) + b(\tau_1)\eta, \quad (5)$$

and substitute into (3).

We obtain

$$\dot{a}(\tau_1) + \dot{b}(\tau_1)\eta = \left( \frac{W(\tau_1)}{T} - (r - q)\eta \right) b(\tau_1), \quad (6)$$

<sup>1</sup>Unlike standard put-call parity, however, arithmetic-average put-call parity is, in general, a model dependent result.

where  $\dot{\cdot}$  denotes differentiation with respect to  $\tau_1$ , and split this up into two ordinary differential equations (ODEs) by comparing powers of  $\eta$ . This gives

$$\dot{a}(\tau_1) = \frac{W(\tau_1)}{T}b(\tau_1), \quad \dot{b}(\tau_1) = -(r-q)b(\tau_1).$$

We can write the solution of these equations as

$$a(\tau_1) = a(0) + b(0) f(\tau_1), \quad b(\tau_1) = b(0) e^{-(r-q)\tau_1},$$

where

$$f(\tau_1) = \frac{1}{T} \int_0^{\tau_1} W(s) e^{-(r-q)s} ds.$$

It is helpful to write down the solution for  $f(\tau_1)$  for the cases of continuous and periodically sampled discrete averaging. For continuous averaging over the life of the option,  $W(\tau_1) = 1$ , we find

$$f(\tau_1) = \frac{1 - e^{-(r-q)\tau_1}}{(r-q)T},$$

with the special case  $f(\tau_1) = \tau_1/T$  when  $r = q$ . In the case of discrete and ‘‘clockwork’’ averaging, i.e. case 3 with  $t_k = kT/N$ ,  $k = 1, \dots, N$ , we set  $\Delta\tau_1 = T/N$  and let  $C(x)$  denote the ceiling of  $x$ —the smallest integer greater than or equal to  $x$ . Then

$$f(\tau_1) = \frac{\Delta\tau_1}{T} \frac{1 - e^{-C(\tau_1/\Delta\tau_1)(r-q)\Delta\tau_1}}{1 - e^{-(r-q)\Delta\tau_1}}$$

with  $f(\tau_1) = \Delta\tau_1 C(\tau_1/\Delta\tau_1)$  when  $r = q$ .

The parity relation for average rate options can now be deduced by noting that, in terms of our similarity variables, a call minus a put has initial condition  $\eta$  everywhere. We know the solution of the master PDE, (3), for this initial condition and deduce that the parity relation for average rate options is

$$\psi_{rc} - \psi_{rp} = f(\tau_1) + e^{-(r-q)\tau_1} \eta. \quad (7)$$

For average strike options the initial condition for a call minus a put is just  $1 - \eta$ , so the corresponding parity relation is

$$\psi_{sc} - \psi_{sp} = 1 - f(\tau_1) - e^{-(r-q)\tau_1} \eta. \quad (8)$$

It is a straightforward exercise to unwind these relations to those for the function  $V$ , and we leave this to the reader.

### 3 A simple derivation of the exact solution for Asian-rate calls as a Laplace transform

From here on we focus on the case of continuous sampling where  $w(t) = 1$ . Our master PDE, (3), serves as a convenient link to the methods developed by Geman and Yor [1] and Vecer [7, 6]. In making the first link we give a more straightforward derivation of the result and, in making the second link, we show how in some cases the Vecer method can be further optimized. The key to this optimization, and the quick derivation of the Geman and Yor model, rely on the concept of reducing the domain on which the master PDE is solved.

#### 3.1 The reduced domain for Asian-rate calls

At first sight it appears that the domain on which we need to solve the master PDE, (3), is  $\tau_1 > 0$  and  $-\infty < \eta < \infty$ . This, however, is an illusion for average-rate options. One can limit attention to  $\eta \leq 0$ , for average-rate options (see also the discussion by Lewis [8]). If we return to our original representation in terms of a call then, based on the balance between  $I/T$  and the strike,  $K$ , we see that if  $I$  ever attains the value  $KT$ , it can never drop below that value subsequently, since it is an integral of the positive quantity  $S$ .

That is,  $I$  is non-decreasing in  $t$ . In terms of the quantity  $\eta$ , we know that if  $\eta$  is positive, then the solution must be the in-the-money solution. So we can just write down the solution by noting that the payoff for positive  $\eta$  is linear in  $\eta$  and seeking a linear solution of the master PDE, (3), for  $\eta \geq 0$ . For a call this is just

$$\psi_{\text{rc}}(\eta, \tau_1) = e^{-(r-q)\tau_1} \eta + f(\tau_1).$$

Then we note that along the at-the-money boundary,  $\eta = 0$ , we have

$$\psi_{\text{rc}}(0, \tau_1) = f(\tau_1). \quad (9)$$

So, the problem for an average rate call is then the master PDE, (3), on the region  $\eta < 0$ ,  $\tau_1 > 0$ , with the boundary condition (9) and the initial condition  $\psi_{\text{rc}}(\eta, 0) = 0$  for  $\eta < 0$ .

### 3.2 A simple derivation of the Geman-Yor formula

We can now give a simple derivation of the Geman-Yor model for Asian call option prices [1], but see also [16]. We start with the PDE for  $\phi$ , for  $\phi$  is closely related to Geman and Yor's function  $C$ . We introduce Geman and Yor's variables (some of their work uses  $q$  for  $\alpha$  and  $h$  for  $\tau_2$ );

$$\nu = \frac{2(r-D)}{\sigma^2} - 1, \quad \tau_2 = \frac{1}{4}\sigma^2 \tau_1, \quad \alpha = -\frac{1}{4}\sigma^2 T \eta, \quad C = \frac{1}{4}\sigma^2 e^{(r-q)\tau_1} T \psi$$

In terms of these variables, the partial differential equation (3) is

$$\frac{\partial C}{\partial \tau_2} = 2\alpha^2 \frac{\partial^2 C}{\partial \alpha^2} - (1 + 2(\nu + 1)\alpha) \frac{\partial C}{\partial \alpha} + 2(1 + \nu)C \quad (10)$$

on the reduced domain  $\alpha > 0$ ,  $\tau_2 > 0$  subject to the initial condition

$$C(\alpha, 0) = 0$$

and the boundary condition

$$C(0, \tau_2) = \frac{(e^{2(1+\nu)\tau_2} - 1)}{2(1 + \nu)}.$$

We now introduce the Laplace transform

$$\hat{C}(\alpha, p) = \int_0^\infty e^{-p\tau_2} C(\alpha, \tau_2) d\tau_2,$$

which satisfies the transformed equation

$$2\alpha^2 \frac{\partial^2 \hat{C}}{\partial \alpha^2} - (1 + 2(1 + \nu)\alpha) \frac{\partial \hat{C}}{\partial \alpha} + (2(1 + \nu) - p) \hat{C} = 0$$

with the boundary condition

$$\hat{C}(0, p) = \frac{1}{p(p - 2(1 + \nu))} \quad \text{for } \text{Re}(p) > \max(2(1 + \nu), 0),$$

where the last restriction on  $p$  ensures the transform of the boundary condition exists. The solution of this problem can be expressed in terms of a pair of confluent hyper-geometric functions,

$$\hat{C}(\alpha, p) = C_1(p) A_1(\alpha, p) + C_2(p) A_2(\alpha, p),$$

where, in terms of  $\mu = \sqrt{2p + \nu^2}$ , we have

$$\begin{aligned} A_1(\alpha, p) &= (2\alpha)^{(2+\nu+\mu)/2} {}_1F_1\left(-\frac{1}{2}(\mu + \nu + 2); 1 - \mu; -\frac{1}{2\alpha}\right) \\ A_2(\alpha, p) &= (2\alpha)^{(2+\nu-\mu)/2} {}_1F_1\left(\frac{1}{2}(\mu - \nu - 2); 1 + \mu; -\frac{1}{2\alpha}\right). \end{aligned}$$

For a valid Laplace transform, we need to choose the solution that is holomorphic in a right half plane. Bearing in mind the series expansion of the hyper-geometric function (see, for example, Abramowitz and Stegun [17], §13.1.2) this excludes the function  $A_1$ . To establish the form of  $C_2$ , we impose the boundary condition. We note (see, for example, [17], §13.1.5) that if  $\operatorname{Re}(z) < 0$  then as  $|z| \rightarrow \infty$

$${}_1F_1(a; b; z) \sim \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a}.$$

Matching to the transformed boundary condition fixes  $C_2$ , and we obtain

$$\hat{C}(\alpha, p) = \frac{(2\alpha)^{\frac{1}{2}(\nu+2-\mu/2)} \Gamma(2 + \frac{1}{2}(\mu + \nu))}{p(p - 2(1 + \nu)) \Gamma(1 + \mu)} {}_1F_1\left(-\frac{1}{2}(2 + \nu - \mu); 1 + \mu; -\frac{1}{2\alpha}\right). \quad (11)$$

This is the Geman and Yor model of the transformed price. In fact, Geman and Yor originally gave their result as an integral—that this integral is a hyper-geometric function is easily demonstrated and the transform  $\hat{C}(\alpha, p)$  can be inverted by direct numerical integration [2]. The current version of the code of [2] is given on the web at [18]. Finally we note that for special values of  $\nu$  the hyper-geometric functions may be simplified to linear combinations of modified Bessel functions. This is discussed briefly in Appendix A and is used later to provide benchmark numerical solutions for the case  $r = q$ .

### 3.3 The continuous Vecer model

As we have worked matters, we have an initial-boundary value problem based on a quadrant within which the master PDE (3) holds,

$$\frac{\partial \psi}{\partial \tau_1} = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \left( \frac{1}{T} - (r - q) \right) \frac{\partial \psi}{\partial \eta},$$

initial condition  $\psi(\eta, 0) = 0$  and boundary condition  $\psi(0, \tau_1) = f(\tau_1)$ . Let us now make the change of independent variables

$$t_1 = \tau_1, \quad y = e^{-(r-q)\tau_1} \eta + f(\tau_1),$$

i.e., use the linear solution of the PDE as our new price-like variable. In terms of these new variables (3) becomes

$$\frac{\partial \psi}{\partial t_1} = \frac{1}{2} \sigma^2 (y - f(t_1))^2 \frac{\partial^2 \psi}{\partial y^2}.$$

This is Vecer's PDE. The initial data for a call remains  $\psi(y, 0) = \max(y, 0)$ . Unfortunately, either the boundary condition which was originally posed on  $\eta = 0$  is mapped to a condition along the time-dependent boundary  $y = f(t_1)$ , or one simply abandons the boundary condition altogether and uses the initial condition posed on the full spatial range. In the latter case, the problem becomes a pure initial value problem which, if solved numerically, involves twice the computation.

## 4 Asymptotic solutions for Asian-rate calls

In this section we consider the low volatility limit of the problem. This is the case which is most problematic when one performs a numerical inversion of the Laplace transform of the solution derived above.

By a low volatility limit we mean the limit  $\sigma^2 T \ll 1$  in which the (non-dimensional) variance of final returns is small. In this limit we are able to find approximate solutions by the method of matched asymptotic expansions, although in our context the self-similar nature of the non-dimensional payoffs

$$\max(\eta, 0) \quad \text{and} \quad \max(\eta - 1, 0)$$

means that the matching conditions between inner and out solutions (see below) are satisfied automatically and need not be explicitly imposed.

The method of matched asymptotics is a well established technique in many areas of applied mathematics, see for example Hinch 1991, [19], and Van Dyke 1978, [20]. In the context of finance it is a less widely used technique. Howison 2005, [21], contains a highly readable overview and examples of the application of

matched asymptotic methods to option pricing problems and, more recently, Howison and co-workers [22, 23] have used the method to obtain quite general continuity corrections for discretely sampled European barrier and Bermudan options.

Before proceeding with the asymptotic analysis, however, it is useful to fully non-dimensionalise the problem. The semi-dimensional form of the problem is (3) with  $W(\tau_1) = 1$ ,

$$\frac{\partial \psi}{\partial \tau_1} + \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \left( \frac{1}{T} - (r - q) \eta \right) \frac{\partial \psi}{\partial \eta} = 0, \quad \tau_1 > 0, \quad -\infty < \eta < \infty,$$

subject to one of the initial conditions

$$\begin{aligned} \psi_{\text{rc}}(\eta, 0) &= \max(\eta, 0), & \psi_{\text{rp}}(\eta, 0) &= \max(-\eta, 0), \\ \psi_{\text{sc}}(\eta, 0) &= \max(\eta - 1, 0), & \psi_{\text{sp}}(\eta, 0) &= \max(1 - \eta, 0), \end{aligned} \tag{12}$$

corresponding to average-rate calls and puts and average-strike calls and puts, respectively.

We introduce a non-dimensional time, based on the option's expiry date,

$$\tau = \frac{\tau_1}{T} = 1 - \frac{t}{T},$$

to make the problem completely non-dimensional, and introduce the non-dimensional parameters

$$\theta = (r - q)T$$

and

$$\epsilon^2 = \frac{1}{2} \sigma^2 T. \tag{13}$$

In what follows we shall assume that  $\epsilon \ll 1$  and use it as the small parameter in our asymptotic series expansions.

In terms of  $\tau$ ,  $\theta$  and  $\epsilon$ , the master PDE (3) becomes

$$\frac{\partial \psi}{\partial \tau} + \frac{1}{2} \epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + (1 - \theta \eta) \frac{\partial \psi}{\partial \eta} = 0, \quad \tau > 0, \quad -\infty < \eta < \infty, \tag{14}$$

with the same initial conditions as given in (12).

The critical observation is that in practice  $\epsilon$  is often small and at worst, in a typical contract, of order unity. For example, for a 10 year Asian contract with  $\sigma = 0.5$ , we have  $\epsilon = 1.118$  and for a two year contract with  $\sigma = 0.25$ , we have  $\epsilon = 0.25$ . So for many practical purposes we may regard  $\epsilon$  as a small and treat it as a perturbation parameter. The analysis which follows exploits this fact.

Given that the payoff is linear in  $\eta$  as  $\eta \rightarrow \pm\infty$  (and the diffusion is apparently small) it seems reasonable to assume that the solution remains linear in  $\eta$ , to a very high degree of approximation, as  $\eta \rightarrow \pm\infty$ .

Financially, this amounts to the assumption that, if the payoff is locally linear for very high and low spot prices and the option is either well in-the-money or well out-of-the-money, then the option's  $\Delta$  will be effectively a function of time only and will not significantly vary as the price of the underlying varies. That is, for a call-like payoff, if the option is deeply in-the-money, then it is going to be exercised and the holder must have the asset ready to deliver at expiry, regardless of the exact price of the asset at expiry, and likewise, if the option is well out-of-the-money it is not going to be exercised and there is no point in holding the asset. This, of course, is not strictly true as there is a small probability that, however deeply in or out-of-the-money the option is, it will change its moniness between now and expiry. This probability, however, is exponentially small for deeply in or deeply out-of-the money options. Thus, to a very high degree of approximation, the  $\Gamma$  of the option is going to be zero as  $\eta \rightarrow \pm\infty$  and the smaller the volatility, the better an approximation this becomes. When the option is at-the-money, however, this approximation breaks down completely because there is a high probability of the option's moniness changing between now and exercise. In this case the diffusion and volatility terms become crucial and certainly cannot be neglected.

With this in mind, we assume an *outer* region where the volatility is effectively irrelevant and an *inner* region where it is not. The outer region, to be made more precise shortly, includes the regions  $\eta \rightarrow \pm\infty$ , and consists of regions where the likelihood of the option's moniness changing is negligible and the term<sup>2</sup>

$$\epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} = \mathcal{O}(\epsilon^2),$$

for  $\tau > 0$ . The inner region, also to be made more precise shortly, is essentially the complement of the outer region and consists of those regions where

$$\epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} = \mathcal{O}(1),$$

for  $\tau > 0$ , that is, regions where the likelihood of the option changing its moniness is significant.

For the sake of brevity we will only present the details for an arithmetic average-rate call option; the result for the rate put follows from Asian put-call parity and the results for the strike-style options follow by a minor modification to the non-dimensional payoff and the “critical-characteristic” defined below.

#### 4.1 The outer region and solution

We start by assuming that if the diffusion term is multiplied by the small number  $\epsilon^2$  then its effect will be, in the main, small. It turns out that this assumption is *not* uniformly valid in  $\eta$ , as we will find when we make an *a posteriori* check.

So, to begin we assume a regular perturbation solution of the form

$$\psi(\eta, \tau) = \Phi_0(\eta, \tau) + \epsilon^2 \Phi_1(\eta, \tau) + \dots$$

We find that, at leading order, the problem for  $\Phi_0$  is the linear, first-order, hyperbolic partial differential equation<sup>3</sup>

$$\frac{\partial \Phi_0}{\partial \tau} + (\theta \eta - 1) \frac{\partial \Phi_0}{\partial \eta} = 0. \tag{15}$$

For reasons which we will give in §4.2.4, there is no point in trying to determine higher order terms in this expansion: for the payoffs mentioned above the outer solution serves chiefly to determine where the inner region lies and this is determined by the leading-order outer solution alone.

The characteristic projections for (15) are defined by

$$\frac{d\eta}{d\tau} = (\theta \eta - 1), \quad \eta(\tau = 0) = \eta_0, \tag{16}$$

and are given by

$$\eta(\tau) = \begin{cases} \eta_0 e^{\theta\tau} + \frac{1}{\theta} (1 - e^{\theta\tau}) & \theta \neq 0, \\ \eta_0 - \tau & \theta = 0. \end{cases}$$

Along these characteristic projections  $\Phi_0$  is constant<sup>4</sup> and so if, in general,

$$\Phi_0(\eta, 0) = F(\eta)$$

then

$$\Phi_0(\eta, \tau) = F\left(e^{-\theta\tau} \eta + \frac{1}{\theta} (1 - e^{-\theta\tau})\right),$$

<sup>2</sup>In fact, if the payoff is linear as  $\eta \rightarrow \pm\infty$ , it turns out that in this region  $\partial^2 \psi / \partial \eta^2$  is exponentially small as a function of  $\eta$ .

<sup>3</sup>This, of course, corresponds to the non-stochastic problem where the asset price grows deterministically at the rate  $r - q$  and the characteristic projections are simply the non-dimensional, zero-volatility price-paths.

<sup>4</sup>Financially, this is a simple consequence of the fact that we have already factored out the risk-free discounting of the option's value at expiry in our non-dimensionalisation.

if  $\theta \neq 0$  and

$$\Phi_0(\eta, \tau) = F(\eta + \tau),$$

if  $\theta = 0$ .

In deriving this expansion, however, we have assumed that

$$\epsilon^2 \frac{\partial^2 \Phi_0}{\partial \eta^2} \ll 1,$$

and for the payoffs we are interested in, this assumption fails along certain characteristics. Specifically, for the average-rate call, where

$$F(\eta) = \max(\eta, 0),$$

we find that  $\partial^2 \psi / \partial \eta^2$  is a delta-function along the *critical characteristic*,

$$\eta^*(\tau) = \begin{cases} \frac{1}{\theta} (1 - e^{\theta\tau}) & \theta \neq 0, \\ -\tau & \theta = 0, \end{cases} \quad (17)$$

passing at  $\tau = 0$  through the point of discontinuity in the slope of the payoff,<sup>5</sup> which in this case is at the origin.

Around this critical characteristic our regular perturbation expansion fails because along the critical characteristic

$$\epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} \gg \epsilon^2,$$

for  $\tau > 0$ . Therefore, we cannot ignore the diffusion effects near this characteristic. We have to introduce a new, “inner”, region around this characteristic in which the diffusion effects balance with the drift effects.

We call the region where the regular expansion given above is valid the “outer region”. For the average-rate call we are interested in, the outer solution is

$$\Phi_0(\eta, \tau) = \max \left( e^{-\theta\tau} + \frac{1}{\theta} (1 - e^{-\theta\tau}), 0 \right), \quad (18)$$

with the obvious limit when  $\theta = 0$ , and it turns out that it is valid in the outer region given by

$$|\eta - \eta^*(\tau)| \gg \epsilon,$$

as we shall see.

Roughly speaking, this regular perturbation expansion is valid when the option is either deeply in-the-money or deeply out-of-the-money and the probability that it will change its moniness prior to expiry is negligible. In this case the effects of volatility are unimportant (which is why we can ignore them at leading order). The critical characteristic corresponds to the option being exactly at-the-money and the inner region is the region where there is a significant probability that the option will change its moniness prior to expiry; here the effects of volatility are crucial and must be considered.

## 4.2 The rate-call with $r = q$

The algebra involved in the analysis of the inner region is much more straightforward when  $\theta = 0$ , i.e.  $r = q$ , and as discussed in Appendix A, the Laplace transform solution can also be simplified, so we shall discuss this case first in order to illustrate the principle of the method.

When  $r = q$ ,  $\theta = 0$  and the non-dimensional form the problem is

$$\frac{\partial \psi}{\partial \tau} = \epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial \psi}{\partial \eta}, \quad \psi(\eta, 0) = \max(\eta, 0). \quad (19)$$

<sup>5</sup>For the strike-call, the critical characteristic is the characteristic projection passing, at time  $\tau = 0$ , through the point of discontinuity in the slope of the payoff  $\max(\eta - 1, 0)$ , namely  $\eta = 1$ , and is given by  $\eta^*(\tau) = e^{\theta\tau} + (1 - e^{\theta\tau})/\theta$  for  $\theta \neq 0$  and  $\eta^*(\tau) = 1 - \tau$  for  $\theta = 0$ .

The leading-order outer solution,  $\Phi_0$ , is given by

$$\Phi_0(\eta, \tau) = \max(\eta + \tau, 0),$$

and it fails near the critical characteristic

$$\eta^*(\tau) = -\tau.$$

Note that the critical characteristic is defined by (16), with  $\eta^*(0) = 0$ .

For the inner expansion, where  $\eta$  is near the critical characteristic, we introduce the scaled<sup>6</sup> price-like variable  $\zeta$  defined by

$$\zeta = \frac{1}{\epsilon}(\eta - \eta^*(\tau)) \quad \text{or} \quad \zeta = \frac{1}{\epsilon}(\eta + \tau).$$

Applying the chain rule and using the fact that  $d\eta^*/d\tau = -1$  when  $\theta = 0$  shows that this new variable eliminates the drift term<sup>7</sup> and reduces the problem to

$$\frac{\partial \psi}{\partial \tau} = (\eta^*(\tau) + \epsilon \zeta)^2 \frac{\partial^2 \psi}{\partial \zeta^2}, \quad -\infty < \zeta < \infty.$$

We now assume a regular asymptotic series in  $\epsilon$ ,

$$\psi = \sum_{k=1}^{\infty} \epsilon^k \varphi_k, \quad (20)$$

for the inner region. In particular, in order that the series *is* asymptotic, we require that

$$\left| \frac{\varphi_{k+1}(\zeta, \tau)}{\varphi_k(\zeta, \tau)} \right| = \mathcal{O}(1)$$

for all  $\tau > 0$  and all  $|\zeta| = \mathcal{O}(1)$ . This gives the following hierarchy of partial differential equations,

$$\begin{aligned} \frac{\partial \varphi_1}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_1}{\partial \zeta^2} &= 0, & k = 1 \\ \frac{\partial \varphi_2}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_2}{\partial \zeta^2} &= 2\eta^*(\tau)\zeta \frac{\partial^2 \varphi_1}{\partial \zeta^2}, & k = 2 \\ \frac{\partial \varphi_k}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_k}{\partial \zeta^2} &= 2\eta^*(\tau)\zeta \frac{\partial^2 \varphi_{k-1}}{\partial \zeta^2} + \zeta^2 \frac{\partial^2 \varphi_{k-2}}{\partial \zeta^2}, & k > 2, \end{aligned} \quad (21)$$

with the corresponding initial conditions

$$\varphi_k(\zeta, 0) = \begin{cases} \max(\zeta, 0) & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases} \quad (22)$$

For the payoffs of interest it is not necessary to explicitly impose matching conditions as  $\zeta \rightarrow \pm\infty$ , as we shall see.

#### 4.2.1 Solution of first-order problem for $r = q$

The first-order problem is

$$\begin{aligned} \frac{\partial \varphi_1}{\partial \tau} &= \eta^*(\tau)^2 \frac{\partial^2 \varphi_1}{\partial \zeta^2} \\ \varphi_1(\zeta, 0) &= \max(\zeta, 0), \end{aligned}$$

<sup>6</sup>If we try an inner variable of the form  $\eta = \eta^*(\tau) + \epsilon^\alpha \zeta$ , we find that only the choice of  $\alpha = 1$  leads to a problem in which time and second price-like derivatives balance in the resulting partial differential equation, i.e.,  $\alpha = 1$  is the distinguished limit.

<sup>7</sup>This is because the characteristic projections are paths which move with the drift and our inner variable effectively moves with one of the characteristic projections.

where  $\eta^*(\tau) = -\tau$ , and the change of variable

$$\hat{t}(\tau) = \int_0^\tau \eta^*(\tau')^2 d\tau' = \frac{1}{3} \tau^3$$

transforms this to

$$\begin{aligned} \frac{\partial \varphi_1}{\partial \hat{t}} &= \frac{\partial^2 \varphi_1}{\partial \zeta^2} \\ \varphi_1(\zeta, 0) &= \max(\zeta, 0). \end{aligned}$$

The solution is easily found to be

$$\begin{aligned} \varphi_1(\zeta, \tau) &= \zeta N\left(\frac{\zeta}{\sqrt{2\hat{t}}}\right) + \sqrt{\frac{\hat{t}}{\pi}} \exp\left(-\frac{\zeta^2}{4\hat{t}}\right) \\ &= \zeta N\left(\sqrt{\frac{3}{2\tau^3}} \zeta\right) + \sqrt{\frac{\tau^3}{3\pi}} \exp\left(-\frac{3\zeta^2}{4\tau^3}\right), \end{aligned}$$

where  $N$  is the standard Normal cumulative density function,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}s^2} ds.$$

#### 4.2.2 Matching the leading-order inner and outer solutions

When  $\zeta = \mathcal{O}(1/\epsilon)$ , we are moving out of the inner region and into the outer region. It is necessary to check that the inner and outer solutions agree in this region. We confirm this here. We find that

$$\begin{aligned} \lim_{\zeta \rightarrow -\infty} \varphi_1(\zeta, \tau) &= \zeta + \mathcal{O}\left(e^{-\zeta^2/\tau^3}\right) \\ \lim_{\zeta \rightarrow -\infty} \varphi_1(\zeta, \tau) &= \mathcal{O}\left(e^{-\zeta^2/\tau^3}\right), \end{aligned}$$

so that for  $\zeta = \mathcal{O}(1/\epsilon)$  we have

$$\epsilon \varphi(\zeta, \tau) = \begin{cases} \eta + \tau + \mathcal{O}(e^{-1/\epsilon^2}) & \zeta > 0, \\ \mathcal{O}(e^{-1/\epsilon^2}) & \zeta < 0, \end{cases}$$

which matches, to all algebraic orders, the outer solution

$$\Phi_0(\eta, \tau) = \max(\eta + \tau, 0).$$

#### 4.2.3 Solution of second-order problem for $r = q$

The key to solving the second and higher order problems is the observation that

$$\frac{\partial \varphi_1}{\partial \zeta} = N\left(\frac{\zeta}{\sqrt{2\hat{t}}}\right)$$

and hence that

$$\frac{\partial^2 \varphi_1}{\partial \zeta^2} = \frac{1}{\sqrt{4\pi\hat{t}}} \exp\left(-\frac{\zeta^2}{4\hat{t}}\right) \equiv \mathcal{G}(\zeta, \hat{t}) = \mathcal{G}\left(\zeta, \frac{1}{3}\tau^3\right) \quad (23)$$

where  $\mathcal{G}(\zeta, \hat{t})$  is the Green's function for the free diffusion equation in  $(\zeta, \hat{t})$ . Importantly, it has the properties that

$$\frac{\partial \mathcal{G}(\zeta, \hat{t})}{\partial \hat{t}} = \frac{\partial^2 \mathcal{G}(\zeta, \hat{t})}{\partial \zeta^2} \quad \text{and} \quad \frac{\partial \mathcal{G}(\zeta, \hat{t})}{\partial \zeta} = -\frac{\zeta}{2\hat{t}} \mathcal{G}(\zeta, \hat{t}). \quad (24)$$

The problem for  $\varphi_2$  is

$$\frac{\partial \varphi_2}{\partial \tau} - \tau^2 \frac{\partial^2 \varphi_2}{\partial \zeta^2} = -2\tau \zeta \mathcal{G} \left( \zeta, \frac{1}{3} \tau^3 \right)$$

with

$$\varphi_2(\zeta, 0) = 0.$$

We seek a solution of the form

$$\varphi_2(\zeta, \tau) = \zeta f(\tau) \mathcal{G} \left( \zeta, \frac{1}{3} \tau^3 \right),$$

where  $f$  is a function to be determined.

On the basis of this assumption and using (24), we find that

$$\frac{\partial \varphi_2}{\partial \tau} - \tau^2 \frac{\partial^2 \varphi_2}{\partial \zeta^2} = \left( \frac{df}{d\tau} + \frac{3}{\tau} f \right) \zeta \mathcal{G}(\zeta, \tau)$$

so the PDE for  $\varphi_2$  is satisfied iff

$$\frac{df}{d\tau} + \frac{3}{\tau} f = -2\tau,$$

which gives

$$f(\tau) = -\frac{2}{5} \tau^2 + \frac{C_2}{\tau^3}$$

The initial condition for  $\varphi_2$  is automatically satisfied since  $\mathcal{G}(\zeta, \tau) \rightarrow 0$  rapidly enough as  $\tau \rightarrow 0$ , no matter what the value of  $C_2$ . If we recall, however, that the series (20) must be asymptotic as  $\tau \rightarrow 0$ , then we must have  $C_2 = 0$  and hence

$$f(t) = -\frac{2}{5} \tau^2. \quad (25)$$

Thus, we are led to

$$\begin{aligned} \varphi_2(\zeta, \tau) &= -\frac{2}{5} \tau^2 \zeta \mathcal{G} \left( \zeta, \frac{1}{3} \tau^3 \right) \\ &= -\frac{1}{5} \zeta \sqrt{\frac{3\tau}{\pi}} \exp \left( -\frac{3\zeta^2}{4\tau^3} \right). \end{aligned}$$

#### 4.2.4 Comments on matching and the outer solution

We find that when  $\zeta = \mathcal{O}(1/\epsilon)$ ,  $\varphi_2(\zeta, \tau)$  is exponentially small. Thus, as we move into the outer region it vanishes to all algebraic orders. In the outer region, the corresponding outer solution,  $\Phi_1(\eta, \tau)$ , formally satisfies the same equation (15) as  $\Phi_0$ , since the second derivative of  $\Phi_0$  is only non-zero along the critical characteristic which is inside the inner region, but with zero initial conditions. Matching it to the inner solution, which vanishes to all algebraic orders, gives us zero boundary conditions and hence the outer solution  $\Phi_1$  vanishes. Similar results apply to all the higher order terms in the outer solution, which is why it is unnecessary to compute them. Moreover, the inner expansion  $\varphi_1$  matches the outer solution  $\Phi_0$  to all algebraic orders in the outer region and the outer solution is therefore effectively redundant, its only use is to determine *where* the inner region lies.

#### 4.2.5 Solution of third-order problem

We will see that the structure of the problem is such that the device employed to obtain the second-order solution may be used again. The PDE for  $\varphi_3$  is given by

$$\frac{\partial \varphi_3}{\partial \tau} - \tau^2 \frac{\partial^2 \varphi_3}{\partial \zeta^2} = -2\tau \zeta \frac{\partial^2 \varphi_2}{\partial \zeta^2} + \zeta^2 \frac{\partial^2 \varphi_1}{\partial \zeta^2}$$

We already know that

$$\frac{\partial^2 \varphi_1}{\partial \zeta^2} = \mathcal{G} \left( \zeta, \frac{1}{3} \tau^3 \right)$$

and after a little calculation using (24) we find that

$$\frac{\partial^2 \varphi_2}{\partial \zeta^2} = -\frac{9}{10} \tau^{-2} \zeta (\zeta^2 - 2\tau^3) \mathcal{G} \left( \zeta, \frac{1}{3} \tau^3 \right).$$

The PDE for  $\varphi_3$  is then

$$\frac{\partial \varphi_3}{\partial \tau} - \tau^2 \frac{\partial^2 \varphi_3}{\partial \zeta^2} = \left( \frac{9}{5} \tau^{-3} \zeta^4 - \frac{13}{5} \zeta^2 \right) \mathcal{G}(\zeta, \tau)$$

As above, we make the assumption of a suitable solution, which this time involves a polynomial in  $\zeta$ ,

$$\varphi_3(\zeta, \tau) = (f_1(\tau) + f_2(\tau)\zeta^2 + f_3(\tau)\zeta^4) \mathcal{G} \left( \zeta, \frac{1}{3} \tau^3 \right).$$

Substituting this into the PDE for  $\varphi_3$  gives the coupled system of ODEs,

$$\begin{aligned} \frac{df_1(\tau)}{d\tau} - 2\tau^2 f_2(\tau) &= 0, \\ \frac{df_2(\tau)}{d\tau} + \frac{6}{\tau} f_2(\tau) - 12\tau^2 f_3(\tau) &= -\frac{13}{5}, \\ \frac{df_3(\tau)}{d\tau} + \frac{12}{\tau} f_3(\tau) &= \frac{9}{5} \tau^{-3}. \end{aligned}$$

These may be solved in sequence using integrating factors and to give

$$\begin{aligned} f_1(\tau) &= -\frac{11}{350} \tau^4, \\ f_2(\tau) &= -\frac{11}{175} \tau, \\ f_3(\tau) &= \frac{9}{50} \tau^{-2}, \end{aligned} \tag{26}$$

where again all arbitrary constants have been set to zero as they lead to stronger singularities in these functions which, in turn, violate the asymptotic condition on (20).

Note that, again, this term in the solution is exponentially small when  $\zeta = \mathcal{O}(1/\epsilon)$ .

#### 4.2.6 The general higher-order term for $r = q$

If, for  $k > 1$ , we make the assumption that

$$\varphi_k = f_k(\zeta, \tau) \mathcal{G} \left( \zeta, \frac{1}{3} \tau^3 \right)$$

then

$$\begin{aligned} f_2(\zeta, \tau) &= -\frac{2}{5} \tau^2 \zeta, \\ f_3(\zeta, \tau) &= -\frac{11}{350} \tau^4 - \frac{11}{175} \tau \zeta^2 + \frac{9}{50} \tau^{-2} \zeta^4, \end{aligned} \tag{27}$$

where again, the condition that the series (20) be asymptotic translates into the condition that  $f_2$  and  $f_3$  have the weakest possible singularities at  $\tau = 0$ .

After some work we find that, for  $k > 3$ ,  $f_k$  must satisfy the recursive system of partial differential equations

$$\begin{aligned} \frac{\partial f_k}{\partial \tau} - \tau^2 \frac{\partial^2 f_k}{\partial \zeta^2} + \frac{3\zeta}{\tau} \frac{\partial f_k}{\partial \zeta} &= -2\tau \zeta \frac{\partial^2 f_{k-1}}{\partial \zeta^2} + \frac{6\zeta^2}{\tau^2} \frac{\partial f_{k-1}}{\partial \zeta} + \left( \frac{3\zeta}{\tau^2} - \frac{9\zeta^3}{2\tau^5} \right) f_{k-1} \\ &+ \zeta^2 \frac{\partial^2 f_{k-2}}{\partial \zeta^2} - \frac{3\zeta^3}{\tau^3} \frac{\partial f_{k-2}}{\partial \zeta} + \left( \frac{9\zeta^4}{4\tau^6} - \frac{3\zeta^2}{2\tau^3} \right) f_{k-2}. \end{aligned}$$

Further solutions of this sequence of PDEs may, in principle, be found as polynomials in  $\zeta$  whose coefficients satisfy ordinary differential equations in  $\tau$ . Working with this scheme<sup>8</sup> gives an equation for  $f_4$ , for example:

$$\frac{\partial f_4}{\partial \tau} - \tau^2 \frac{\partial^2 f_4}{\partial \zeta^2} + \frac{3\zeta}{\tau} \frac{\partial f_4}{\partial \zeta} = -\frac{81}{100} \frac{\zeta^7}{\tau^7} + \frac{297}{70} \frac{\zeta^5}{\tau^4} - \frac{93}{28} \frac{\zeta^3}{\tau} + \frac{11}{70} \tau^2 \zeta,$$

<sup>8</sup>The fourth and fifth-order terms were computed using the symbolic capabilities of Mathematica

with the (least singular) solution

$$f_4 = -\frac{27}{500} \frac{\zeta^7}{\tau^6} + \frac{144}{875} \frac{\zeta^5}{\tau^3} - \frac{1}{300} \zeta^3 + \frac{4}{175} \tau^3 \zeta.$$

Similarly the (least singular) fifth-order solution is can be shown to be

$$f_5 = \frac{243}{20000} \frac{\zeta^{10}}{\tau^{10}} - \frac{1377}{14000} \frac{\zeta^8}{\tau^7} + \frac{13317}{98000} \frac{\zeta^6}{\tau^4} - \frac{29}{2695} \frac{\zeta^4}{\tau} + \frac{2049}{431200} \tau^2 \zeta^2 + \frac{2049}{1078000} \tau^5.$$

Although more terms may be found, those given above more than suffice for our purposes. They are all exponentially small outside the inner region.

#### 4.2.7 Summary complete to fifth-order for $r = q$

For the purposes of numerical evaluation, it is convenient to gather together the following summary of the first five terms in our asymptotic solution for  $r = q$ . Firstly,  $\tau$ ,  $\epsilon$  and  $\hat{t}$  are defined in terms of the original, financial, variables by

$$\tau = \frac{T-t}{T}, \quad \epsilon^2 = \frac{1}{2} \sigma^2 T, \quad \hat{t} = \frac{1}{3} \tau^3 = \frac{(T-t)^3}{3T^3}, \quad (28)$$

and the Green's function for the diffusion equation in  $(\zeta, \hat{t})$  is

$$\mathcal{G}(\zeta, \hat{t}) = \frac{1}{\sqrt{4\pi\hat{t}}} \exp\left(-\frac{\zeta^2}{4\hat{t}}\right).$$

Our inner asymptotic solution for the non-dimensional problem (14) is

$$\psi(\zeta, \tau) = \epsilon \varphi_1(\zeta, \tau) + \epsilon^2 \varphi_2(\zeta, \tau) + \epsilon^3 \varphi_3(\zeta, \tau) + \epsilon^4 \varphi_4(\zeta, \tau) + \epsilon^5 \varphi_5(\zeta, \tau) + \dots$$

and the first five terms in this series, for  $r = q$  and  $\theta = 0$ , are given by

$$\begin{aligned} \varphi_1 &= \zeta \text{N}\left(\frac{\zeta}{\sqrt{2\hat{t}}}\right) + \sqrt{\frac{\hat{t}}{\pi}} \exp\left(-\frac{\zeta^2}{4\hat{t}}\right) \\ &= \zeta \text{N}\left(\sqrt{\frac{3}{2\tau^3}} \zeta\right) + \sqrt{\frac{\tau^3}{3\pi}} \exp\left(-\frac{3\zeta^2}{4\tau^3}\right) \\ \varphi_2 &= -\frac{2}{5} \tau^2 \zeta \mathcal{G}\left(\zeta, \frac{1}{3} \tau^3\right) \\ &= -\frac{1}{5} \sqrt{\frac{3\tau}{\pi}} \zeta \exp\left(-\frac{3\zeta^2}{4\tau^3}\right) \\ \varphi_3 &= \left(-\frac{11}{350} \tau^4 - \frac{11}{175} \tau \zeta^2 + \frac{9}{50} \frac{\zeta^4}{\tau^2}\right) \mathcal{G}\left(\zeta, \frac{1}{3} \tau^3\right) \\ &= \left(-\frac{11}{350} \tau^4 - \frac{11}{175} \tau \zeta^2 + \frac{9}{50} \frac{\zeta^4}{\tau^2}\right) \sqrt{\frac{3}{4\pi\tau^3}} \exp\left(-\frac{3\zeta^2}{4\tau^3}\right) \\ \varphi_4 &= \left(-\frac{27}{500} \frac{\zeta^7}{\tau^6} + \frac{144}{875} \frac{\zeta^5}{\tau^3} - \frac{1}{300} \zeta^3 + \frac{4}{175} \tau^3 \zeta\right) \mathcal{G}\left(\zeta, \frac{1}{3} \tau^3\right) \\ \varphi_5 &= \left(\frac{243}{20000} \frac{\zeta^{10}}{\tau^{10}} - \frac{1377}{14000} \frac{\zeta^8}{\tau^7} + \frac{13317}{98000} \frac{\zeta^6}{\tau^4} - \frac{29}{2695} \frac{\zeta^4}{\tau} + \frac{2049}{431200} \tau^2 \zeta^2 + \frac{2049}{1078000} \tau^5\right) \mathcal{G}\left(\zeta, \frac{1}{3} \tau^3\right) \end{aligned}$$

It turns out that for many of our test problems, with practical parameters of interest, the fifth-order solution is essentially exact, agreeing, for example, with the contour integral solution to six significant figures. Four significant figure accuracy is achievable if we keep terms up to and including  $\varphi_3$ .

### 4.3 The general case: $r \neq q$

When  $r \neq q$ , the non-dimensional problem for a rate-call is

$$\begin{aligned} \frac{\partial \psi}{\partial \tau} &= \epsilon^2 \eta^2 \frac{\partial^2 \psi}{\partial \eta^2} + (1 - \theta \eta) \frac{\partial \psi}{\partial \eta}, \quad \tau > 0, \quad -\infty < \eta < \infty, \\ \psi(\eta, 0) &= \max(\eta, 0), \quad -\infty < \eta < \infty. \end{aligned}$$

The leading-order outer solution is given by (18) and it also is only valid for  $|\eta - \eta^*(\tau)| \gg \epsilon$ . To find the inner expansion we again use the inner variable  $\zeta$ , defined by

$$\zeta = \frac{1}{\epsilon} (\eta - \eta^*(\tau)),$$

where in this case  $\eta^*(\tau)$  is defined by

$$\frac{d\eta^*}{d\tau} = \theta \eta^* - 1, \quad \eta^*(0) = 0 \quad (29)$$

and given by (17),

$$\eta^*(\tau) = \frac{1}{\theta} (1 - e^{\theta\tau}).$$

Applying the chain rule and using (29) which defines the critical characteristic we find that in terms of  $\zeta$  and  $\tau$  the problem simplifies to

$$\begin{aligned} \frac{\partial \psi}{\partial \tau} &= (\eta^*(\tau) + \epsilon \zeta)^2 \frac{\partial^2 \psi}{\partial \zeta^2} - \theta \zeta \frac{\partial \psi}{\partial \zeta}, \quad \tau > 0, \quad -\infty < \zeta < \infty, \\ \psi(\zeta, 0) &= \epsilon \max(\zeta, 0), \quad -\infty < \zeta < \infty. \end{aligned}$$

We again assume the regular perturbation expansion (20),

$$\psi = \sum_{k=1}^{\infty} \epsilon^k \varphi_k,$$

and obtain the hierarchy of PDEs

$$\begin{aligned} \frac{\partial \varphi_1}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_1}{\partial \zeta^2} + \theta \zeta \frac{\partial \varphi_1}{\partial \zeta} &= 0, \quad k = 1, \\ \frac{\partial \varphi_2}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_2}{\partial \zeta^2} + \theta \zeta \frac{\partial \varphi_2}{\partial \zeta} &= 2\eta^*(\tau) \zeta \frac{\partial^2 \varphi_1}{\partial \zeta^2}, \quad k = 2, \\ \frac{\partial \varphi_k}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_k}{\partial \zeta^2} + \theta \zeta \frac{\partial \varphi_k}{\partial \zeta} &= 2\eta^*(\tau) \zeta \frac{\partial^2 \varphi_{k-1}}{\partial \zeta^2} + \zeta^2 \frac{\partial^2 \varphi_{k-2}}{\partial \zeta^2}, \quad k > 2, \end{aligned}$$

with the respective initial conditions

$$\varphi_k(\zeta, 0) = \begin{cases} \max(\zeta, 0) & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

#### 4.3.1 Solution of first-order problem for $r \neq q$

The problem for the first-order term is

$$\begin{aligned} \frac{\partial \varphi_1}{\partial \tau} &= \eta^*(\tau)^2 \frac{\partial^2 \varphi_1}{\partial \zeta^2} - \theta \zeta \frac{\partial \varphi_1}{\partial \zeta}, \\ \varphi_1(\zeta, 0) &= \max(\zeta, 0). \end{aligned}$$

To solve it, we introduce the new variables

$$\hat{t} = \hat{t}(\tau), \quad \text{and} \quad x = \hat{f}(\tau) \zeta$$

where  $\hat{t}(\tau)$  and  $\hat{f}(\tau)$  are yet to be determined. From the chain rule we have

$$\frac{\partial}{\partial \tau} = \frac{d\hat{t}}{d\tau} \frac{\partial}{\partial \hat{t}} + \zeta \frac{d\hat{f}}{d\tau} \frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial \zeta} = \hat{f}(\tau) \frac{\partial}{\partial x}.$$

The PDE for first-order term is then

$$\frac{d\hat{t}}{d\tau} \frac{\partial \varphi_1}{\partial \hat{t}} = \eta^*(\tau)^2 \hat{f}(\tau)^2 \frac{\partial^2 \varphi_1}{\partial x^2} - \zeta \left( \frac{d\hat{f}}{d\tau} + \theta \hat{f} \right) \frac{\partial \varphi_1}{\partial x}$$

and we can reduce this to the pure diffusion equation, in  $(x, \hat{t})$ , by choosing

$$\frac{d\hat{f}}{d\tau} + \theta \hat{f} = 0 \quad \text{and} \quad \frac{d\hat{t}}{d\tau} = \left( \eta^*(\tau) \hat{f}(\tau) \right)^2.$$

Further, if we insist that

$$\hat{f}(0) = 1, \quad \hat{t}(0) = 0,$$

then, apart from replacing  $\zeta$  by  $x$ , the initial condition remains unaltered. We find that this implies that

$$\begin{aligned} \hat{f}(\tau) &= e^{-\theta\tau}, \\ \hat{t}(\tau) &= \frac{4e^{-\theta\tau} - e^{-2\theta\tau} + 2\theta\tau - 3}{2\theta^3} \\ &= \frac{1}{3}\tau^3 - \frac{1}{4}\theta\tau^4 + \frac{7}{60}\theta^2\tau^5 - \frac{1}{24}\theta^3\tau^6 + \frac{31}{2520}\theta^4\tau^7 - \frac{1}{320}\theta^5\tau^8 + \mathcal{O}(\theta^6\tau^9), \end{aligned}$$

the series expansion showing that that our scaled time is consistent with the  $\theta = 0$  case and also being useful for numerical evaluation when  $\theta\tau \ll 1$  (see below).

We may now solve the diffusion equation,

$$\frac{\partial \varphi_1}{\partial \hat{t}} = \frac{\partial^2 \varphi_1}{\partial x^2},$$

subject to the initial condition

$$\varphi_1(x, 0) = \max(x, 0),$$

to find that, in this case,

$$\varphi_1 = x \text{N} \left( \frac{x}{\sqrt{2\hat{t}(\tau)}} \right) + \sqrt{\frac{\hat{t}(\tau)}{\pi}} \exp \left( -\frac{x^2}{4\hat{t}(\tau)} \right).$$

where  $\text{N}(\cdot)$  is the normal cumulative density. In this case it is clearly more concise to leave  $\varphi_1$  expressed in terms of  $x$  and  $\hat{t}(\tau)$ .

Note that as we let  $\zeta \rightarrow \pm\infty$  we recover the outer solution to all algebraic orders.

As before, it is important that

$$\frac{\partial^2 \varphi_1}{\partial x^2} = \frac{1}{\sqrt{4\pi\hat{t}(\tau)}} \exp \left( -\frac{x^2}{4\hat{t}(\tau)} \right) \equiv \mathcal{G}(x, \hat{t}(\tau)).$$

### 4.3.2 Solution of second-order problem for $r \neq q$

The problem for  $\varphi_2$  is

$$\frac{\partial \varphi_2}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_2}{\partial \zeta^2} + \theta \zeta \frac{\partial \varphi_2}{\partial \zeta} = 2\eta^*(\tau) \zeta \frac{\partial^2 \varphi_1}{\partial \zeta^2},$$

$$\varphi_2(\zeta, 0) = 0$$

and, in terms of  $x$  and  $\hat{t}$ , it becomes

$$\frac{\partial \varphi_2}{\partial \hat{t}} - \frac{\partial^2 \varphi_2}{\partial x^2} = \frac{2}{\eta^*(\tau) e^{-\theta \tau}} x \mathcal{G}(x, \hat{t}),$$

$$\varphi_2(x, 0) = 0.$$

We proceed in a manner similar to the case  $r = q$ , and seek a solution in the form

$$\varphi_2(x, \hat{t}) = x g(\hat{t}) \mathcal{G}(x, \hat{t}),$$

where  $g(\hat{t})$  has the lowest possible singularity as  $\hat{t} \rightarrow 0$ . On this assumption we have

$$\frac{\partial \varphi_2}{\partial \hat{t}} - \frac{\partial^2 \varphi_2}{\partial x^2} = \left( \frac{dg}{d\hat{t}} + \frac{1}{\hat{t}} g \right) x \mathcal{G}(x, \hat{t}),$$

so the PDE is satisfied iff

$$\frac{dg}{d\hat{t}} + \frac{1}{\hat{t}} g = \frac{2}{\eta^*(\tau) e^{-\theta \tau}} \quad (30)$$

To solve this ODE for  $g(\hat{t})$ , let  $h(\tau) = \hat{t}(\tau)g(\tau)$  and revert to  $\tau$  as an independent variable. The ODE for  $h$  is then

$$\frac{dh}{d\tau} = \frac{dh}{d\hat{t}} \frac{d\hat{t}}{d\tau} = \frac{dh}{d\hat{t}} (\eta^*(\tau) e^{-\theta \tau})^2 = 2\hat{t}(\tau) \eta^*(\tau) e^{-\theta \tau}$$

and hence may be integrated, with the initial condition that  $h(0) = 0$ , to give

$$h(\tau) = \frac{e^{-3\theta \tau} \left( -6e^{2\theta \tau} (2\theta \tau - 5) - 15e^{\theta \tau} + e^{3\theta \tau} \left( -6\theta \tau (\theta \tau - 3) - 17 \right) + 2 \right)}{6\theta^5}.$$

So, we finally arrive at the least singular solution

$$\varphi_2 = x \frac{h(\tau)}{\hat{t}(\tau)} \mathcal{G}(x, \hat{t}(\tau)).$$

We note that in the limit  $\theta \rightarrow 0$  this is consistent with the result (25) found earlier for  $\varphi_2$  when  $\theta = 0$ , since

$$\frac{h(\tau)}{\hat{t}(\tau)} = -\frac{2}{5} \tau^2 + \frac{7}{60} \theta \tau^3 - \frac{229}{8400} \theta^2 \tau^4 + \frac{181}{33600} \theta^3 \tau^5 + \mathcal{O}(\theta^4 \tau^6).$$

Although no approximation is needed at this stage, this and the previous series expansions are noted also because  $0 < \tau \leq 1$  by construction and, in practice,  $\theta = (r - q)T$  is usually small. These power series representations are often actually more useful than the exact representations of the solutions. This is because the exact representations involve differences and, worse still ratios of differences, of functions which cancel up to relatively high powers of  $\theta$  as  $\theta$  becomes small, leading to considerable rounding error.

We also note that as  $\zeta \rightarrow \pm\infty$  this term in the asymptotic expansion vanishes exponentially fast.

### 4.3.3 Solution of third-order problem for $r \neq q$

The PDE for  $\varphi_3$  is in general given by

$$\frac{\partial \varphi_3}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_3}{\partial \zeta^2} + \theta \zeta \frac{\partial \varphi_3}{\partial \zeta} = 2\eta^*(\tau) \zeta \frac{\partial^2 \varphi_2}{\partial \zeta^2} + \zeta^2 \frac{\partial^2 \varphi_1}{\partial \zeta^2},$$

subject to the initial condition  $\varphi_3(\zeta, 0) = 0$ . We already know that

$$\zeta^2 \frac{\partial^2 \varphi_1}{\partial \zeta^2} = x^2 \frac{\partial^2 \varphi_1}{\partial x^2} = x^2 \mathcal{G}(x, \hat{t}(\tau))$$

and we can deduce that

$$\frac{\partial^2 \varphi_2}{\partial \zeta^2} = e^{-2\theta\tau} \frac{\partial^2 \varphi_2}{\partial x^2} = e^{-2\theta\tau} \frac{x(x^2 - 6\hat{t}(\tau))}{4\hat{t}^2(\tau)} \frac{h(\tau)}{\hat{t}(\tau)} \mathcal{G}(x, \hat{t}(\tau)).$$

The PDE for  $\varphi_3$  can therefore be written as

$$\frac{\partial \varphi_3}{\partial \tau} - \eta^*(\tau)^2 \frac{\partial^2 \varphi_3}{\partial \zeta^2} + \theta \zeta \frac{\partial \varphi_3}{\partial \zeta} = \left( x^2 \left( 1 - \frac{3h(\tau) \eta^*(\tau) e^{-\theta\tau}}{\hat{t}^2(\tau)} \right) + x^4 \frac{h(\tau) \eta^*(\tau) e^{-\theta\tau}}{2\hat{t}^3(\tau)} \right) \mathcal{G}(x, \hat{t}(\tau)),$$

and, regarding  $\varphi_3$  as a function of the variables  $x$  and  $\hat{t}$ , this becomes

$$\frac{\partial \varphi_3}{\partial \hat{t}} - \frac{\partial^2 \varphi_3}{\partial x^2} = \left( x^2 \left( 1 - \frac{3h(\tau) \eta^*(\tau) e^{-\theta\tau}}{\hat{t}^2(\tau)} \right) + x^4 \frac{h(\tau) \eta^*(\tau) e^{-\theta\tau}}{2\hat{t}^3(\tau)} \right) \frac{\mathcal{G}(x, \hat{t})}{(\eta^*(\tau) e^{-\theta\tau})^2}.$$

As before, we make the assumption of a suitable solution, in the form of the product of a polynomial in  $x$  with  $\hat{t}$  dependent coefficients and  $\mathcal{G}$ ,

$$\varphi_3 = (f_1(\hat{t}) + f_2(\hat{t}) x^2 + f_3(\hat{t}) x^4) \mathcal{G}(x, \hat{t}).$$

On differentiating and substituting into the PDE, we find that

$$\frac{\partial \varphi_3}{\partial \hat{t}} - \frac{\partial^2 \varphi_3}{\partial x^2} = \left( \left( \frac{df_3}{d\hat{t}} + \frac{4f_3(\hat{t})}{\hat{t}} \right) x^4 + \left( \frac{df_2}{d\hat{t}} + \frac{2f_2(\hat{t})}{\hat{t}} - 12f_3(\hat{t}) \right) x^2 + \left( \frac{df_1}{d\hat{t}} - 2f_2(\hat{t}) \right) \right) \mathcal{G}(x, \hat{t}(\tau)).$$

Matching powers of  $x$  gives us three first-order ODEs for  $f_1(\hat{t})$ ,  $f_2(\hat{t})$  and  $f_3(\hat{t})$ .

As  $\hat{t}(\tau)$  is defined by

$$\frac{d\hat{t}(\tau)}{d\tau} = \eta^*(\tau)^2 e^{-2\theta\tau}, \quad \hat{t}(0) = 0,$$

we can revert to  $\tau$  as our independent time variable, introduce suitable integrating factors, and show that these three ODEs may be written as

$$\begin{aligned} \frac{d(\hat{t}(\tau)^4 f_3(\tau))}{d\tau} &= \frac{1}{2} \hat{t}(\tau) h(\tau) \eta^*(\tau) e^{-\theta\tau}, \\ \frac{d(\hat{t}(\tau)^2 f_2(\tau))}{d\tau} &= (1 + 12 f_3(\tau) \eta^*(\tau)^2 e^{-2\theta\tau}) \hat{t}(\tau)^2 - 3h(\tau) \eta^*(\tau) e^{-\theta\tau}, \\ \frac{df_1(\tau)}{d\tau} &= 2 f_2(\tau) \eta^*(\tau)^2 e^{-2\theta\tau}, \end{aligned}$$

subject to minimum singularity conditions at  $\tau = 0$ . These may be integrated directly, in sequence.

First, for  $f_3(\tau)$ , we obtain

$$f_3(\tau) = \frac{F(\tau)}{\hat{t}(\tau)^4},$$

where

$$\begin{aligned} F(\tau) &= \frac{1}{2} \int_0^\tau h(\tau') \hat{t}(\tau') \eta^*(\tau') e^{-\theta\tau'} d\tau' \\ &= \frac{1}{4} \int_0^\tau h(\tau') \frac{dh(\tau')}{d\tau'} d\tau' \\ &= \frac{1}{8} h(\tau)^2. \end{aligned}$$

Expanding in powers of  $\theta = (r - q)T$  we find that

$$\begin{aligned} f_3(\tau) &= \frac{h(\tau)^2}{8\hat{t}(\tau)^4} \\ &= \frac{9}{50}\tau^{-2} + \frac{33}{200}\frac{\theta}{\tau} + \frac{6731}{112000}\theta^2 + \frac{209}{22400}\tau\theta^3 - \frac{57}{6272000}\tau^2\theta^4 - \frac{2851}{18816000}\tau^3\theta^5 \\ &+ \mathcal{O}(\tau^4\theta^6) \end{aligned}$$

which is consistent with the formula found earlier for  $\theta = 0$ .

For  $f_2(\tau)$  and  $f_1(\tau)$ , direct integration leads to very complicated closed-form expressions with the property that both the numerator and denominator behave as high powers of  $\theta$  as  $\theta \rightarrow 0$ . It is simpler and better to simply state, and use, the resulting series expansions for these functions, which are

$$\begin{aligned} f_2(\tau) &= -\frac{11}{175}\tau - \frac{87}{5600}\tau^2\theta - \frac{151}{504000}\tau^3\theta^2 + \frac{71}{192000}\tau^4\theta^3 - \frac{10849}{1552320000}\tau^5\theta^4 - \frac{12311}{993484800}\tau^6\theta^5 \\ &+ \mathcal{O}(\tau^7\theta^6) \end{aligned}$$

and

$$\begin{aligned} f_1(\tau) &= -\frac{11}{350}\tau^4 + \frac{53}{2800}\tau^5\theta - \frac{1543}{216000}\tau^6\theta^2 + \frac{4217}{2016000}\tau^7\theta^3 - \frac{4843981}{9313920000}\tau^8\theta^4 + \frac{4295503}{37255680000}\tau^9\theta^5 \\ &+ \mathcal{O}(\tau^{10}\theta^6). \end{aligned}$$

These expressions are consistent with the zero drift case (26).

Note again that as  $\zeta \rightarrow \pm\infty$ ,  $\varphi_3$  vanishes exponentially.

#### 4.4 Summary of the third-order solution for $r \neq q$

It is convenient to gather together the following summary of the various functions for the inner solution when  $r \neq q$ . We first remind the reader of the basic transformations,

$$\begin{aligned} \theta &= (r - q)T, \quad \tau = \frac{T - t}{T}, \quad \epsilon^2 = \frac{1}{2}\sigma^2 T, \\ \eta &= \frac{I - K T}{S T}, \quad V = S e^{-q(T-t)} \psi, \end{aligned}$$

the further changes of variable

$$\begin{aligned} \eta^*(\tau) &= \frac{1}{\theta}(1 - e^{\theta\tau}), \quad \eta = \eta^*(\tau) + \epsilon\zeta = \eta^*(\tau) + \epsilon e^{\theta\tau} x, \\ x &= e^{-\theta\tau} \zeta = \frac{1}{\epsilon}(\eta - \eta^*(\tau)) e^{-\theta\tau}, \quad \hat{t}(\tau) = \frac{1}{2\theta^3}(2\theta\tau - 3 + 4e^{-\theta\tau} - e^{-2\theta\tau}), \end{aligned}$$

the auxiliary functions defined by

$$\begin{aligned} \mathcal{G}(x, \hat{t}) &= \frac{1}{\sqrt{4\pi\hat{t}}} \exp\left(-\frac{x^2}{4\hat{t}}\right), \\ h(\tau) &= \frac{1}{6\theta^5} e^{-3\theta\tau} \left(2 - 15e^{\theta\tau} - 6(2\theta\tau - 5)e^{2\theta\tau} + (-6\theta\tau(\theta\tau - 3) - 17)e^{3\theta\tau}\right), \end{aligned}$$

and the asymptotic expansion

$$\psi = \epsilon\varphi_1 + \epsilon^2\varphi_2 + \epsilon^3\varphi_3 + \dots$$

The first, second and third-order terms in this asymptotic expansion are given by

$$\begin{aligned}\varphi_1 &= x N\left(\frac{x}{\sqrt{2\hat{t}(\tau)}}\right) + \sqrt{\frac{\hat{t}(\tau)}{\pi}} \exp\left(-\frac{x^2}{4\hat{t}(\tau)}\right) \\ \varphi_2 &= \frac{h(\tau)}{\hat{t}(\tau)} x \mathcal{G}(x, \hat{t}(\tau)) \\ \varphi_3 &= (f_1(\tau) + f_2(\tau)x^2 + f_3(\tau)x^4) \mathcal{G}(x, \hat{t}(\tau)),\end{aligned}$$

where the functions  $f_3$ ,  $f_2$  and  $f_1$  are given by the series expansions

$$\begin{aligned}f_3(\tau) &= \frac{h(\tau)^2}{8\hat{t}(\tau)^4} \\ &= \frac{9}{50}\tau^2 + \frac{33}{200}\theta\tau + \frac{6731}{112000}\theta^2 + \frac{209}{22400}\tau\theta^3 - \frac{57}{6272000}\tau^2\theta^4 - \frac{2851}{18816000}\tau^3\theta^5 \\ &\quad + \mathcal{O}(\tau^4\theta^6) \\ f_2(\tau) &= -\frac{11}{175}\tau - \frac{87}{5600}\tau^2\theta - \frac{151}{504000}\tau^3\theta^2 + \frac{71}{192000}\tau^4\theta^3 - \frac{10849}{1552320000}\tau^5\theta^4 - \frac{12311}{993484800}\tau^6\theta^5 \\ &\quad + \mathcal{O}(\tau^7\theta^6) \\ f_1(\tau) &= -\frac{11}{350}\tau^4 + \frac{53}{2800}\tau^5\theta - \frac{1543}{216000}\tau^6\theta^2 + \frac{4217}{2016000}\tau^7\theta^3 - \frac{4843981}{9313920000}\tau^8\theta^4 + \frac{4295503}{37255680000}\tau^9\theta^5 \\ &\quad + \mathcal{O}(\tau^{10}\theta^6)\end{aligned}$$

Hence, we may determine the contract value to third order in  $\epsilon = \sigma\sqrt{T/2}$ . It is perhaps helpful to take a step back from the details and observe what has been achieved. We have been able to determine the contract value as a series in the volatility using a combination of some changes of variables, one reference to the Normal distribution, some polynomials and some exponential functions. It is now trivial to code this up in any computational environment, and work out the results. This is in effect an analogue of the Black-Scholes formula, at least in terms of its complexity, for an Asian rate option. We shall shortly make a comparison of the results with the known exact solution expressed as a contour integral. Before doing so, we shall introduce another low volatility approximation based on the SDE. This will help us evaluate the quality of the asymptotic series developed above when the contour integral is difficult to evaluate, i.e. when  $\epsilon$  is *very* small.

## 5 Asymptotic solution of the SDE

As a further check on our methodology, we now develop an alternative perturbation solution method based directly on the corresponding (risk-neutral) SDE, (4). For continuous sampling over the life of the option this becomes

$$d\eta_t = \left(\frac{1}{T} - (r - q)\eta_t\right) dt + \sigma \eta_t dW_t,$$

where  $W_t$  is a standard Brownian motion. We also suppose that at  $t = 0$  we are given  $\eta_0$ . To treat this in the low volatility limit we let

$$\eta_t = \eta_t^0 + \sigma \eta_t^1 \tag{31}$$

The SDE for the first term,  $\eta_t^0$ , is the deterministic ODE

$$d\eta_t^0 = \left(\frac{1}{T} - (r - q)\eta_t^0\right) dt$$

with the elementary solution

$$\eta_t^0 = \begin{cases} \eta_0 e^{-(r-q)t} + \frac{1 - e^{-(r-q)t}}{(r-q)T} & \text{if } r \neq q, \\ \eta_0 + \frac{t}{T} & \text{if } r = q. \end{cases}$$

The *exact* SDE for the second term is the SDE

$$d\eta_t^1 = -(r-q)\eta_t^1 dt + (\eta_t^0 + \sigma \eta_t^1) dW_t$$

subject to the initial condition  $\eta_0^1 = 0$ . This SDE can be simplified in the usual way by setting

$$\tilde{\eta}_t^1 = e^{(r-q)t} \eta_t^1$$

and substituting for the known form of  $\eta_t^0$ . For  $r \neq q$ , we obtain<sup>9</sup>

$$d\tilde{\eta}_t^1 = \left( \eta_0^0 + \frac{e^{(r-q)t} - 1}{(r-q)T} + \sigma \tilde{\eta}_t^1 \right) dW_t$$

which has the formal solution

$$\tilde{\eta}_t^1 = A_t + \sigma B_t + \mathcal{O}(\sigma^2),$$

where

$$A_t = \int_0^t dW_s \left( \eta_0^0 + \frac{e^{(r-q)s} - 1}{(r-q)T} \right),$$

$$B_t = \int_0^t dW_s \int_0^s dW_u \left( \eta_0^0 + \frac{e^{(r-q)u} - 1}{(r-q)T} \right).$$

It is clear that  $A_t^1$  is normally distributed with zero mean and variance  $\Sigma^2$ , given by

$$\Sigma^2 = \int_0^t ds \left( \eta_0^0 + \frac{e^{(r-q)s} - 1}{(r-q)T} \right)^2.$$

If we recall that

$$\eta_t = \eta_t^0 + \sigma (A_t + \sigma B_t + \mathcal{O}(\sigma^2)) e^{-(r-q)t},$$

it follows that if we neglect terms quadratic and higher order in  $\sigma$ ,  $\eta_t$  is normally distributed with mean  $\eta_t^0$  and variance  $e^{-2(r-q)t} \sigma^2 \Sigma^2$ . We can now write down the value of  $\psi$  based on the Feynman-Kac formula, to this order in  $\sigma$ . Elementary calculations show that at initiation (at time zero) of an option maturing at time  $T$ , the value is given by

$$\psi_0 = \eta_T^0 N_c \left( -\frac{\eta_T^0 e^{(r-q)T}}{\sigma \Sigma} \right) - e^{-(r-q)T} \sigma \Sigma N'_c \left( \frac{\eta_T^0 e^{(r-q)T}}{\sigma \Sigma} \right)$$

where  $N_c$  denotes the *complementary* cumulative density for a standard normal distribution. Note that in this formula the quantity  $\eta_T^0$  denotes the solution of a deterministic ODE evaluated at maturity<sup>10</sup>.

## 6 Comparison of results

In this section we define an extension of part the useful family of test problems introduced by Fu, Madan and Wang (abbreviated henceforth to FMW)[3]. First we consider the seven cases introduced by FMW. These all have zero dividend yield and positive  $r$ , so fall into the first category we analyzed. The other parameters are all given as follows:

<sup>9</sup>The results for  $r = q$  follow by taking the limit  $r \rightarrow q$  in the obvious manner.

<sup>10</sup>It is not therefore a stochastic quantity.

FMW No.	$S$	$K$	$r$	$\sigma$	$T$
1	1.9	2	0.05	0.5	1
2	2.0	2	0.05	0.5	1
3	2.1	2	0.05	0.5	1
4	2.0	2	0.02	0.1	1
5	2.0	2	0.18	0.3	1
6	2.0	2	0.0125	0.25	2
7	2.0	2	0.05	0.5	2

The results are as follows for these six cases. The results cited from Vecer are those in his RISK article [6] to 6 decimal places, noted as already agreeing with the results of [2, 5]. The Linetsky data is high precision data supplied by V. Linetsky and based on his work in [5]. The FMW-MC100 data is Monte Carlo data for their “MC100” analysis from the paper by FMW [3]. The GYShaw results are those from the GY model run with the Shaw’s code in [2] with a longer contour. Note that the cases are numbered differently by FMW and Linetsky, hence the first two columns “F” and “L” label their case numbers for ease of comparison. The results labelled “MAE2” and “MAE3” respectively are the results from the asymptotics as far as  $\varphi_2$  and  $\varphi_3$  respectively.

F	L	GYShaw	LVSDE	MAE2	MAE3	Linetsky	Vecer	FMW
1	4	0.1931737903	0.1996	0.194472	0.193188	0.1931737903	0.193174	0.196
2	5	0.2464156905	0.2428	0.247732	0.246382	0.2464156905	0.246416	0.249
3	6	0.3062203648	0.2935	0.307604	0.306139	0.3062203648	0.306220	0.309
4	1	0.0559860415	0.0556	0.055997	0.055986	0.0559860415	0.055986	0.0565
5	2	0.2183875466	0.2104	0.218663	0.218369	0.2183875466	0.218388	0.220
6	3	0.1722687410	0.1710	0.172741	0.172263	0.1722687410	0.172269	0.172
7	7	0.3500952190	0.3404	0.353704	0.349909	0.3500952190	0.350095	0.348

We note that three of the approaches based on very different non-asymptotic methods agree to many significant figures, so there is now little doubt as to the correct answer to these problems. The two methods for which more details are given agree to 10 decimal places. The low-volatility SDE column gives an idea of the accuracy of the method in cases where one would not necessarily expect it to be very accurate, but it does moderately well. The important observation, however, is the accuracy of the third-order asymptotic results, which typically give four significant figure accuracy.

### 6.1 Low Volatility Test Problems

In order to probe the difficult case of low volatility, we now introduce some variants of FMW Case 4 as follows. The “Low-vol S” column corresponds to the stochastic approximation introduced in Section 4. The GYShaw method is as before, with GYS-Mellin denoting the use of an asymptotic expansion to treat the low volatility case via a Mellin transform method, as discussed in [24]. Note that the MAE3 results appear now to be essentially exact as far as can be tested.

Case	$S$	$K$	$r$	$\sigma$	$T$
4	2.0	2	0.02	0.1	1
4A	2.0	2	0.02	0.05	1
4B	2.0	2	0.02	0.01	1
4C	2.0	2	0.02	0.005	1
4D	2.0	2	0.02	0.001	1

The results are as follows

Case	GYShaw	GYS–Mellin	Low–Vol SDE	MAE2	MAE3
4	0.0559860	0.0559853	0.0555956	0.0559970	0.0559860
4A	0.0339412	0.0339485	0.0337509	0.0339426	0.0339412
4B	NA	0.0199383	0.0199188	0.0199277	0.0199278
4C	NA	unstable	0.0197357	0.0197357	0.0197357
4D	NA	0.0197353	0.0197353	0.0197353	0.0197353

## 6.2 Test problems for $q > r$

We now define analogues of the FMW test problems for the case where  $q > r$ . The simplest approach is to take the original set and but also to set  $q = 2r$ , so that we exactly reverse the risk-neutral drift. The parameters are, using an asterisk to denote the reflected drift in the case numbers:

FMW No.	$S$	$K$	$r$	$q$	$\sigma$	$T$
1*	1.9	2	0.05	0.1	0.5	1
2*	2.0	2	0.05	0.1	0.5	1
3*	2.1	2	0.05	0.1	0.5	1
4*	2.0	2	0.02	0.04	0.1	1
5*	2.0	2	0.18	0.36	0.3	1
6*	2.0	2	0.0125	0.025	0.25	2
7*	2.0	2	0.05	0.1	0.5	2

The results are as follows. In this case given that there is not a published set of high-precision values, we confine attention to six significant figures.

FMW No.	GYShaw	MAE2	MAE3
1*	0.147562	0.148852	0.147618
2*	0.191747	0.193010	0.191760
3*	0.242316	0.243595	0.242283
4*	0.0357853	0.0357961	0.0357854
5*	0.0522607	0.0524892	0.0522755
6*	0.145308	0.145771	0.145310
7*	0.240495	0.243819	0.240564

### 6.2.1 Low volatility and $q > r$ tests

To see what happens for low volatilities, we consider the test problems used previously with the volatility tending to zero and also set  $q = 2r$ . The parameters are therefore as follows:

Case	$S$	$K$	$r$	$q$	$\sigma$	$T$
4*	2.0	2	0.02	0.04	0.1	1
4A*	2.0	2	0.02	0.04	0.05	1
4B*	2.0	2	0.02	0.04	0.01	1
4C*	2.0	2	0.02	0.04	0.005	1
4D*	2.0	2	0.02	0.04	0.001	1

The corresponding results are as follows

Case	GYS-full	Low-Vol SDE	MAE2	MAE3
4*	0.0357853	0.0361954	0.0357961	0.0357854
4A*	0.0140247	0.0142171	0.0140261	0.0140248
4B*	NA	0.000199435	0.000190211	0.000190254
4C*	NA	$4.33117 \times 10^{-7}$	$3.78827 \times 10^{-7}$	$3.7991 \times 10^{-7}$
4D*	NA	$\mathcal{O}(10^{-68})$	$\mathcal{O}(10^{-70})$	$\mathcal{O}(10^{-70})$

### 6.3 Test problems for $q = r$

This case is a particularly stringent one as we have up to five orders of perturbation theory at our disposal and we can compare with a detailed contour integral solution expressed in terms of Bessel functions. We now define analogues of the FMW test problems for the case where  $r = q$ . The simplest thing to do is to take the original set and but also to set  $r = q$ , so that we zero the risk-neutral drift of  $S$ . The parameters are, using a double asterisk to denote the zeroed drift in the case numbers,

FMW No.	$S$	$K$	$r$	$q$	$\sigma$	$T$
1**	1.9	2	0.05	0.05	0.5	1
2**	2.0	2	0.05	0.05	0.5	1
3**	2.1	2	0.05	0.05	0.5	1
4**	2.0	2	0.02	0.02	0.1	1
5**	2.0	2	0.18	0.18	0.3	1
6**	2.0	2	0.0125	0.0125	0.25	2
7**	2.0	2	0.05	0.05	0.5	2

The results are as follows. We confine attention to six significant figures. These exact results are labelled “CIBess” as the use the contour integral of the Bessel function form of the hyper-geometric function that is given in the Appendix in (32).

FMW No.	CIBess	MAE2	MAE3	MAE4	MAE5
1**	0.169202	0.170494	0.169238	0.169192	0.169201
2**	0.217815	0.219096	0.217805	0.217805	0.217815
3**	0.272924	0.274251	0.272869	0.272914	0.272925
4**	0.0451431	0.0451537	0.0451431	0.0451431	0.0451431
5**	0.115188	0.115432	0.115188	0.115188	0.115188
6**	0.158380	0.158846	0.158378	0.158378	0.158380
7**	0.291315	0.294737	0.291264	0.291264	0.291316

It should be noted that the MAE3 results are all correct to at least 4 significant figures or better, and the MAE5 data is exact for all practical purposes.

#### 6.3.1 Low volatility and $q = r$ tests

To see what happens for low volatilities, we consider the test problems used previously with the volatility tending to zero and also set  $q = r$ . The parameters are therefore as follows:

Case	$S$	$K$	$r$	$q$	$\sigma$	$T$
4**	2.0	2	0.02	0.02	0.1	1
4A**	2.0	2	0.02	0.02	0.05	1
4B**	2.0	2	0.02	0.02	0.01	1
4C**	2.0	2	0.02	0.02	0.005	1
4D**	2.0	2	0.02	0.02	0.001	1

The corresponding results are as follows

Case	CIBess	Low-Vol SDE	MAE2	MAE3
4**	0.0451431	0.0451537	0.0451537	0.0451431
4A**	0.0225755	0.0225769	0.0225769	0.0225755
4B**	NA	0.00451537	0.00451537	0.00451536
4C**	NA	0.00225769	0.00225769	0.00225768
4D**	NA	0.000451537	0.000451537	0.000451537

We note that in these cases given that  $\zeta = 0$  for all cases where  $S = K$  and  $r = q$ , the MAE4 results are the same as MAE3. Where we know the exact result we see that the MAE3 values are good to at least 6 significant figures. Going to MAE5 makes no difference to the results with the given precision.

## 7 Conclusion

We have developed the partial differential equation approach to Asian options in such a way as to allow both an easy derivation of the exact solution for the Rate Call as a Laplace transform and as a power series in

$$\epsilon = \sigma \sqrt{\frac{T}{2}}$$

When carried out to third-order the expansion approach agrees with the Laplace transform method to typically four significant figures for real-world volatilities and a range of values of  $(r - q)T$ . The expansion method is essentially exact as  $\epsilon \rightarrow 0$ . When  $r = q$  we have a very accurate fifth-order expansion which gives excellent agreement with the contour integral solution optimized for this case. The expansion approach gives a formula that is little more complicated than the original Black-Scholes model for European options and may be readily implemented in any computational environment. An on-line Mathematica notebook demonstrating some of these calculations is available [18]. These models may be used stand-alone or for checking numerical implementations of e.g. the Vecer model. Our model is essentially an accurate analytical solution of Vecer's PDE.

The methodology developed here may in principle be applied to many other financial problems of interest. Problems that immediately come to mind are the extension of the present work to include:

- the rate-call with discrete sampling;
- the inclusion of discrete dividends;
- the strike-call.

Within the PDE framework it is very clear how this methodology can be iterated through higher powers of  $\epsilon$ . Indeed we have demonstrated this very explicitly to high order in the case of zero risk-neutral drift:  $r = q$ . It would be interesting to consider how precisely this would take place within the SDE framework. While the SDE can be formally iterated in powers of  $\sigma$ , the question arises as to how one could get explicit formulae for expectations, within such an expansion?

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## Appendix A - reduction of the hyper-geometric contour integral to Bessel form for special values of $r - q$ .

The Laplace transform solution of (11) can be represented in a number of equivalent ways. It is in fact useful to recast the transform back as an integral, though not in the form given by GY. Three representations are interesting. These representations will not be justified here - they can be deduced from various identities and results in Gradshteyn and Ryzik [25] and Abramowitz and Stegun [17]. We can write, using some well known integral identities:

$$\hat{C}(\alpha, p) = \frac{\Gamma\left(\frac{1}{2}(\mu + \nu + 4)\right)}{\Gamma\left(\frac{1}{2}(\mu - \nu - 2)\right)} \frac{2^{\nu+3}}{p(p - 2\nu - 2)} \int_0^\infty \frac{1}{x^{\nu+3}} J_\mu(x) e^{-\alpha x^2/2} dx.$$

Applying a Kummer transformation and another identity leads to the simpler form

$$\hat{C}(\alpha, p) = \frac{\alpha^{\nu+3} e^{-\frac{1}{2\alpha}}}{p(p - 2\nu - 2)} \int_0^\infty x^{\nu+3} I_\mu(x) e^{-\alpha x^2/2} dx.$$

Finally, we note our third representation, in terms of a Mellin transform:

$$\hat{C}(\alpha, p) = \frac{1}{2\pi i} \frac{1}{p(p - 2\nu - 2)} \frac{\Gamma(b)}{\Gamma(a)} \int_{-i\infty}^{i\infty} (2\alpha)^{-z} \frac{\Gamma(z)\Gamma(a-z)}{\Gamma(b+z)} dz,$$

where  $a = \mu/2 - \nu/2 - 1$ ,  $b = \mu/2 + \nu/2 + 2$ . Any of these representations make it clear that there is a collapse in complexity when  $\nu = -3$ ! This might seem like an odd candidate for a special case - it does in fact correspond to zero drift in the variable  $u = 1/\eta$ . In this case, we can carry out the integration in both the first two representations to obtain a simpler representation:

$$\hat{C}(\alpha, p) = \frac{e^{-\frac{1}{4\alpha}}}{p(p+4)} \sqrt{\frac{\pi}{2\alpha}} I_{\frac{\mu}{2}}\left(\frac{1}{4\alpha}\right), \quad \mu = \sqrt{2p+9}.$$

There are also simplifications when  $\nu = -2 - 1, 0, 1, 2, 3, 4, \dots$ , when the hyper-geometric function can be reduced to a finite sum of Bessel functions. Only the first few of these will be noted here. Note that  $\nu = -1$  is particularly important as it corresponds to the situation  $r = q$ , and fills in the case where  $\epsilon = \infty$  where the asymptotic analysis cannot be used. All the noted integer cases for  $\nu$  can be derived from that above for  $\nu = -3$  and the following that is valid for  $\nu = -2$ :

$$\hat{C}(\alpha, p) = \frac{e^{-\frac{1}{4\alpha}}}{p(p+2)} \sqrt{\frac{\pi}{2\alpha}} \frac{1}{2} \left( I_{\frac{\mu}{2}-\frac{1}{2}}\left(\frac{1}{4\alpha}\right) + I_{\frac{\mu}{2}+\frac{1}{2}}\left(\frac{1}{4\alpha}\right) \right), \quad \mu = \sqrt{2p+4}.$$

In general, for the sequence  $\nu = 2k - 3, k = 0, 1, 2, \dots$

$$\hat{C}(\alpha, p) = 2^k \alpha^{2k} \frac{e^{-\frac{1}{2\alpha}}}{p(p+4-4k)} \left(-\frac{d}{d\alpha}\right)^k \left( e^{\frac{1}{4\alpha}} \sqrt{\frac{\pi}{2\alpha}} I_{\frac{\mu}{2}}\left(\frac{1}{4\alpha}\right) \right).$$

Applying this result with  $k = 1$  we obtain an important result for  $\nu = -1$ ,  $r = q$ , which after the use of some Bessel identities reduces to:

$$\hat{C}(\alpha, p) = \frac{e^{-\frac{1}{4\alpha}}}{p^2} \sqrt{\frac{\pi}{8\alpha}} \left( (1 + 2\alpha(\mu + 1)) I_{\frac{\mu}{2}}\left(\frac{1}{4\alpha}\right) + I_{\frac{\mu}{2}+1}\left(\frac{1}{4\alpha}\right) \right), \quad \mu = \sqrt{2p+1}. \tag{32}$$

For the even sequence  $\nu = 2k - 2, k = 0, 1, 2, \dots$  we obtain the analogous result:

$$\hat{C}(\alpha, p) = 2^k \alpha^{2k+1} \frac{e^{-\frac{1}{2\alpha}}}{p(p+2-4k)} \left(-\frac{d}{d\alpha}\right)^k \left\{ e^{\frac{1}{4\alpha}} \sqrt{\frac{\pi}{2\alpha}} \frac{1}{2\alpha} \left( I_{\frac{\mu-1}{2}}\left(\frac{1}{4\alpha}\right) + I_{\frac{\mu+1}{2}}\left(\frac{1}{4\alpha}\right) \right) \right\}.$$

What is interesting about these expressions is that they reveal the essential problem of inversion - we need to understand the Laplace inverse of a Bessel function whose *index* is the (square root of) the Laplace transform variable. These expressions allow a formal analytical inverse to be extracted for such special cases in terms of Kontorovich-Lebedev transforms, as discussed in [26]. We will not discuss the analytics of this here as our use of these Bessel representations is merely to facilitate the numerical computation of the contour integral when  $r = q$ . This is carried out using numerical inversion of the representation (32). Such integrations proceed up to an order of magnitude faster than direct use of the original hypergeometric function, when carried out in Mathematica.

## Appendix B - comments on deforming the contour

The acceleration of the contour integration for inversion may also be considered from the point of view of *contour deformation*. It is commonplace to assume that the Bromwich contour may be deformed into the left half plane, for the purposes of either acceleration of numerical inversion (e.g. by some variation of the Talbot method) or for analytical consideration and, hopefully, extraction of an analytical form for the inverse. In the initial presentation of the inversion given in [2], this was not presented as some numerical experiments had revealed this to be highly pathological as  $\sigma \rightarrow 0$ . The pathology of analytical representations based on a deformation and separation out of the residue contribution was noted by one of us [26] for the special case  $\nu = -3$ , where a representation of the inverse as a Kontorovich-Lebedev transform was given and the zero volatility limit pathology noted. The matter has also been revisited by Lewis [8] who argued that the general case should also be considered by pulling out the residues and doing the remaining integral along the branch cut.

We now understand that there are two reasons why such a deformation approach is, in general, not appropriate. While it might offer some acceleration for higher volatilities, there is one practical objection and one objection of principle. The practical issue is that for out of the money options the branch cut integral must come close to cancelling the contribution of the residues, so it is sub-optimal to separate it out and work it out separately. The more serious issue of principle can be seen by taking the Fourier transform of the zero volatility solution. This was first discussed in a Mathematica notebook [24] given limited circulation as a working paper where the low volatility case was analyzed using Mellin transforms. The fact the completion in the left half plane fails in explained in Section 7 of [24], but given its direct relationship to the characteristic solution used here and the fact that some authors may be tempted to pursue this route we consider it useful to explain why this will never work for small volatilities.

The explanation of this problem can be done by either

1. taking the limit as  $\sigma \rightarrow 0$  of the hyper-geometric Laplace transform and looking at its inversion;
2. starting from the zero-volatility solution given here and taking its Laplace transform.

It is an elementary exercise left to the reader to take route 2. We shall quote the result for the limiting form of the transform as in 1, the details of which are on-line in the 2002 working paper [24], and focus on the valid locations of the inversion contour, which is the main point at issue. Note that the discussion is for the case  $r > q$ , so that  $\nu > -1$ . In [24] it was shown, by two different methods, including the Mellin representation, that the asymptotic form of the Laplace transform as  $\sigma \rightarrow 0$  is

$$\hat{c}(\alpha, p) = \frac{(1 + 2\alpha(\nu + 1))^{1 - \frac{p}{2\nu + 2}}}{p(p - 2(\nu + 1))} = \frac{(1 + 2\alpha(\nu + 1))}{p(p - 2(\nu + 1))} e^{-p\tau_c},$$

where

$$\tau_c = \frac{\log(1 + 2\alpha(\nu + 1))}{2\nu + 2}.$$

This has an inverse which is computable exactly by direct use of tables or using the Bromwich integral inversion formula. It is important to note that the contour of completion that we introduce in order to apply the calculus of residues must be in the *right* half-plane if  $\tau < \tau_c$ , when the answer zero is obtained. This also explains why numerical inversion by uncritical application of the method of Talbot-Rizzardi can give unreliable answers, as that method typically involves a contour deformation to the left half-plane. Similarly, *any approach replying on deformation to the left half plane will fail in general*. This only works for  $\tau > \tau_c$  and then the inversion gives a non-zero result. Putting the two results together gives the inverse as

$$\frac{1}{2\nu + 2} \max \left( e^{(2\nu + 2)\tau} - 1 - (2\nu + 2)\alpha, 0 \right).$$

This is in indeed the correct zero volatility solution and can be written out in terms of the original financial variables as

$$e^{-r(T-t)} \max \left( \frac{I_0 + \frac{S(e^{(r-q)(T-t)} - 1)}{r-q}}{T - t_0} - K, 0 \right).$$

This is why the contour integral representations used in [2, 4, 24, 18] stick to the original, vertical, Bromwich contour. Although the calculations may take an exceedingly long time when  $\sigma$  is small, or even fail to complete, we can be sure that answers that do arise from this procedure will be correct.