Share Price Movements in the Post-Credit-Crunch environment

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December 2, 2008

Abstract

The market events of 2007-2008 have reinvigorated the search for realistic share price models that capture greater likelihoods of extreme movements. In this paper we model the medium-term log-return dynamics in a market containing both fundamental and technical traders. This is done in a simple way based on a Poisson trade arrival model with variable size orders. With simplifying assumptions we are led to a novel SDE mixing arithmetical and geometric Brownian motions. Various dynamics and equilibria are possible depending on the balance of trades. Under mean-reverting circumstances we arrive naturally at an equilibrium fat-tailed return distribution with a Pearson Type IV form. Under less restrictive assumptions still richer dynamics are possible. One special case leads to a natural hyperbolic variation of the OU SDE. The phenomenon of variance explosion is identified that gives rise to much larger price movements that might have a priori been expected, so that "25σ" events can become more commonplace. We exhibit a solution of the Fokker-Planck equation for a special case that shows how such variance explosion can hide beneath a standard Gaussian facade. This is one member of an extended class of "inverse-hyperbolic-normal" distributions with a rich and varied structure, capable of describing a wide range of market behaviours.

Keywords: Market microstructure, fundamental trader, technical trader, Student distribution, t-distribution, Skew-Student, Pearson Type IV, Fokker-Planck equation, stochastic differential equation, partial differential equation, credit crunch, variance explosion.
Working Paper: Version 1.2 (December 2008)

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1 Introduction

'Technical analysis is an anathema to the academic world.'

Burton Malkiel, in "A Random Walk Down Wall Street".

The fat-tailed and non-normal distribution of share-price returns has been well known for decades. Mandelbrot [12] and Fama [8] noted the excess kurtotic nature of equity returns in the 1960s. The events of 2007-2008 have made it very clear that extreme share-price movements are much more likely than in a simple log-normal model associated with geometric Brownian motion. Despite the fat-tailed behaviour being known for 40 years many market participants cling to the Gaussian or near Gaussian picture, and attempt to explain away the large movements while remaining within that picture. The CFO of Goldmans, in a *Financial Times* article, famously attempted to excuse the implosion of Goldman's hedge funds with the comment[1]

"We were seeing things that were 25-standard deviation moves, several days in a row".

These were not events that can be written off as un-modellable due to their extreme unlikelihood - people are just thinking of the wrong distribution based on an unrealistic world-view, while simultaneously discounting the possibility of both major shifts in the mean and sudden increases in variance. For example, the likelihood of a single 25σ or worse event, without allowing for a major shift in the distributional mean or variance, is

- about 6×10^{-138} in a Gaussian picture;
- about 4×10^{-6} in a Student-t picture (four degrees of freedom, as estimated for global indices in [9]);
- about 1/625 for a toss of a very unfair coin where the probability of a head is 1/625.[2]

The huge losses in the markets in the fallout of the credit crunch were due to several factors, which include not only the very real loss in value of financial institutions who might, for example have substantial CDS exposure or exposure to genuine economic downturn, but also to the fear of traders panicking to offload falling assets. These mass share dumps were punctuated by occasional bottom-fishing exercises.

But long before the recent panic, participants in financial markets traded according to diverse strategies. Two common approaches are the so-called "fundamental" and "technical" trading strategies. Traders in the first group are largely interested in measure of value and traders in the second group are interested more in the dynamics of the price (and volume) history. In this

[1]David Viniar, Goldman's chief financial officer, as quoted by Peter Larsen, Financial Times August 13 2007.

[2]This example is based, with thanks, on an anonymous observation at `http://worldbeta.blogspot.com/2007/08/really-with-seth-and-amy-part-ii.html`

note the impact of having these two types of trades is analyzed employing a simple trade arrival model with Poisson characteristics, with variable size orders. In general we obtain a large class of price evolution models.

The approach presented here has many conceptual links with other approaches. In particular, Brody, Hughston and Macrina ('BHM') [5] have introduced the notion of *information-based asset pricing*, paying rather more attention to the filtration component of processes than has hitherto been developed. In the model presented here we package information received since the start of a trading period into the price history since the trading start, on the basis that market participants who believe in technical trading are at work.[3] Our linearized model results in an SDE that is, at least at the conceptual level, a linear price-information correction to standard Brownian motion of log-returns. It remains to be seen whether this notion can be given a rigorous mathematical basis within the BHM framework.

We consider the dynamics on a medium time scale which may be considered as assessing the distribution of returns on a daily basis as the aggregate of many intra-day trades. Under certain simplifying assumptions, corresponding to linearization of price impact functions, linearization of technical trading criteria, and a "many trade" limit, we are lead to Various dynamics and equilibria arise from this SDE depending on the balance of trades. Under mean-reverting circumstances we are lead naturally to an equilibrium fat-tailed return distribution with a Student t or skew-Student form, with the latter defined within the framework of "Pearson diffusions" defined by Forman and Sørensen [10]. The construction of the standard Student distribution from an SDE has also been recently considered by Steinbrecher and Shaw [20]. The model is capable of still richer dynamics, and in more generality leads to a hyperbolic extension of the OU process. In general form a non-linear diffusion with or without jumps is possible. One effect of note is the simultaneous explosion in the variance coupled to the emergence of a non-Gaussian distribution, and such distributions hiding their character under a Gaussian mask.

1.1 Analogies from the physical world

We will work with an SDE with a combination of arithmetical and geometric Brownian motions. This SDE has many interesting features, and has an interesting history in the physics literature[4]. A very closely related analysis has been given quite recently in the plasma physics literature [19]. However, the history of hybrids of both arithmetical and geometric Brownian motions dates back to the 1979 paper by Schenzle and Brand [16]. The work in [16] makes progress on the eigenfunctions of the Fokker-Planck equation that may allow further development of the solutions provided here in Sections 7-9.

[3]This of course makes no judgement as to the wisdom of technical trading. In fact all we really need is the presence of price-sensitive orders.

[4]I am grateful for Dr. G. Steinbrecher for these insights.

Also, discrete models in the form of a linear Langevin equation:

$$x(t+1) = b(t)x(t) + f(t) \ , \tag{1}$$

where b and f are both random, were considered in a thread of research originating in 1997 with Takayasu *et al*, who established conditions for the realization of power law behaviour [21].

1.2 Plan of article

The plan of this work is as follows. In Section two the underlying dynamics of a market containing both fundamental and technical traders is considered. In Section three the model is simplified and linearized, and approximated to a tractable model. In general a model with a combination of jumps and pure Brownian motion is possible, but we focus on the non-jump case for further analysis. In Section four the possibility of a Student t equilibrium is identified. In Section five the notion of a "hyperbolic OU" process is established, and in Section six the possibility of Pearson Type IV is considered. Some initial conclusions about the fully dynamical case are given in Section seven. In particular the phenomenon of "variance explosion" is identified. We also demonstrate the nature of the full distribution by reference to an explicitly solvable special case. Section eight explores other closed-form distributional solutions and gives a Laplace transform solution for a large class of cases. Section nine offers a preliminary market classification based on this approach. Section ten offers conclusions and speculations.

2 Two types of market participant

Consider a market containing an asset with share price S_t at time t. We consider a trading period $t \in [0, T]$. The log-return, x_t, on the asset is given by

$$x_t = \log\left(\frac{S_t}{S_0}\right) \ , \tag{2}$$

and we shall be concerned with time intervals over which $S_t \sim S_0(1 + x_t)$ so that no distinction is made between log and linear returns.

Our approach is to consider that the agents trading in this asset have two different types of motivation.

One set of agents comprise those acting independently of the current value of x_t. These will include, but will not necessarily be limited, to those trading on fundamentals, such as the dividend or earnings yield based on the price S_0. Other such traders may be engaged in a portfolio rebalance, or re-hedging a derivative position.

The second set of agents trade on the basis of the value of x_t. These are our technical traders relative to the time period under consideration. Technical trading is based on a number of different algorithms, which may be classified according to a "Rumsfeld" scheme: price momentum, mean reversion (known knowns) through unknown unknowns such as black box unpublished models being run by covert trading operations. We do not know the totality of such agents but we do know that they care about x_t, or possibly about the history $\{x_s | 0 \leq s \leq t\}$. In this paper I shall simplify and consider traditional momentum and mean-reversion traders who act on the value of x_t. There may also be longer term technical traders focusing on the ratio of S_0 to some much older price, who happen to be at work in the given time interval, but for our purposes they will not count as technical as they are not reacting specifically to x_t. There may be other participants who would not wish to be considered as technical traders, e.g those carefully achieving a position by a sequence of trades, but to the extent that their willingness to trade depends on x_t they are technical. Neither shall we enter the debate as to whether technical trading makes any sense, whether price increments are independent or whether there are material serial correlations etc. etc. *It will suffice that people who believe in technical trading* **are** *trading.* In this sense Malkiel's famous objection is irrelevant.

2.1 Fundamental "buy" orders

Let us now consider a time interval $(t, t + \Delta t) \subset [0, T]$ with $\Delta t << T$, and that orders may be effected in lots of size L. *We consider first* **buy** *orders based on* **fundamental** *trading.*

Let Y be the integer-valued random variable denoting the number of such trades arriving in time Δt, and let $N_i, i = 1, \ldots Y$, be the integer-valued random variable denoting the number of lots in each buy order. The number of shares, M_B, in the total collection of buy orders is $M = L \times Z$, where the random variable Z is given by

$$Z = \sum_{i=0}^{Y} N_i \ . \tag{3}$$

We assume that the number of trades is independent of the size of each trade, and that the trade sizes are independent and identically distributed[5]. Then the probability generating function (PGF) $f_Z(s)$ of Z is related to the PGFs of Y and N by

$$f_Z(s) = f_Y(f_N(s)) \ , \tag{4}$$

from which elementary PGF theory tells us that

$$E[Z] = E[Y]E[N] = E[Y]\overline{n} \ , \tag{5}$$

[5]These *are* assumptions and ones that might reasonably be questioned. One might consider, for example, that in a significant market downturn, there is correlation between having larger trades and having more trades. But even in this case we can imagine many small investors selling as well. Our goal is to get a model of price feedback, and the elegant compositional relationship for PGFs that these assumptions enable allow us to proceed more easily, if not in complete generality.

where $E[N] = \overline{n}$, and

$$Var[Z] = Var[Y]\overline{n}^2 + E[Y]Var[N] \ . \tag{6}$$

We shall now assume that the buy variable Y either follows a Poisson process with arrival rate λ_B, or, more loosely, that the process is sufficiently Poisson-like that we can write

$$E[Y] = Var[Y] = \lambda_B \Delta t \ . \tag{7}$$

In fact what we really need is only that this last equation holds. Y following a Poisson process is sufficient but not necessary. So then we have

$$E[Z] = \lambda_B \Delta t E[N] = \lambda_B \Delta t \overline{n} \ , \tag{8}$$

$$Var[Z] = \lambda_B \Delta t (E[N]^2 + Var[N]) = \lambda_B \Delta t (\overline{n}^2 + Var[N]) = \lambda_B \Delta t E[N^2] \ . \tag{9}$$

It follows that

$$E[M_B] = L\lambda_B \Delta t \overline{n} \ , \tag{10}$$

and that the standard deviation of M_B, $sd(M_B)$, is

$$sd(M_B) = L\sqrt{\lambda_B \Delta t E[N^2]} \ . \tag{11}$$

2.2 Fundamental "sell" orders

This proceeds in the same way. With similar assumptions, including that the variable N is similarly distributed, we end up with mean of the number of sales in the sell orders as

$$E[M_S] = L\lambda_S \Delta t \overline{n} \ , \tag{12}$$

and that the standard deviation of M_S, $sd(M_S)$, is

$$sd(M_S) = L\sqrt{\lambda_S \Delta t E[N^2]} \ . \tag{13}$$

2.3 Aggregation of fundamental trades

We define

$$M_F = M_B - M_S \tag{14}$$

as the net buy volume. The random variable M_F has mean

$$E[M_F] = L(\lambda_B - \lambda_S)\Delta t \overline{n} \ . \tag{15}$$

and its variance depends on the correlation between arrival rates of buy and sell trades. If we assume that fundamental buyers and sellers are acting on rather different motivations, which seems reasonable, then we might assume independence and hence that

$$Var[M_F] = L^2(\lambda_B + \lambda_S)\Delta t E[N^2] \ . \tag{16}$$

2.4 The technical traders

We consider now a second group of traders who only trade in response to a return created within the period under consideration. We shall model these trades in the same way as for the fundamental case, except now the trade arrival rates, instead of $\lambda_{B,F}$ are now $\mu_B(x)$ and $\mu_S(x)$ where the buy and sell μ_i have the property that $\mu_i(0) = 0$ - we assume that in the absence of a price movement there are no technical trades. Treating the variation in trade size in an identical fashion, we have additional net buying pressure M_T which is a random variable with

$$E[M_T] = L(\mu_B(x) - \mu_S(x))\Delta t\overline{n} \ , \tag{17}$$

and a variance which again depends on the degree of correlation between the buy and sell actions. In this case we shall assume that the buy and sell actions are perfectly correlated, which makes sense for example if a trader is pursuing a mean-reverting strategy:

$$Var[M_T] = L^2(\mu_B(x) - \mu_S(x))\Delta t E[N^2] \tag{18}$$

It is natural to assume that the technical traders are operating independently from the fundamental traders and this will be done.

2.5 The return impact function

We introduce a log-return impact function $\mathcal{I}(q)$ that is a function with $\mathcal{I}(0) = 0$ such that the aggregate buy and sell orders of both types create a log-return impact of the form

$$\Delta x = \mathcal{I}(M_F + M_T) \ . \tag{19}$$

2.6 Summary of discrete model

In the time interval Δt the return changes by

$$\Delta x = \mathcal{I}(M_F + M_T) \ , \tag{20}$$

where $\mathcal{I}(q)$ is the return impact of a net order to buy q shares. The variable M_F is a random variable with mean

$$E[M_F] = L(\lambda_B - \lambda_S)\Delta t\overline{n} \tag{21}$$

and variance

$$Var[M_F] = L^2(\lambda_B + \lambda_S)\Delta t E[N^2] \ . \tag{22}$$

The variable M_T is independent of M_F, and has mean

$$E[M_T] = L(\mu_B(x) - \mu_S(x))\Delta t\overline{n} \tag{23}$$

and variance

$$Var[M_T] = L^2(\mu_B(x) - \mu_S(x))\Delta t E[N^2] \ . \tag{24}$$

3 Linearization and the SDE

In order to proceed to a continuum representation we shall now make some simplifying assumptions. These are

1. Linearization of the return impact;
2. Linearization of the μ functions;
3. a Brownian motion model.

We now consider each of these assumptions.

3.1 Impact linearization

For the return impact function all we know for sure is that no orders mean no price impact, i.e. $\mathcal{I}(0) = 0$. In general $\mathcal{I}(q)$ may be a very complicated function. We think it should not decrease as q increases. The structure of the order book may give it a staircase character[6] However, we shall assume that on certain scales we may linearize it - a staircase may look like a smooth line viewed from a "long way" away - in order to capture the gross feature of the model. So for some unknown constant ω we write

$$\Delta x = \omega \times (M_F + M_T) \ . \tag{25}$$

The notion of a linear model of price impact was employed by Almgren and Chriss[1] as a component of both temporary and permanent price impact. It should be appreciated that this is very much in the same spirit as the assumptions made in the Bakstein-Howison model [3], where the essential features of the order book are reduced to a spread and liquidity parameter pair, where liquidity is the reciprocal of the slope of the order book. This idea is a powerful one which may also be used to analyze derivatives, as considered e.g. by Mitton [13].

3.2 Technical trade linearization

The appearance of trades depending on price movements within the time period under consideration will also have a complicated dependence. The existence of limit orders at various price thresholds will also create a staircase effect. We shall linearize this in the same way as the price impact, and make the replacement:

$$\mu_B(x) - \mu_S(x) \to -\mu \times x \ , \tag{26}$$

where μ is an effective constant parameter that captures the gross slope of the function[7] We put in a minus sign due to the nature of limit orders coming

[6] Note we always work based on mid-prices so the central step is taken as absent.

[7] In both this case and the treatment of price impact we are *not* making differentiability assumptions and using a power series - the idea in both cases is that a complex staircase may be grossly idealized as a sloping plane.

into play to act against the direction of price movement. One would expect the effect of μ to be negative on balance due to the effects of profit taking as well, *unless the system is being overwhelmed by momentum trades.* The appearance of the latter effect will be analyzed in detail later in this paper.

3.3 Process approximation

The trade arrival model is at this stage purely of Poisson type with a variable trade size. There are a number of ways in which this might be managed, depending on the details and frequency of trade arrivals. If there are a sparse number of large trades, this will be best viewed as a pure jump process. We might also have a situation wherec a large number of trades of moderate size together with sporadic large ones. This will generate in effect a jump diffusion. In the following simplification we assume that there are a large number of trades of moderate size so that we do not consider the jumps. The other cases will be investigated elsewhere. It is also possible, indeed likely, that the net price impacts of the two types of trade and their associate volatilities can be time dependent - this model does not rule out stochastic volatility at all. However, in what follows we shall confine attention to constant parameters and the pure-diffusion view - even this subset of possibilities will demonstrate a rich structure through the emergence of a *hybrid arithmetic-geometric* stochastic process.

In this context we now approximate the Poisson-type trade(s) arrival model(s) by independent Brownian motions centred on the mean arrival rate. We are lead finally to a discrete-time stochastic evolution equation for the return:

$$\Delta x = \omega L \left[\overline{n}[(\lambda_B - \lambda_S) - \mu x]\Delta t + s_1 \Delta W_1 + s_2 x \Delta W_2 \right] , \tag{27}$$

where

$$s_1 = \sqrt{(\lambda_B + \lambda_S)E[N^2]} , \quad s_2 = \sqrt{\mu E[N^2]} . \tag{28}$$

We now take the continuum limit and finally arrive at the SDE

$$dX_t = (\mu_1 - \mu_2 X_t)dt + \sigma_1 dW_{1t} + \sigma_2 X_t dW_{2t} , \tag{29}$$

where

$$\begin{aligned} \mu_1 &= \alpha \overline{n}(\lambda_B - \lambda_S) , \\ \mu_2 &= \alpha \mu \overline{n} , \\ \sigma_1 &= \alpha \sqrt{(\lambda_B + \lambda_S)E[N^2]} , \\ \sigma_2 &= \alpha \sqrt{\mu E[N^2]} , \end{aligned} \tag{30}$$

and $\alpha = L\omega$ is the return impact of trading one lot of shares. The SDE given above is the basic description where we show the explicit contribution separately of the fundamental and technical trades. We can of course reduce it to an SDE with a single noise term as follows. If ρ is the correlation between the two Brownian motions, then we can write the SDE as

$$dX_t = (\mu_1 - \mu_2 X_t)dt + \sqrt{\sigma_1^2 + X_t^2\sigma_2^2 + 2\rho\sigma_1 X_t\sigma_2^2}\, dW_t \ . \tag{31}$$

This is one of the class of "Pearson diffusions" considered by Forman and Sørensen [10]. It has a notable special case that we now consider.

4 The Student equilibrium model

A particular case of interest is obtained by considering $\mu_1 = 0 = \rho$, so that we obtain the SDE

$$dX_t = -\mu_2 X_t dt + \sqrt{\sigma_1^2 + X_t^2\sigma_2^2}\, dW_t \ . \tag{32}$$

In the equilibrium situation, the quantile ODE [20] associated with this SDE reduces to

$$\frac{\partial^2 Q}{\partial u^2}\left(\frac{\partial Q}{\partial u}\right)^{-2} = \frac{2(\sigma_2^2 + \mu_2)Q}{\sigma_1^2 + \sigma_2^2 Q^2} \ . \tag{33}$$

Bearing in mind the results of [20] we see that we have a quantile function for a Student distribution with

$$Q = \frac{\sigma_1}{\sqrt{\sigma_2^2 + 2\mu_2}} w(u) \ , \tag{34}$$

where $w(u)$ is the standard Student quantile with degrees of freedom

$$\nu = 1 + 2\frac{\mu_2}{\sigma_2^2} \ . \tag{35}$$

So it is clear that we need $\mu_2 > 0$ for this to be a Student distribution. This of course corresponds to the requirement that the underlying SDE mean-revert to the origin, and this mean-reversion condition in turn allows an equilibrium to establish. This equilibrium origination of the standard Student distribution arises naturally in plasma physics [19]. The faster the mean-reversion rate is compared to the multiplicative volatility, the closer the system is to the normally distributed limit. The Student distribution also arises naturally in the modelling of asset returns [9, 17].

In more generality we obtain a dynamic hyperbolic generalization of an OU process. This has been argued by Forman and Sørensen to be, in equilibrium, a natural candidate for a skew-Student model. In complete generality a still richer class of diffusions with or without jumps is possible. We do of course need to find the full time-dependent solution of the Fokker-Planck equation for the full SDE - this is not yet known.

5 The "hyperbolic O-U" SDE

While we do not have a characterization of the full time-dependent aspects, some insight can be gained by reducing one form of the hybrid SDE to standard form. First we do some scalings to standardize Eqn.(31). We let

$$X_t = \frac{\sigma_1}{\sigma_2\sqrt{\nu}} Y_t \ , \quad \Sigma_0 = \sigma_2\sqrt{\nu} \ . \tag{36}$$

Then the SDE for Y_t is

$$dY_t = -\frac{\Sigma_0^2}{2}\left(1 - \frac{1}{\nu}\right) Y_t dt + \Sigma_0 \sqrt{1 + \frac{Y_t^2}{\nu}} dW_t \ , \tag{37}$$

and this is a simple two-parameter form of the problem. We can re-cast this by setting

$$Y_t = \sqrt{\nu}\sinh(Z_t) \ , \tag{38}$$

and this leads to

$$dZ_t = -\frac{\Sigma_0^2}{2}\tanh(Z_t)dt + \frac{\Sigma_0}{\sqrt{\nu}} dW_t \ , \tag{39}$$

or equivalently

$$dZ_t = -\frac{\sigma_2^2}{2}\nu\tanh(Z_t)dt + \sigma_2 dW_t \ . \tag{40}$$

The original variable X_t is then given in terms of Z_t as simply:

$$X_t = \frac{\sigma_1}{\sigma_2}\sinh(Z_t) \ . \tag{41}$$

One can explore the full Fokker-Planck equation based on either of equations (36) or (39). Equation (39) reveals the essential nature of the process: for small Z_t and small times, the process is essentially OU in character. But the mean-reversion in the tails levels off and becomes much weaker.

6 Pearson type IV: skew-Student equilibria

We now turn to the more general case, where the SDE is

$$dX_t = (\mu_1 - \mu_2 X_t)dt + \sqrt{\sigma_1^2 + X_t^2\sigma_2^2 + 2\rho\sigma_1 X_t \sigma_2^2}\, dW_t \ . \tag{42}$$

This time, from the results of [20] we obtain the equilibrium quantile ODE as

$$\frac{\partial^2 Q}{\partial u^2}\left(\frac{\partial Q}{\partial u}\right)^{-2} = \frac{2[(\rho\sigma_1\sigma_2 - \mu_1) + (\sigma_2^2 + \mu_2)Q)]}{(\sigma_1^2 + \sigma_2^2 Q^2 + 2\rho\sigma_1\sigma_2 Q)} \ . \tag{43}$$

This is then related to the logarithmic derivative of the density function $f(x)$ as

$$-\frac{1}{f(Q)}\frac{df(Q)}{dQ} = \frac{2[(\rho\sigma_1\sigma_2 - \mu_1) + (\sigma_2^2 + \mu_2)Q)]}{(\sigma_1^2 + \sigma_2^2 Q^2 + 2\rho\sigma_1\sigma_2 Q)} \ . \tag{44}$$

We can solve this ODE and find that, after careful normalization,

$$f(x) = k\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{-(\nu+1)/2} \exp\left[-\nu_2 \tan^{-1}\left(\frac{x-\lambda}{a}\right)\right] , \tag{45}$$

where the parameters are given by

$$\begin{aligned}
a &= \frac{\sigma_1}{\sigma_2}\sqrt{1-\rho^2} , \\
\lambda &= -\rho\frac{\sigma_1}{\sigma_2} , \\
m &= 1+\frac{\mu_2}{\sigma_2^2} , \\
\nu &= 1+2\frac{\mu_2}{\sigma_2^2} , \\
\nu_2 &= \frac{2(\mu_1\sigma_2+\rho\sigma_1\mu_2)}{\sigma_1\sigma_2^2\sqrt{1-\rho^2}} , \\
k &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{a\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\left|\frac{\Gamma\left(\frac{\nu+1+i\nu_2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right|^2 .
\end{aligned} \tag{46}$$

This is the class Pearson Type IV distribution, which is one candidate for a choice of "skew-Student" distribution, with a rich variety of skewness and kurtosis in the structure. A useful guide to the properties of the Type IV Pearson is given by Heinrich [11], who uses the parameter m above, and his ν is our ν_2, i.e. his density is

$$f(x) = k\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{-m} \exp\left[-\nu \tan^{-1}\left(\frac{x-\lambda}{a}\right)\right] . \tag{47}$$

Transferring the results of [11] to our own notation[8] we can identify the mean, provided $\nu > 1$, as

$$E[X] = \lambda - \frac{a\nu_2}{\nu-1} . \tag{48}$$

The variance exists provided $\nu > 2$ and is then

$$\Sigma^2 = E[X^2] - E[X]^2 = \frac{a^2((\nu-1)^2+\nu_2^2)}{(\nu-1)^2(\nu-2)} . \tag{49}$$

The third moment can be calculated provided $\nu > 3$ and leads to the normalized skewness as

$$\frac{E[(X-E[X])^3]}{\Sigma^3} = \frac{-4\nu_2}{\nu-3}\sqrt{\frac{\nu-2}{(\nu-1)^2+\nu_2^2}} . \tag{50}$$

The fourth moment exists provided $\nu > 4$ and may be expressed through *excess* kurtosis, which is

$$\frac{E[(X-E[X])^4]}{\Sigma^4} - 3 = \frac{6\,(\nu_1-3)\,(\nu_1-1)^{\,2} + 6\,(5\nu_1-11)\,\nu_2^2}{(\nu_1-4)\,(\nu_1-3)\,((\nu_1-1)^{\,2}+\nu_2^2)} . \tag{51}$$

[8] Users of the standard student 't' are perhaps more used to working with the degrees of freedom parameter ν.

When $\nu_2 = 0$ the skewness is zero and the excess kurtosis reduces to the well-known expression for the pure Student distribution: $6/(\nu-4)$. We verified the translation of these expressions to our notation by the computation of the moments by simple numerical integration for numerous parameter values.

7 Towards the full dynamics

The achievement of an equilibrium is not realistic for most trading periods, especially during a panic of the credit-crunch period. The pure equilibrium analysis above is meant to indicate how a rich variety of distributional types may emerge from a simple model, and no more. We now turn to the full dynamics.

7.1 Dynamic moment evolution

A partial characterization of the full dynamical situation may be given in terms of the evolution of the moments. Let us set

$$e_n = E[X_t^n] \ . \tag{52}$$

Then elementary analysis gives us the sequential families of ODEs:

$$\frac{de_n}{dt} + (\mu_2 n - \frac{1}{2}n(n-1)\sigma_2^2)e_n = \frac{1}{2}n(n-1)\sigma_1^2 e_{n-2} + (\mu_1 n + n(n-1)\rho\sigma_1\sigma_2)e_{n-1} \ , \tag{53}$$

with $e_0 \equiv 1$. The evolution of the mean e_1 is therefore governed by

$$\frac{de_1}{dt} + \mu_2 e_1 = \mu_1 \tag{54}$$

and, as in an ordinary OU process, evolves according to

$$e_1 = X_0 e^{-\mu_2 t} + \frac{\mu_1}{\mu_2}(1 - e^{-\mu_2 t}) \ . \tag{55}$$

With our conventions $X_0 = 0$ at the start of the trading period so we then have

$$e_1 = \frac{\mu_1}{\mu_2}(1 - e^{-\mu_2 t}) \ , \tag{56}$$

which will settle down to μ_1/μ_2 if $\mu_2 > 0$ and grows exponentially otherwise.

7.2 The explosion of variance

Solution for the higher moments is straightforward but leads to rather unwieldy formulae in general. In the special case where $\rho = 0 = \mu_1$ the variance $V(X_t)$ may be written in the tractable form

$$V(X_t) = \frac{\sigma_1^2}{\sigma_2^2(\nu-2)}\left[1 - e^{-\sigma_2^2(\nu-2)t}\right] \sim \sigma_1^2 t - \frac{1}{2}\sigma_1^2\sigma_2^2(\nu-2)t^2 + O(t^3) \ , \tag{57}$$

where, as before $\nu = 1 + 2\mu_2/\sigma_2^2$. Once the market has kicked off (the behaviour near $t = 0$ being always Gaussian), the market dynamics are thus critically dependent on the sign of $\nu - 2$. If the strength of the mean-reverting trades is such that $\mu_2 > \sigma_2^2/2$ the market settles down. If instead $\mu_2 < \sigma_2^2/2$ the variance grows exponentially. Note that there is a region $0 < \mu_2 < \sigma_2^2/2$ where the average level stabilizes but the variance does not. If $\mu_2 < 0$ both the average level and variance grow exponentially. If $\nu > 2$ the distribution settles down to the Student equilibrium already analysed, the with the condition $\nu > 2$ guaranteeing a finite variance.

The exponential growth in variance is not a new idea - it has been present in the *price evolution* model of geometric Brownian motion for decades. What is being suggested here is that technical trade effects in the evolution of the *log*-return lead to a variance explosion in the log-returns due to these being a hybrid arithmetic-geometric hybrid.

We can now return to the famous 25σ issue. If one's perception is that normal Gaussian behaviour is to be expected, with the classical variance $\sigma_1^2 t$ in the log-returns, then the inclusion of technical trading effects via the hybrid model causes the actual variance to differ by a ratio, that we call the *variance explosion factor*

$$V_E(t) = \frac{V(t)}{\sigma_1^2 t} \ . \tag{58}$$

In general the variance explosion factor may be found by solving the ODE for e_2. In the special case considered above we have

$$V_E(t) = \frac{1}{\sigma_2^2 t(\nu - 2)} \left[1 - e^{-\sigma_2^2(\nu-2)t} \right] \ . \tag{59}$$

If the markets are minded to settle to equilibrium this ratio tends to zero. But if $\nu < 2$ we have instead

$$V_E(t) = \frac{1}{\alpha t} \left[e^{(\alpha t)} - 1 \right] \ ; \qquad \alpha = \sigma_2^2(2 - \nu) > 0 \ . \tag{60}$$

To return to the likelihood of 25σ events, we note that if

$$\alpha \geq 6.4746 \ , \text{ i.e. } \sigma_2^2(2 - \nu) > 6.4746 \ , \tag{61}$$

then $V_E(t) > 100$ and a 25σ event based on an initial perception of variance $\sigma_1^2 t$ is no less likely than a $2.5\tilde{\sigma}$ event with the right variance (and with a different, non-Gaussian, distribution). Such events may then occur repeatedly if the technical market effects are strong enough, even without incorporating the additional effect of net price pressure due to fundamental trades ($\mu_1 \neq 0$).

We see that a variety of different dynamics are possible, with quite small shifts in the mean-reversion strength of technical trades making a dramatic difference to the return distribution.

7.3 Dynamic distributional aspects

Having identified the impact of the arithmetic-geometric hybrid on the moments we now need to understand the full shape of the distribution. If the distribution is significantly fat tailed then this can also amplify the likelihood of extreme movements.

To get at this in general we must analyse the full Fokker-Planck equation in the form

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[-(\mu_1 - \mu_2 x)f(x,t) + \frac{1}{2}\frac{\partial}{\partial x}\left[(\sigma_1^2 + x^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 x)f(x,t))\right]\right] , \tag{62}$$

with the initial condition $f(x,0) = \delta(x)$. One route to this is via the Laplace transform with respect to time. So let

$$\tilde{f}(x,p) = \int_0^\infty f(x,t)e^{-pt}dt \ . \tag{63}$$

Then the Fokker-Planck equation gives us, suppressing the independent variables,

$$p\tilde{f} - \delta(x,0) = \frac{\partial}{\partial x}\left[-(\mu_1 - \mu_2 x)\tilde{f} + \frac{1}{2}\frac{\partial}{\partial x}\left[(\sigma_1^2 + x^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 x)\tilde{f})\right]\right] , \tag{64}$$

This is now a Green's function computation on the transform. For $x > 0$ and $x < 0$ we need two independent solutions of

$$p\tilde{f} = \frac{\partial}{\partial x}\left[-(\mu_1 - \mu_2 x)\tilde{f} + \frac{1}{2}\frac{\partial}{\partial x}\left[(\sigma_1^2 + x^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 x)\tilde{f})\right]\right] , \tag{65}$$

with the junction condition that $\tilde{f}$ is continuous at $x = 0$, and a jump in the first derivative. This condition, integrating about zero, is

$$\frac{\partial \tilde{f}}{\partial x}(0+,p) - \frac{\partial \tilde{f}}{\partial x}(0-,p) = -\frac{2}{\sigma_1^2} \ . \tag{66}$$

7.3.1 The Gaussian case

In order to make this approach clearer, we first pursue the standard Gaussian problem where $\sigma_2 = 0 = \mu_2$. The solution of the transformed ODE vanishing as $x \to \pm\infty$ and with the correct junction condition at zero is

$$\tilde{f}(x,p) = \frac{e^{\mu_1 x/\sigma_1^2}}{\sqrt{\mu_1^2 + 2p\sigma_1^2}} \begin{cases} \exp\left[-\frac{x}{\sigma_1^2}\sqrt{\mu_1^2 + 2p\sigma_1^2}\right] & \text{if } x > 0, \\ \exp\left[+\frac{x}{\sigma_1^2}\sqrt{\mu_1^2 + 2p\sigma_1^2}\right] & \text{if } x < 0. \end{cases} \tag{67}$$

Inversion of the two cases leads to the single well-known formula

$$f(x,t) = \frac{1}{\sqrt{2\pi\sigma_1^2 t}} \exp\left[-(x - \mu_1 t)^2/(2\sigma_1^2 t)\right] \ . \tag{68}$$

7.3.2 A dynamic Student distribution

The next case of interest is when $\rho = 0 = \mu_1$, for which we previously demonstrated a Student t equilibrium under certain circumstances. The Laplace transform of the Fokker-Planck equation is, for $x \neq 0$,

$$\frac{1}{2}(\sigma_1^2 + \sigma_2^2 x^2)\tilde{f}''(x,p) + (\mu_2 + 2\sigma_2^2)x\tilde{f}'(x,p) + (\mu_2 + \sigma_2^2 - p)\tilde{f}(x,p) = 0 \ . \quad (69)$$

This equation may be simplified somewhat by setting

$$\tilde{f}(x,p) = (\sigma_1^2 + \sigma_2^2 x^2)^{-(1+\mu_2/\sigma_2^2)} g(x,p) \ , \quad (70)$$

and the ODE for $g(x,p)$ is then

$$(\sigma_1^2 + \sigma_2^2 x^2) g''(x,p) - 2x\mu_2 g'(x,p) - 2pg(x,p) = 0 \ . \quad (71)$$

We have already worked out the equilibrium case when $p = 0$ and g is constant in x. The management of such an equation is straightforward in the special case $\mu_2 = -\sigma_2^2/2$, as discussed in [15]. We shall use the change of independent variables indicated in [15] to treat the general case, and indeed this is almost the same change of variables that took us to the hyperbolic OU picture. We introduce $z(x)$ with the condition that

$$\frac{dz}{dx} = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 x^2}} \quad (72)$$

and fix the arbitrary constants so that

$$z = \frac{1}{\sigma_2} \sinh^{-1}\left(\frac{\sigma_2 x}{\sigma_1}\right) \ . \quad (73)$$

Our equation for g expressed in terms of z is then just

$$\frac{d^2 g}{dz^2} - (2\mu_2 + \sigma_2^2)\frac{1}{\sigma_2}\tanh(\sigma_2 z)\frac{dg}{dz} - 2pg = 0 \ , \quad (74)$$

or in terms of the degrees of freedom parameter $\nu = 1 + 2\mu_2/\sigma_2^2$,

$$\frac{d^2 g}{dz^2} - \nu\sigma_2 \tanh(\sigma_2 z)\frac{dg}{dz} - 2pg = 0 \ . \quad (75)$$

The easy case when $\nu = 0$ leads to

$$\tilde{f}(x,p) = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 x^2}} \frac{1}{\sqrt{2p}} \begin{cases} e^{-\sqrt{2p}z(x)} & \text{if } z, x > 0, \\ e^{+\sqrt{2p}z(x)} & \text{if } z, x < 0. \end{cases} \quad (76)$$

and the inversion gives us

$$f(x,t) = \frac{1}{\sqrt{2\pi t(\sigma_1^2 + \sigma_2^2 x^2)}} \exp\left\{ \frac{-1}{2\sigma_2^2 t}[\sinh^{-1}(\sigma_2 x/\sigma_1)]^2 \ , \right\} \quad (77)$$

which is the density arising from a change of variables $z \to x$ on the z-density

$$\frac{1}{\sqrt{2\pi t}} \exp(-z^2/(2t)) \ , \tag{78}$$

and is a significantly fatter-tailed object (while in the neighbourhood of $x = 0$ resembling a Gaussian with variance $\sigma_1^2 t$!) The presence of significant momentum trading has fattened the tails in this case. We shall now explore the properties of the PDF given for the special case hybrid given by Equation (76).

7.4 The hidden menace

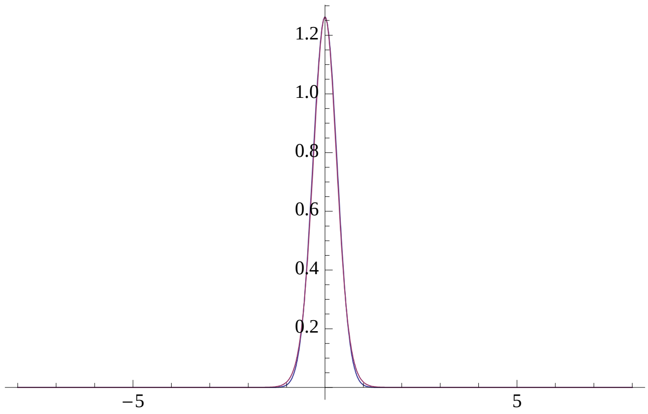

Figure 1: PDFs for Gaussian and special case hybrid, $t = 0.1$.

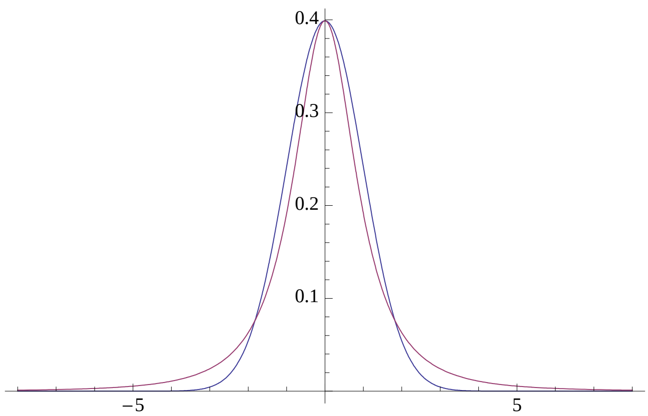

Figure 2: PDFs for Gaussian and special case hybrid, $t = 1$.

Even the very special case solved for exhibits some interesting features. At the start of the trading period the distribution of log-returns is barely distinguishable from Gaussian, as shown in Figure 1 for the parameters $\sigma_1 = \sigma_2 = 1, t = 0.1$, where the hybrid is overlaid with the Gaussian.

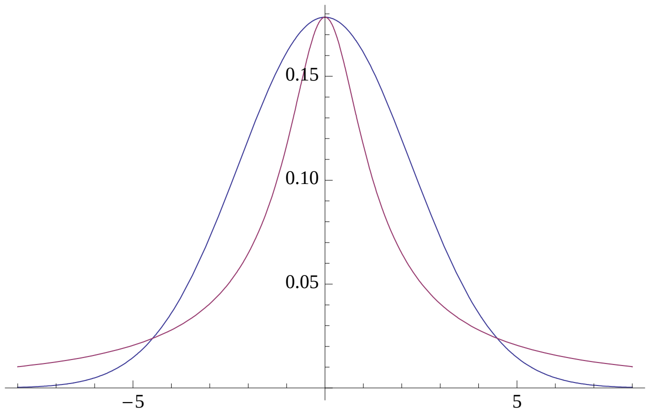

Figure 3: PDFs for Gaussian and special case hybrid, $t = 5$.

However, as time passes the hybrid distribution spreads out more, in a manner consistent with the variance explosion formula. In Figure 2 and Figure 3 we show the hybrid PDF overlaid with the Gaussian at times $t = 1$ and $t = 5$ respectively. At *all* times the probability of a very small move remains at the Gaussian level - the PDF osculates the Gaussian at the origin. The overall behaviour represents the hidden menace of these processes. It starts off looking Gaussian with variance $\sigma_1^2 t$; the probability of a very small movement remains near the Gaussian value, yet dependent on the size of σ_2 the probability of extreme movements grows exponentially in time.

8 More general solutions

We now look at the case where $\nu \geq 0$. We go back to Eqns. (73-75), and change independent variable to

$$u = \sinh^{-1}(\sigma_2 x/\sigma_1) = \sigma_2 z \ . \tag{79}$$

Setting $s = 2p/\sigma_2^2$, and letting $'$ now denote d/du, Eqn. (75) may be reorganized as

$$(e^u + e^{-u})(g''(u) - sg(u)) = \nu(e^u - e^{-u})g'(u) \ . \tag{80}$$

Previously we considered the case $\nu = 0$, where we could write down a solution decaying as $x \to +\infty$ as a single exponential in $e^{-\sqrt{s}u}$. When $\nu \neq 0$ we have to proceed differently, but the presence of the two exponentials in the coefficients suggests a solution approach. To tidy up a little we make a further change of variable to

$$w = e^{-u} = \left(\frac{\sigma_2 x}{\sigma_1} + \sqrt{1 + \frac{\sigma_2^2 x^2}{\sigma_1^2}}\right)^{-1} \ . \tag{81}$$

So with $u = -\log w$ we have $d/du = -wd/dw$ and our ODE may be rewritten as

$$\left(\frac{1}{w} + w\right)\left[w^2\frac{d^2g}{dw^2} + w\frac{dg}{dw} - sg\right] = \nu\left(w - \frac{1}{w}\right)w\frac{dg}{dw} \ . \tag{82}$$

We seek a power series solution in the form

$$g = \sum_{k=0}^{\infty} a_k w^{k+\gamma} \ , \quad a_0 \neq 0 \ . \tag{83}$$

After some standard manipulations we find an indicial equation in the form

$$\gamma^2 + \nu\gamma - s = 0 \ , \tag{84}$$

and the required root, for $\nu \geq 0$, to get the right behaviour as $x \to \infty$, $w \to 0$, is

$$\gamma = \sqrt{s + \frac{\nu^2}{4}} - \frac{\nu}{2} \ . \tag{85}$$

The resulting recurrence relation simplifies to

$$a_{k+2}(k+2)(k+2+2\gamma+\nu) = -a_k(k-\nu)(k+2\gamma) \ . \tag{86}$$

After some experimentation with hypergeometric functions and some work with *Mathematica* we are able to recognize the solution in the form

$$g = a_0(p) w^{\gamma} \, {}_2F_1\left(\gamma, -\frac{\nu}{2}; \gamma + \frac{\nu}{2} + 1; -w^2\right) \ . \tag{87}$$

This representation of the solution has the nice property that we can see that the hypergeometric function reduces to a polynomial if ν is an even integer. The case $\nu = 0$ has already been exhibited. Before discussing other such simple cases we must complete the solution and determine a_0. This involves the application of the jump condition on the derivative at the origin, assuming an even solution. After some algebra we find that

$$a_0(p) = \frac{\sigma_1^{\nu}}{\sigma_2 \Omega(\nu, \gamma)} \ , \tag{88}$$

where

$$\Omega(\nu, \gamma) = \frac{d}{dw}\left[w^{\gamma} \, {}_2F_1\left(\gamma, -\frac{\nu}{2}; \gamma + \frac{\nu}{2} + 1; -w^2\right)\right]\Bigg|_{w=1} \ . \tag{89}$$

After some use of *Kummer's identity* and variations (specifically identities 15.1.21 and 15.1.22 from [2]), we are lead to

$$\Omega(\nu, \gamma) = \frac{2^{1-\gamma}\sqrt{\pi}\,\Gamma\left(\gamma + \frac{\nu}{2} + 1\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}(\gamma + \nu + 1)\right)} \ . \tag{90}$$

We arrive at a closed form for the Laplace transform of the density as

$$\tilde{f}(x, p) = \frac{\sigma_1^{\nu} 2^{\gamma-1} w^{\gamma} \Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}(\gamma+\nu+1)\right) {}_2F_1\left(\gamma, -\frac{\nu}{2}; \gamma + \frac{\nu}{2} + 1; -w^2\right)}{\sqrt{\pi}\sigma_2 \Gamma\left(\gamma + \frac{\nu}{2} + 1\right)\left(\sigma_1^2 + x^2\sigma_2^2\right)^{\frac{1}{2}(\nu+1)}} \ . \tag{91}$$

We remind the reader that in the use of this expression,

$$\begin{aligned} s &= \frac{2p}{\sigma_2^2} \ , \\ \gamma &= \sqrt{s + \frac{\nu^2}{4}} - \frac{\nu}{2} \ , \\ w &= \left(\frac{\sigma_2 x}{\sigma_1} + \sqrt{1 + \frac{\sigma_2^2 x^2}{\sigma_1^2}} \right)^{-1} \ . \end{aligned} \tag{92}$$

It is now easy to check the known special case, when $\nu = 0$, for the hypergeometric function simplifies leading to

$$\tilde{f}(x,p) = \frac{w^\gamma}{\gamma \sigma_2 \sqrt{\sigma_1^2 + x^2 \sigma_2^2}} \ , \quad \gamma = \sqrt{s} \ . \tag{93}$$

We now also have a new family of relatively simple cases when ν is an even integer. For example, when $\nu = 2$ we have

$$\tilde{f}(x,p) = \frac{\sigma_1^2}{2\sigma_2 \left(\sigma_1^2 + x^2 \sigma_2^2\right)^{3/2}} \left(\frac{w^\gamma}{\gamma} + \frac{w^{\gamma+2}}{\gamma+2} \right) \ , \quad \gamma = \sqrt{s+1} - 1 \ . \tag{94}$$

In the case $\nu = 4$ we have

$$\begin{aligned} \tilde{f}(x,p) &= \frac{\sigma_1^4}{4\sigma_2 \left(\sigma_1^2 + x^2 \sigma_2^2\right)^{5/2}} \left(\frac{(3+\gamma) w^\gamma}{\gamma(2+\gamma)} + \frac{2w^{\gamma+2}}{\gamma+2} + \frac{(\gamma+1) w^{\gamma+4}}{(\gamma+2)(\gamma+4)} \right) \ , \\ \gamma &= \sqrt{s+4} - 2 \ . \end{aligned} \tag{95}$$

We now turn to an exploration of the new simple case given by $\nu = 2$.

8.1 A "chameleon" distribution

In this sub-section we shall introduce a one-parameter family of dynamic distributions with the following interesting properties:

- The distributions arise from stochastic differential equations;
- The mean is identically zero;
- The variance is of the form $\sigma_1^2 t$;
- The behaviour is initially Gaussian, in standard form.
- The distribution tends to a non-Gaussian steady-state.

This is just a matter of specializing the analysis above to the case $\nu = 2$. Writing out the solution for the transform more explicitly, we have

$$f(x,p) = \frac{\sigma_1^2}{2\sigma_2(\sigma_1^2 + \sigma_2^2 x^2)^{3/2}} \left[\frac{e^{-[\sqrt{s+1}-1]|u(x)|]}}{\sqrt{s+1}-1} + \frac{e^{-[\sqrt{s+1}+1]|u(x)|]}}{\sqrt{s+1}+1} \right] \ , \tag{96}$$

where

$$s = \frac{2p}{\sigma_2^2}\ , \quad u(x) = \sinh^{-1}(\sigma_2 x/\sigma_1)\ . \tag{97}$$

This may be inverted in closed form, making careful use of identity 29.3.88 from [2] and some standard Laplace transform identities. After some careful simplifications we are lead to the following density function:

$$\begin{aligned} f(x,t) = & \frac{\sigma_1 \exp[-\frac{u(x)^2}{2t\sigma_2^2} - \frac{t\sigma_2^2}{2}]}{\sqrt{2\pi t}\,(\sigma_1^2 + x^2\sigma_2^2)} \\ & + \frac{\sigma_2\sigma_1^2}{2\,(\sigma_1^2 + x^2\sigma_2^2)^{3/2}}\left[\Phi\left(\frac{|u(x)| + t\sigma_2^2}{\sqrt{t}\sigma_2}\right) - \Phi\left(\frac{|u(x)| - t\sigma_2^2}{\sqrt{t}\sigma_2}\right)\right]\ , \end{aligned} \tag{98}$$

where Φ is the standard normal CDF. This is the probability density function for our "chameleon distribution". We have obtained it by solving the Fokker-Planck equation from an SDE. Its mean is zero and is variance satisfies

$$V(X) = \sigma_1^2 t\ , \quad \forall \sigma_2,\ t\ . \tag{99}$$

Its asymptotic behaviour as $t \to 0$ is obtained by considering just the first exponential part of the expression, which we see tends to

$$f(x,t) \sim \frac{\sigma_1 \exp[-\frac{u(x)^2}{2t\sigma_2^2}]}{\sqrt{2\pi t}\,(\sigma_1^2 + x^2\sigma_2^2)} \sim \frac{\exp[-\frac{x^2}{2t\sigma_1^2}]}{\sqrt{2\pi t}\sigma_1}\ , \tag{100}$$

where the last approximation arises as the support of the distribution contracts about the origin, allowing us to expand the arcsinh and denominator. So the distribution starts off in standard Gaussian form. For $t \to \infty$ we just note that the first term tends to zero and the second line tends to

$$f(x,t) \sim \frac{\sigma_2\sigma_1^2}{2\,(\sigma_1^2 + x^2\sigma_2^2)^{3/2}}\ , \tag{101}$$

which is the density of a scaled Student t distribution with two degrees of freedom. This of course has infinite variance. We have therefore demonstrated the list of conditions claimed at the start of this Section. Parametrized by σ_2, there are infinitely many SDEs of the form

$$dX_t = -\frac{\sigma_2^2}{2} X_t dt + \sqrt{\sigma_1^2 + \sigma_2^2 X_t^2} dW_t\ , \tag{102}$$

whose variance is given by the standard formula $\sigma_1^2 t$, normally associated with the simple case

$$dX_t = \sigma_1 dW_t\ . \tag{103}$$

This is a useful reminder that *linear evolution of variance does not in any way imply the elementary Brownian model.* Although we have produced a rather exotic density function, the underlying dynamics as given by Eqn. (102) are very simple. Changing variables to hyperbolic coordinates via $X_t = \sigma_1/\sigma_2 \tanh(Z_t)$ gives us the nice form

$$dZ_t = -\sigma_2^2 \frac{\nu}{2} \tanh(Z_t) dt + \sigma_2 dW_t\ . \tag{104}$$

8.2 Further generalizations

The still more general case of the Fokker-Planck equation in the full situation, with asymmetry, is under investigation. Other mathematical properties of the Pearson diffusions have been investigated by Forman and Sørensen [10].

9 States of the market

Having looked at some typical dynamics, we look at the overall picture. Our model has four parameters, and we now give them names:

- σ_1, the fundamental volatility;
- σ_2, the technical volatility;
- μ_1, the fundamental drift;
- μ_2, the technical drift;

While we have not incorporated μ_1 into any detailed analysis thus far, its interpretation is clear. The main influences on the state of the market are the parameters μ_2, σ_1, σ_2. In fact, it is the balance between these parameters that matters. A critical quantity is

$$\nu = 1 + \frac{2\mu_2}{\sigma_2^2} \ . \tag{105}$$

If an equilibrium is achieved, this is the degrees of freedom of the associated Student distribution that results. But now we see that it plays an esential dynamical role:

- if $\nu < 2$ the variance explodes exponentially;
- if $\nu = 2$ the variance remains in Gaussian form, but any member of the chameleon family may exist;
- $\nu > 2$ the variance tends to a constant.

The circumstances under which the distribution attains an equilibrium are subtly different. We know that when $\nu = 0,\ m = 1/2$ the solution is eternally dynamic and the PDF has been calculated explicitly. When $m > 1/2, \nu / > 0$, the equilibrium solution exists and has a normalizable PDF in Pearson IV form [11]. So there is a range $0 < \nu < 2$ where the equilibrium exists but the variance explodes. It would be useful to get a better grip on the dynamic Cauchy case that sites in the middle of this zone with $\nu = 1 = m$. This case corresponds to $\mu_2 = 0$.

The volatiities themselves play a different role. Inspection of the hyberbolic OU equation indicates that it is σ_2 that sets the time-scales, and the ratio σ_2/σ_1 determines the scale on which asset price movements are affected by the price feedback. So in the absence of fundamental drift the market state is best characterized by the triple:

- $\nu = 1 + \frac{2\mu_2}{\sigma_2^2}$ - determines the market condition;
- σ_2 defines the time-scale;
- σ_2/σ_1 defines the asset price scale.

Since all but the first are scaling variables, we can observe that when $\mu_1 = 0$, the parameter ν is critical. Historically this has been the elementary 'degrees of freedom" parameter for static Student distributions. It now plays a critical dynamical role. If we set $\tau = \sigma_2^2 t$ the underlying dimensionless SDE is

$$dZ_\tau = -\frac{\nu}{2}\tanh(Z_\tau)d\tau + dW_\tau \ . \tag{106}$$

with all other parameters removed by transformation. We have explicit time-domain solutions for the resulting PDF for $\nu = 0, 2$ and the Laplace transform for other values.

10 Conclusions and further work

We have developed a simple hybrid trading model that mixes both fundamental and technical trades. The general form of the model allows for jumps and Brownian motion together, but the theory here has been more fully developed for the pure Brownian case. We obtain a hybrid process for the log-return that is a composite of arithmetical and geometric types, and that is capable of exhibiting a variety of behaviours depending on the relative strength of the fundamental and technical components.

Even when the market settles down we obtain a rich family of distributions including fat-tailed Student and skew-Student models as a special case. It is therefore possible to attribute some of the skewness and fat-tailed behaviour in asset returns to a simple composite model of traders acting on different logic.

In the fully dynamic context a the likelihood of extreme events is greatly increased by the use of a hybrid arithmetical-geometric process for the log-returns. Such hybrids can exhibit the phenomenon of variance explosion at the same time as hiding their non-Gaussian nature very effectively. Work on the full time-dependent form, with all parameters non-zero, is in progress.

Another thread of research involves the use of multivariate extensions of these models, where the coupled distributions arise through coupled diffusions of Pearson type. In this way multivariate distributions with marginals drawn from the Pearson family may be constructed, as discussed in [18].

Acknowledgments

Dr G. Steinbrecher is responsible for providing information on the existing literature on hybrid processes in theoretical physics. Dr A. Macrina gave many useful comments on earlier versions of this article. I also wish to thank R. Wilson and A. Healey for several useful remarks.

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