Preface

Why this book?

Since 1985, I have been fortunate to have taught the theory of complex variables for several courses in both the USA and the UK. In the USA I lectured a course on advanced calculus for engineers and scientists at MIT, and in the UK I have given tutorials on the subject to undergraduate students in mathematics at both Cambridge and Oxford. Indeed, draft versions of this text have been inflicted on my students at Balliol and, more recently, at St. Catherine's over the last fourteen years. Few topics have given me such pleasure to teach, given the rich yet highly accessible structure of the subject, and it has at times formed the subject of my research, notably in its development into twistor theory, and latterly in its applications to financial mathematics. A parallel thread of my work has been in the applications of computer algebra and calculus systems, and in particular *Mathematica*®, to diverse topics in applied mathematics. This book is in part an attempt to use *Mathematica* to illuminate the topic of complex analysis, and draws on both these threads of my experience.

The book attempts also to inject some new mathematical themes into the topic and the teaching of it. These themes I feel are, if not actually missing, under-emphasized in most traditional treatments. It is perfectly possible for students to have had a formal training in mathematics that leaves them unaware of many key and/or beautiful topics. If we take the beginning of the historical time-line to supply our first example, many students will not be aware of how to solve a cubic equation, despite this procedure being one of the key early developments in this field. Having invented complex numbers to cope with a general quadratic, early algebraists found that the cubic could also be solved. This is of paramount importance, not just for the elegance of the solution, but also because it is the first indication that the fundamental theorem of algebra might be a possible theorem! If we take a more recent example, those students that do consider the applications to basic physics will almost always emerge with the entirely mistaken notion that complex variable methods are limited to a few very special problems in two-dimensional electrostatics or fluid dynamics. Similarly, the Möbius transform will often be presented only as a neat trick for mapping shapes around the complex plane, and its profound links to relativistic physics, via its equivalence to the Lorentz transformation, are ignored.

So in addition to providing illuminations and visualizations with *Mathematica*, I have tried to put back and indeed add some of the topics that I feel students ought to know. In particular, and unusually for a text targeted at undergraduate mathematicians or graduate students in other disciplines, this book includes a friendly introduction to the theory of spinors and twistors, thereby unlocking the applications of complex functions to problems in three and four dimensions.

But you do not have to accept this particular set of views to make good use of this book. It is perfectly possible to use this text to teach a standard course in complex analysis, ignore my idiosyncratic additions, and take the *Mathematica* elements as purely an embedded tool that has been used to generate some of the pictures!

Mathematica makes its appearance in many different ways. In several chapters it is there purely to provide, literally, illustrations. In some places it is used as a checking tool, for example when calculating residues and integrals. In other chapters it is fundamental, and indeed in Part II, it is the centre of a set of investigations into the solving of equations by iteration. In places its rich library of special functions, the ability to evaluate them over the complex plane, to do calculus with them, come to the fore. It is particularly valuable when applied to topics in conformal mapping. Finally, *Mathematica's* wonderful graphics are universally useful.

How this text is organized

It is best to think of the material of this book as being grouped informally into five parts. These are as follows:

Part I Basic complex number theory and history

Attention will be focused on three topics, each of which constitutes one chapter:

- Chapter 1: Why you need complex numbers;
- Chapter 2: Complex algebra and geometry;
- Chapter 3: Cubics, quartics and visualization of complex roots.

The idea of this part of the book is to explain how and why complex numbers were introduced, and then to go on to discuss elementary properties of the complex number system. This material is at a level normally to be found in final year high school programs or introductory college level. Chapter 3 should be regarded as optional, but is highly recommended for any students with an interest in the history of the subject. It covers the treatment of cubics and quartics, which is not usually taught in modern courses, and also includes some material on the visualization of roots of polynomials.

Part II Iterated mappings

Part I showed how to define complex numbers and how to use them to solve low-order polynomial equations. The methods used to treat the quadratic, cubic and quartic equations are the classical techniques that have been known for some time – many hundreds of years in some cases.

Now that computer systems are available, you can explore, either through this text, or directly yourselves through the use of these *Mathematica* notebooks, the rich structure that is obtained by the application of *iterative* equation-solving techniques to these same simple polynomial systems. This idea originates with A. Cayley in the 19th century, who although able to understand quickly the complex structure of Newton–Raphson methods when applied to a quadratic, was frustrated by the corresponding problem with a cubic. In getting to grips with Cayley's problem, we shall quickly encounter some of the most beautiful objects in modern mathematics – chaotic systems and fractals.

This part of the text consists of material that is not part of a *traditional* course on complex analysis. It may be skipped by those using this text to pursue such a traditional route, who should proceed to Part III. Part III does not rely on any of the material in Part II.

In Part II, all of the systems that you will see can be regarded as special cases of the general first-order iterated map:

$$z_{n+1} = f(z_n)$$

You will be able to explore how this works for various choices of the function f. One way or other, f is to be associated with the solution of a low-order polynomial equation. The association of the iterated map with the polynomial equation can take place in several ways, and two will be considered here.

The first approach will involve polynomial (or even transcendental) equations of the form

$$g(z) = 0$$

and you will explore the Newton–Raphson iteration scheme given by the choice

$$f(z) = z - \frac{g(z)}{g'(z)}$$

The second scheme will involve a polynomial equation that is already written (for example, by simply isolating the linear term, if there is one) in the form

$$z = f(z)$$

and you will explore the 'cobwebbing' solution scheme based on iteration of this representation.

Attention will be focused on four topics, each of which constitutes one chapter. Of these four topics, the first is specifically Newton–Raphson. The next two may be regarded as being associated with the cobwebbing method. The fourth topic is a complex extension of the cobwebbing method with symmetry. In order of presentation, the topics are:

- Chapter 4: Newton–Raphson iteration and complex fractals;
- Chapter 5: A complex view of the real logistic map;
- Chapter 6: The Mandelbrot set;
- Chapter 7: Symmetric chaos in the complex plane.

In Chapter 4 the solution of a low-order polynomial equation will be reconsidered in the complex plane using Newton–Raphson iteration, as part of an investigation of Cayley's problem (Cayley, 1879). This is a standard technique for solving real non-linear equations – our purpose here is to explore what happens when Newton–Raphson is applied in the complex plane, and to use the computer to understand why Cayley was defeated by the cubic!

In this part of the text we will engage occasionally engage in 'fashionable number crunching', as chaos theory was once famously described. The desire to produce some spectacular pictures is never far from one's mind. But not all mathematics has to be useful, and the uncovering of beauty is a worthwhile goal in itself. So in this part of the book I shall indulge shamelessly in some fashionable number crunching - this is sometimes referred to somewhat pompously as 'experimental mathematics'. But *good* experiments should be designed to test some theory about what should happen, and we can use complex numbers, to some extent, to provide a framework for first formulating a hypothesis regarding what may happen in a simple real non-linear system.

The logistic map, as developed by May (1976), is the place where this experimentation will commence for real, with Chapter 5. This is usually regarded as a real mapping, so what is it doing here? The point is that we shall not just indulge in computation, but shall attempt to predict, through the machinery of complex numbers, what should happen in a certain experiment. It will turn out that the period-doubling behaviour of the logistic map is in fact a simple and predictable result that requires nothing more than 'end of high school' mathematics. The experimentation will serve to confirm our hypotheses about it. What is surprising and fascinating is the transition to chaos that follows, and there are indeed many properties of the logistic map that are still not properly understood.

It is admittedly very hard to extend this approach to more complicated non-linear systems, so we shall then rely more substantially on numerical experiments for our other chapters. In Chapter 6 we shall extend the cobwebbing concept to the complex plane using the simple quadratic (Mandelbrot) map. Finally, in Chapter 7, we shall revisit the logistic map again, constructing complex versions of it possessing various types of symmetry, leading to the recently developed concepts of symmetric chaos. This leads to some stunning imagery, discovered by Field and Golubitsky. Their text (Field and Golubitsky, 1992) is one of the most beautiful books I have ever seen. Here we will see how some of their work can be readily investigated using *Mathematica*.

Part III Traditional complex analysis

By the beginning of Part III you will have seen how to define complex numbers and how to use them to solve simple polynomial equations by both classical solution methods (and by modern iterative techniques if you have worked through Part II). In Part III you begin the study of complex functions from a formal point of view. Your goal is to understand the calculus of complex functions – differentiation, integration, series (just as in the real case) – and the very *special* results that apply to *complex* differentiable functions, in manifest distinction to the real case. The plan of this part of the text is as follows:

- Chapter 8: Complex functions;
- Chapter 9: Sequences, series and power series;
- Chapter 10: Complex differentiation;
- Chapter 11: Paths and complex integration;
- Chapter 12: Cauchy's theorem;
- Chapter 13: Cauchy's integral formula and its remarkable consequences;
- Chapter 14: Laurent series, zeroes, singularities and residues;
- Chapter 15: Residue calculus: integration, summation and the argument principle.

There are various ways of presenting and ordering this material, and it is worth explaining the particular approach taken here. Our approach is to give a first introduction to standard functions in Chapter 8, by extension of their definitions for real variables. Next, in Chapter 9, we assume some basic results from real analysis related to sequences and series. A summary of results about sequences and series are presented without formal proof. Students of pure mathematics should consult a good calculus or basic real analysis text for background on this (a comprehensive text is the book by Rudin, 1976). Then we define power series for complex functions, and establish their convergence within a circle of convergence. Then, in Chapter 10, differentiability is introduced. The approach to complex differentiability is based on the notion of a local linear approximation to a function – equivalent to the notion that there is a tangent to a complex curve. The definition quite frequently given, based on the quotient formula, is given as an aside. There are several very good reasons for this approach. First, *the quotient formula for the derivative does not work for functions of two or more real or complex variables*, so if we were to take this approach we could not sensibly relate complex differentiation to differentiation of functions of two real variables, nor can we make a generalization to functions of several complex (or real) variables without starting again with the linear approximation approach. I think it is better to do it properly in the first place. Second, the standard properties of derivatives such as the product, ratio and chain rules are really easy to write down within the linear approximation framework. Once differentiability has been defined, the differentiability of a power series within the circle of convergence is then established immediately. We then redefine our standard basic functions in terms of power series – their differentiability properties are then obvious.

Chapter 10 also includes a discussion of the theorem that the author has tentatively called the 'Ahlfors-Struble' theorem. This is the means by which one can recover a holomorphic function from its real part alone (or from just the imaginary part) by a purely *algebraic* method. This idea seems to have been rediscovered several times over the years. It is a very powerful technique when linked to the symbolic power of *Mathematica* and I have also given a discussion of the history, to justify crediting the result to Ahlfors and Struble, in Section 10.10.

Next, in Chapter 11, paths and integrals along paths are defined. Chapter 12 introduces the key theorem of this section – Cauchy's theorem – that certain integrals vanish identically. This is the key to the magic that follows, and a standard approach to the consequences of Cauchy's theorem is given in Chapters 13–15, culminating in the evaluation of certain integrals and series by the calculus of residues. Some of the material here can be augmented by other texts and I would recommend Rudin (1976), particularly as it also proceeds in a manner that makes the multi-variable case straightforward. I also suggest that geometrically-minded students look at Needham's (1997) beautiful book, *Visual Complex Analysis*.

Part IV Standard applications

In this part of the book you explore the basic applications of the material. Most first courses in complex variable theory include at least some of these topics, though the transform material may also find its way into other applied mathematics courses, and the basic applications to two-dimensional physics could also be useful in courses on potential theory and/or fluid dynamics. This part begins with basic conformal mapping – more advanced conformal maps are revisited in Chapter 21. Similarly, numerical issues with transforms are deferred to Chapter 20. You should note that Chapters 17–18 also discuss more advanced topics in contour integration, including the development and application of Jordan's lemma for semicircles. The plan of this part of the book is as follows:

- Chapter 16: Conformal mapping I: simple mappings and Möbius transforms;
- Chapter 17: Fourier transforms;
- Chapter 18: Laplace transforms;
- Chapter 19: Elementary applications to two-dimensional physics.

The novel features in this part of the book include the use of *Mathematica* to visualize conformal maps and their applications to potential flow. The generalization of the convolution theorem for Laplace transforms due to Efros is also presented, and the discussion of fluid dynamics includes a discussion of viscous flow and the biharmonic equation in complex form.

Part V Advanced applications

In this part of the book you may explore material that is not so frequently presented in introductory complex variable texts. There are five topics:

- Chapter 20: Numerical transform techniques;
- Chapter 21: Conformal mapping II: the Schwarz–Christoffel transformation;
- Chapter 22: Tiling the Euclidean and hyperbolic planes;
- Chapter 23: Physics in three and four dimensions I;
- Chapter 24: Physics in three and four dimensions II;.

The first three of these chapters have been added because the integration of the presentation with *Mathematica* allows a full treatment of some issues that require a combination of numerical/advanced analytical and graphical methods on a computer. With a computer one can explore the numerical treatment of transforms, the beautiful applications of the Schwarz–Christoffel transformation, and produce stunning hyperbolic tilings! Finally, in the last two chapters, you can see how complex numbers are very useful for doing physics and geometry in more than two dimensions. For example, you will discover that the Möbius transformation is not just a dry device for mapping circles and lines, but is really the mapping at the heart of Einstein's theory of special relativity. You will discover how complex numbers may be used to solve non-linear partial differential equations such as arise for the shape of a soap bubble, in a formalism – Penrose's theory of twistors – that links the nineteenth century work of Weierstrass to modern minimal surface and string theory. In the last chapter you will see at last the true power of holomorphic functions in solving the 3-D Laplace equation and the wave equation in four dimensions, again through Penrose's theory of twistors.

Some suggestions on how to use this text

In the end this is up to you, the reader, whether you are student or teacher. But in writing this material I have had several possible course threads in my mind. Let's look at a few possibilities.

A basic computer-enhanced course on complex numbers and solving equations

This might consist of Chapters 1–7. The unifying theme is the solution of equations. In the first few chapters the emphasis is on solving polynomial equations by traditional attempts at factorization, whereas in Chapters 4–7 we look at iterative methods of solution and the consequences.

A traditional mathematics course on complex analysis

As a minimum this would consist of Chapters 8–15, with parts of Chapters 1–3 for less well prepared students, and some portions of Chapters 16–19, 21–22 depending on the scope of the presentation.

For physics and engineering

Students taking a serious mathematics component could use Chapters 8–15 together with material from 16–19 and 23–24.

For a numerical programme

Students studying numerical methods could draw on material from Chapters 4–6, with 7 for fun, a review of 17–18 and then 20–21.

Material for specific courses in physics and engineering

It is hopefully evident that some topics may be useful for parts of other programmes. Obvious cases include courses on potential theory, whether in electrostatics, gravity or fluids, which frequently dip into complex variable theory. This material is available here, notably in Chapter 19, but it is to be hoped that those who dip into Chapter 19 also take a good look at Chapters 23 and 24!

Motivational mathematics

Many of the topics developed here could also be used as motivational material, perhaps for students not taking specialist mathematics, physics or engineering programs, but on more general courses. In my view, having an appreciation of the beauty, indeed the art, of mathematics is a vital component of an advanced education. The material in Part II could be extensively drawn on for such a program, in addition to snippets from other chapters.

Playing

Everybody should play! You can have fun just trying out the *Mathematica* implementations in many of the chapters. You can have even more fun by coming up with better ways of doing things than the author has done here and letting the author know.

About the enclosed CD

The enclosed CD contains three directories, entitled ‘Notebooks’, ‘MathLink’ and ‘Goodies’.

The Notebooks directory contains electronic copies of all the chapters of the text, in the form of *Mathematica* notebooks. These have been finalized in *Mathematica* 5.1 and therefore should open directly in V5.1 or later. If you are using an earlier version you will get a warning that you can ignore and open the file anyway. If you are using version 4.x or even 3.x you may find that some things do not quite work as in the text. The results from `Integrate` have now stabilized but differed in earlier versions, so you should watch out for that, particularly in the sections where you check a contour integral or work out a Schwarz–Christoffel map. There are other minor stylistic issues, such as `Conjugate` appearing in output form as the whole word 'Conjugate' in older versions, whereas now the output form is a simple star!

The MathLink directory contains *MathLink* code in the form of (a) source for any system, (b) binaries for some systems. The source consists of `.tm` files and `.c` files. A lack of resources prevents me from making immediately useful binaries for every operating system.

The Goodies directory contains encrypted information pertinent to *Mathematica* technologies beyond version 5.2 that will be made available once such technology is officially released. See below for more details on this. First I need to remark on *kernel* compatibility issues in general.

The author is unable to offer support on the code or *MathLink* issues. But I do wish to receive bug reports on kernel operation. This code started off as working in *Mathematica* 2.2, and has been updated for compatibility with 3.x, 4.x, 5.x. As *Mathematica* has been updated it has become increasingly difficult to retain total compatibility with older versions, as noted above. The evolution of the software has in fact resulted in better code for this book, as I have been driven to write code that relies less on a trick that might work in one version, and more towards code that uses the fundamental principles of *Mathematica* .

Please let me know if you find anything that does NOT work under versions 5.1 or 5.2. You are strongly encouraged to use version 5.x or later, until a new version is released. I have made an effort to explain where there is a significant different between the way this book works between major versions. As for *Mathematica* technologies beyond Version 5.2, I cannot comment on any of the details of unreleased software, but you should see the author's web site at King's College London:

`www.mth.kcl.ac.uk/staff/w_shaw.html`

and the CUP website for the book at

`www.cambridge.org/0521836263`

for updates when a new version is released, including a key to unlock the encrypted material in the 'Goodies' section of the CD. If you are using a *Mathematica* technology beyond version 5.2, please do not send me bug reports until you have first checked the CD and *then* the on-line information, as the author will do his best to ensure that the book as distributed together with the CD is future-proof.

Exercises and solutions

Each chapter ends with a collection of exercises. These consist of some requiring traditional thought and pen and paper analysis, others where you can additionally check the results with *Mathematica* and others that are entirely based on *Mathematica*. Questions entirely based on *Mathematica* are indicated by a polyhedral *Mathematica* icon, while those having some partial or optional involvement of *Mathematica* have the icon bracketed. Similarly sections of the book based primarily on the software are prefixed with the same icon.

Some problems are elementary exercises based on the material, while others are more open-ended investigations that do do not have a 'correct answer'. The author intends to make a 'Solutions to Selected Exercises' available on-line to educators at the earliest opportunity, and information of the progress on this will be available from the web sites noted above.

Acknowledgements

I need to start with the mathematics staff of Manshead School who, during the 1970s, inspired me with a love of mathematics. Then the inspirational teaching of A. F. Beardon and T. W. Körner at Cambridge got me hooked on complex analysis, and J. M. Stewart taught me how to apply it. Sir R. Penrose, F.R.S. showed me what could be done with complex variables and relativistic physics, and has provided me with more inspiration than anyone has a right to have. M. Perry set me on the course of looking at twistor models of string theory, and some of the material in Chapter 23, especially the twistor solution of the relativistic string (minimal surface) equations, arose from these studies.

Particular topics presented here have benefited from contributions from particular people. I am particularly grateful to L.N. Trefethen, F.R.S. for providing me with background material on modern approaches to conformal mapping, and to Vanessa Thomas for allowing me to use her work on tiling the hyperbolic plane. In addition, N. Hitchin, F.R.S. taught me how properly to use twistor methods in 3-D. The material of Chapter 7 was inspired by the beautiful book by M. Field and M. Golubitsky. The material in the final two chapters (23 and 24) owes a great deal to R. Penrose, who has educated me in all sorts of other matters including the hyperbolic tilings of Chapter 22, the fine details of fractals (Chapter 6) as well as everything to do with twistors. I am also grateful to J. Ockendon, F.R.S. for making me aware of the advanced applications of complex variables to fluids, so that this text, unusually for a first course, includes a discussion of viscous (biharmonic) flow. He also provoked me in several ways, most constructively by suggesting I look into ways of deducing the full structure of holomorphic functions from a purely algebraic treatment of their real (or imaginary) parts. I published a short educational note on this method (Shaw, 2004) and requested information on the history of the method. I am grateful to H. Boas, B. Margolis and others for responding to this request, and also providing some suggestions that lead to the improved *Mathematica* implementation given in Section 10.10.

Particular thanks go to N. Woodhouse and S. Howison for professional support during a difficult time coinciding with the latter stages of preparing this book. S. Howison, B. Hambly, J. Dewynne, A. Ilhan and C. Reisinger have all kept me sane with their considerable support on my real job of running a graduate programme in mathematical finance in Oxford.

On the *Mathematica* side I am indebted to S. Wolfram and the staff of Wolfram Research for providing this wonderful software, and to C. Wolfram, T. Wickham Jones and the UK team for particular support on numerous activities. I have used various programming devices borrowed from several other people over the years, and I am sure that ideas developed by P. Abbott and R. Maeder have found their way into several bits of code here and there! I also thank numerous unnamed individuals in technical support for help over the years, in patiently answering my questions. Gratitude is expressed to T. Gayley for compiling my *MathLink* binaries to run under a certain operating system whose name I shall not speak.

I also have to thank many students of Balliol and St Catherine's Colleges, Oxford for using and correcting various versions of this material. There were too many contributions to name everybody, but special thanks are due to Rosie Bailey, Robin Oliver-Jones and Elizabeth Lang. I also thank Frances Kirwan and Keith Hannabuss for allowing me to teach this topic at Balliol.

Several others are mentioned at specific places in the text. I apologize to anyone I have left out.

There are others I need to thank whose names I do not know. These are the anonymous reviewers who have patiently and thoroughly commented on this text during various revisions, and made numerous helpful comments. I think I agreed with all they said, and were it not for them a few important things would have been left out, and chapters presented in a more haphazard order.

This text was initially inspired by some discussions with Karen Mosman about ten years ago. Since then I have taken far too long to finish it. But it is better for the wait, in part because computer speed has rather changed since I started. First drafts of chapters were worked out on a 66 MHz machine, and the final draft was edited on a 1.4 GHz machine (already 'one-third speed' by standards as of late 2005). This means that many ideas for interactive work now work, well, interactively, rather than over a coffee break. I also learnt a lot from others about how to do things better in the meantime. *Mathematica* also got much better over time, particularly in regard to typesetting and the consistency of symbolic integration. These comments will hopefully go some way to appeasing my CUP overall editor, David Tranah, whose patience has been tested, and to whom I am indebted for his support and views. David Hemsley took on the task of editing the manuscript, and set me numerous challenges in formatting the book. I have not managed to do all the things he wished – typesetting glitches that remain are all my fault. Production was taken on by Jayne Aldhouse's team at CUP.

To finish this book I have neglected many people, notably my patient wife Helen and my impatient son Benjamin, who is worryingly interested in numbers. It is to them that this text is dedicated.