Singular Integrals and Geometric Measure Theory: towards a solution of David–Semmes problem

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The boundedness of the Riesz operator (whose kernel is the gradient of the fundamental solution for Laplacian in $\mathbb{R}^d$) in L^2 with respect to $d-1$ dimensional Hausdorff measure must imply the rectifiability of this measure. This statement became known as David–Semmes problem. Guy David and Steven Semmes devoted two books to it. But it has been proved only for $d = 2$, first by Mattila–Melnikov–Verdera for the case of homogeneous set, and later by Tolsa in a non-homogeneous situation. The non-homogeneous situation for $d = 2$ also involves relations between beta numbers of Peter Jones and Menger's curvature. However, Menger's curvature is "cruelly missing" (by the expression of Guy David) in dimensions $d > 2$. In a recent work of Nazarov–Tolsa–Volberg the conjecture of David and Semmes has been validated. The proof (which does not involve Menger's curvature) gives a new and much different proofs of the above mentioned results also in the case $d = 2$. It is a long and not-so-easy paper. But we will split it to small pieces and try to present the main ideas of the proof. The material does not require any special knowledge of Singular Integrals Theory, and should be quite accessible to graduate students who have attended the course on functional analysis and know what a subharmonic function is. The result can be cast in the language of the existence of bounded harmonic vector fields in certain (infinitely connected) domains. In fact, our result is a certain co-dimension 1 claim. In higher co-dimensions the problem (which we will explain) rests open.