Rectifiability and Singular Integrals: solving David–Semmes' problem

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Meinen Dank für eine Gelegenheit, hier zu sprechen. Durch Erlaubnis von Studenten baldigst ich gebrauch eine Gelegenheit zu kommunizieren einer Untersuchung über Verbindung von Singulären Integrale und Lipschitz Geometry. Das ist ein Gegenstand welches David und Semmes demselben längere Zeit geschenkt haben.

Unsere Beweis auf tiefliegenden Ergebnis des David–Semmes beruhen, wie Sie sehen werden. Indess zwei neue Ideen wurden dazu beigetragen.

We are interested in the following singular Riesz transforms:

$$R^s\phi(x) = \int R^s(x-y)f(y)\,d\mu(y)$$

understood as a Calderón-Zygmund operator. Here $x, y \in \mathbb{R}^{d+1}$, $s \in (0, d+1]$,

$$R^s(x) = \frac{x}{|x|^{s+1}}\,, R^s(x) = (R^s_1, \ldots, R^s_{d+1})\,, R^s_j(x) = \frac{x_j}{|x|^{s+1}}\,.$$

and μ is an Ahlfors–David (AD) regular measure in $\mathbb{R}^{d+1}$ meaning that

$$c\, r^s \le \mu(B(x,r)) \le C\, r^s$$

for all x in support of μ and all $r \le \mathrm{diam}\, E$, where $E := \mathrm{supp}\,\mu$.

frame 2.

Conjecture

If operator R_s (this is actually $d+1$ operators) is bounded in $L^2(\mu)$ then

- *1) s is integer;*
- *2) if $s = m$ is already integer, then support E of μ is m-rectifiable.*

Definition

Set E in $\mathbb{R}^{d+1}$ is called m-rectifiable, if there are $\{\Gamma_n\}_{n=1}^{\infty}$ Lipschitz images of R^m, so that $\mathcal{H}^m(E \setminus \bigcup_{n=1}^{\infty} \Gamma_n) = 0$.

The conjecture belongs to David and Semmes. For a special case $s = d$ it got a lot of attention. In particular because of its relations with regularity of solutions of Laplace equation. For a long time it is remained open even for the case $d = 1$ (and $1 < s < 2$). For $d = 1, s = 1$ it was done by Mattila–Melnikov–Verdera, Tolsa.... .
For $s = 1$ Menger's curvature tool was available. It is "cruelly missing" for $s > 1$.
We present here a case of arbitrary d and $s = d$. That is the case of co-dimension 1.
The case $0 < s \leq 1$ can be treated using Menger's curvature. This has been done by Laura Prat, Xavier Tolsa.
The case $d < s \leq d + 1$ was solved by Eiderman–Nazarov–Volberg recently. New tools of Riesz energy were needed. We start with these tools here.

We skip index $s=d$, that is we write
$R := R^d(x) = (R_1, \dots, R_{d+1})\,, R_j := R_j^d(x) = \frac{x_j}{|x|^{d+1}}$.
Given a hyperplane H on which $x_{d+1} = \text{const}$ we consider
$R^H := R^s(x) = (R_1^s, \dots, R_d^s)$ and notice that operator R^{H*} acts on vector fields: let $\psi = (\psi_1, \dots, \psi_d)$ be an $L^p(m_d)$ vector function on H. Then $R^{H*}\psi = R_1(\psi_1\, dm_d) + \dots R_d(\psi_d\, m_d)$, where m_d is Lebesgue measure on H.

Riesz Energy. We wish to give the estimate from below for the expression

$$\mathcal{E}(f, E) := \int_H (Rf)^2(x) f(x)\, dm_d(x)\,,$$

where $E \subset H$ and $0 \le f \le 1$ function supported on E. We want to give the estimate from below of $\mathcal{E}(f, E)$ in terms of
$|E| := \mathcal{H}^d(E) = m_d(E) < \infty$ and

$$\mathrm{mass} := \mathrm{mass}(f\, dm_d) = \mathrm{mass}(f\, d\mathcal{H}^d) := \int_H f\, dm_d\,.$$

frame 5.

Theorem

$\mathcal{E}(f, E) \geq c_d \frac{(\mathrm{mass})^5}{|E|^4}$, *where* $c_d > 0$.

To do that we want first the following vector field ψ on H:

- $\int_H |\psi| \, dm_d \leq C_1 < \infty$;
- $\int_H |\psi|^2 \, dm_d \leq C_2 < \infty$;
- $R^{H*}\psi(x) = 1$, m_d a. e. on E .

To do this put $\psi_0 = 0$, $\phi_0 = \chi_E$,

$$\chi_E R^*(\psi_{n+1} - \psi_n) = \phi_n - \phi_{n+1}$$

and

$$\psi_{n+1} - \psi_n = \chi_{\{R\phi_n > A^{-n}\}} R\phi_n \,,$$

where $A := 2 + \varepsilon$ will be chosen momentarily. Here we use R for R^H temporarily

frame 6.

Then

$$\phi_{n+1} = \chi_E R^*(\chi_{\{R\phi_n \le A^{-n}\}} R\phi_n)$$

By induction (using that $m_d(E) < \infty$) $\|\phi_{n+1}\|_2 \le C\|\phi_{n+1}\|_4 \le \left(\int |R\phi_n|^2 A^{-2n}\right)^{1/4} \le C2^{-n/2}A^{-n/2} \le 2^{-n-1}$.
Automatically ψ_n converges in $L^2(H, m_d)$ (see the previous slide). But also in $L^1(H, m_d)$. In fact,

$$\int_H |\psi_{n+1} - \psi_n| \, dm_d \le C\|\phi_n\|_2 |\{R\phi_n > A^{-n}\}|^{1/2} \le C\, 2^{-n}(2^{-2n}A^{2n})^{1/2} =$$

$= C\, 4^{-n}A^n \le C\, q^n$, and $q < 1$ if $A < 4$. Hence, $\psi := \lim_n \psi_n$ is in $L^1(H) \cap L^2(H)$. As $\psi_0 = 0, \phi_0 = \chi_E$, we use the previous slide:

$$\chi_E R^* \psi_N = \chi_E - \phi_N\,.$$

Going to the limit we get $R^*\psi = 1$ on E.

frame 7.

Suppose that $\mathcal{E}(f, E) < \lambda \cdot \mathrm{mass}$ with a very small λ. Let $H = \{x_{d+1} = 0\}$. Consider new measure

$$d\nu := f(x) dm_d \times \delta^{-1} \chi_{[0,\delta]} dx_{d+1} \,.$$

Lemma

$\int |R^H \nu|^2 \, d\nu \to \mathcal{E}(f, E)$ *when* $\delta \to 0$.

In fact, notice that given intervals I containing 0 and of length δ we have for almost every $x \in E \subset H$:

$$\lim_{\delta \to 0} \sup_I \left| \frac{1}{|I|} \int_I (R^H f)(x, x_{d+1}) \, dx_{d+1} - (R^H f)(x, 0) \right| = 0 \,.$$

Moreover this convergence is dominated by $L^2(H)$ majorant.

frame 8.

Therefore,

$$\lim_{\delta\to 0}\sup_{x_{d+1}\in[0,\delta]}\left|\int_E |(R^H f)(x,x_{d+1})|^2\,dm_d - \int_E |(R^H f)(x,0)|^2\,dm_d\right| = 0\,,$$

which immediately means that

$$\int |R^H\nu|^2\,d\nu \to \mathcal{E}(f,E)\,.$$

Lemma is proved. Hence we can assume that

$$\mathcal{E}(\nu) := \int |R^H\nu|^2\,d\nu < \lambda\cdot \mathrm{mass}(\nu)\,. \tag{1}$$

frame 9.

Now we will estimate the **Riesz Energy** $\mathcal{E}(\nu)$ from below. For that purpose introduce functional on functions $a \in L^\infty(\nu)$:

$$\mathcal{H}(a) := \lambda \|a\|_\infty \mathrm{mass}(\nu) + \int |R^H(a\,d\nu)|^2\, a d\nu \to \ \mathrm{minimum}$$

under the assumptions $a \geq 0$, $\mathrm{mass}(a\,d\nu) = \mathrm{mass}(\nu)$. The minimum is attained. In fact, let $\{a_k\}$ be a minimizing sequence.

$$\lambda \|a_k\|_\infty \mathrm{mass} \leq \mathcal{H}(a_k) \leq \mathcal{H}(1) = \lambda\, \mathrm{mass} + \mathcal{E}(\nu) < 2\lambda\, \mathrm{mass}$$

by the assumption of the previous slide. Therefore, $\|a_k\|_\infty \leq 2$. WLOG $a_k \to a \in L^\infty(\nu)$ weakly. So

- $\|a\|_\infty \leq \liminf_k \|a_k\|_\infty$.
- $R^H(a_k\, d\nu)$ are uniformly in any $L^p(\nu)$ ($p = 4$, say).
- For every compact subset $S \subset \mathrm{supp}\, \nu$ we can conclude that $R^H(a_k\, d\nu)(x)$ converge to $R^H(a\, d\nu)(x)$ uniformly for $x \in S$.

This last assertion follows from the observation that the set $\{R^H(x-\cdot)\}_{x\in S}$ is a continuos image of the compact set S into $L^1(\nu)$, and hence, it is compact in $L^1(\nu)$. Integrating it with $a_k(x)$ that converges weakly to a in $L^\infty(\nu)$, we obtain the uniform convergence on S. The existence of a minimizer a, $\|a\|_\infty \le 2$ and $\mathcal{H}(a) \le 2\lambda\,\mathrm{mass}$ is very important. Denote

$$\nu_a := a\,d\nu$$

and let U be a set, where $a > 0$. Denote

$$\nu_{a_t} := a(1 - t\chi_U)\nu\,.$$

$$\mathcal{H}(a_t) = \mathcal{H}(a) - t\Big[\int_U |R^H\nu_a|^2\,d\nu_a + 2\int_U R^{H*}\left[(R^H\nu_a)d\nu_a\right]d\nu_a\Big] + o(t^2)\,.$$

The mass of ν_{a_t} is $(\mathrm{mass} - t\nu_a(U))$, therefore a_t is not admissible. To make it admissible consider $\frac{\mathrm{mass}}{\mathrm{mass} - t\nu_a(U)}a_t = \left(1 - t\frac{\nu_a(U)}{\mathrm{mass}}\right)^{-1}a_t$.

Then

$$\mathcal{H}(a) \le \mathcal{H}\Big(\frac{\mathrm{mass}}{\mathrm{mass} - t\nu_a(U)} a_t\Big) \le \Big(1 - t\frac{\nu_a(U)}{\mathrm{mass}}\Big)^{-3} \mathcal{H}(a_t) \le \mathcal{H}(a) +$$
$$t\Big[3\frac{\nu_a(U)\mathcal{H}(a)}{\mathrm{mass}} - \Big(\int_U |R^H\nu_a|^2\, d\nu_a + 2\int_U R^{H*}\big[(R^H\nu_a)d\nu_a\big]\, d\nu_a\Big)\Big] + o(t^2)$$

This immediately implies:

$$\int_U |R^H\nu_a|^2\, d\nu_a + 2\int_U R^{H*}\big[(R^H\nu_a)d\nu_a\big]\, d\nu_a \le 3\nu_a(U)\frac{\mathcal{H}(a)}{\mathrm{mass}}\,.$$

This holds for every U on which a is strictly positive. We use also $\mathcal{H}(a) \le 2\lambda\,\mathrm{mass}$. Then pointwisely

$$|R^H\nu_a|^2 + 2R^{H*}\big[(R^H\nu_a)d\nu_a\big] \le 6\lambda$$

on $O := \{x \in \mathbb{R}^{d+1} : a > 0\}$. But all functions here are continuous (this is why we replaced $f\, dm_d$ by “mollified” ν). So this holds on $\mathrm{clos}\, O$.

frame 12.

However, $R^H \mu$ is harmonic outside of the support of μ for any μ. In our case $\mu = \nu_a$ and $\mathrm{supp}\, \nu_a = \mathrm{clos}\, O$. All functions above are subharmonic and continuos. **Maximal Principle** shows now that

$$|R^H \nu_a|^2 + 2R^{H*}\left[(R^H \nu_a) d\nu_a\right] \leq 6\lambda \tag{2}$$

is true **everywhere** in $\mathbb{R}^{d+1}$. In particular, it is true on H on which ψ lives. Integrate (2) with respect to $|\psi|\, dm_d$. We remember:

- $\int_H |\psi|\, dm_d \leq C_1 < \infty$;
- $\int_H |\psi|^2\, dm_d \leq C_2 < \infty$;
- $R^{H*}\psi(x) \geq 1$, m_d a. e. on $N(E)$.

From the very beginning we can think that E is bounded (somehow) and open. Then we can mollify ψ to keep first two claims and to to have the third one to hold in a small neighborhood $N(E)$ of E in $\mathbb{R}^{d+1}$.

We get

$$\int |R^H(\nu_a)|^2|\psi|\,dm_d \le 6C_1\lambda + \Big|\int R^H(|\psi|\,dm_d)\cdot R^H(\nu_a)d\nu_a\Big|$$

$$\le 6C_1\lambda + \sqrt{2}\mathcal{H}(a)^{1/2}\Big(\int |R^H(|\psi|\,dm_d)|^2\,d\nu\Big)^{1/2}$$

The last integral can be taken "layer" by "layer" as $d\nu = d\,dm_d \times \delta^{-1}dx_{d+1}$. On each layer we use that R^H is bounded in $L^2(m_d)$. Hence, we continue

$$\int |R^H(\nu_a)|^2|\psi|\,dm_d \le 6C_1\lambda + 2\lambda^{1/2}\mathrm{mass}^{1/2}\Big(\int_H |\psi|^2\,dm_d\Big)^{1/2}.$$

Temporarily normalize by $|E|\le 1 \Rightarrow \mathrm{mass}\le 1$. Then Cauchy inequality gives $\mathrm{mass} = \mathrm{mass}(\nu_a) \le |\int R^{H*}\psi\,d\nu_a| =$

$$|\int R^H(\nu_a)\psi\,dm_d| \le \int |R^H(\nu_a)||\psi|\,dm_d \le C(\lambda+\lambda^{1/2})^{1/2} \le C\lambda^{1/4}.$$

Therefore, using the assumption $|E| \leq 1$ we finally get the estimate on λ from below $\lambda \geq c(\mathrm{mass})^4 = c(\int d\, dm_d)^4$. This gives us immediately the following estimate on the Riesz energy from below (see (1) with minimal λ):

$$\mathcal{E}(f, E) \geq c\Big(\int f\, dm_d\Big)^5 .$$

To get rid of the assumption $|E| \leq 1$ we just use the scaling invariance to get

$$\mathcal{E}(f, E) \geq c\Big(\frac{\int f\, dm_d}{|E|}\Big)^4 \int f\, dm_d = \frac{\mathrm{mass}^5}{|E|^4} . \qquad (3)$$

Theorem 3 is proved.

frame 15. David-Semmes lattice

Let μ be a d-dimensional AD regular measure in $\mathbb{R}^{d+1}$. Let $E = \mathrm{supp}\mu$. then there exists a family $\mathcal{D}$ of sets $Q \subset \mathbb{R}^{d+1}$ with the following properties:

- The family $\mathcal{D}$ is the union of families $\mathcal{D}_k$ (families of level k cells), $k \in \mathbb{Z}$.
- If $Q', Q'' \in \mathcal{D}_k$, then either $Q' = Q''$ or $Q' \cap Q'' = \varnothing$.
- Each $Q' \in \mathcal{D}_{k+1}$ is contained in some $Q \in \mathcal{D}_k$ (necessarily unique due to the previous property).
- The cells of each level cover E, i.e., $\cup_{Q \in \mathcal{D}_k} Q \supset E$ for every k.
- For each $Q \in \mathcal{D}_k$, there exists $z_Q \in Q \cap E$ (the "center" of Q) such that
$$B(z_Q, 2^{-4k-3}) \subset Q \subset B(z_Q, 2^{-4k+2}) \,.$$
- For each $Q \in \mathcal{D}_k$ and every $\varepsilon > 0$, we have
$$\mu\{x \in Q : \mathrm{dist}(x, \mathbb{R}^{d+1} \setminus Q) < \varepsilon 2^{-4k}\} \leq C\varepsilon^\gamma \mu(Q); C, \gamma = C, \gamma(d, reg)$$

frame 16. Sketch of construction

Since all cells in $\mathcal{D}_k$ have approximately the same size 2^{-4k}, it will be convenient to introduce the notation $\ell(Q) = 2^{-4k}$ where k is the unique index for which $Q \in \mathcal{D}_k$.

Let Z_k be a maximal 2^{-4k}-separated set in $E = \mathrm{supp}\mu$. Then $\{B(z, 2^{-4k})\}_{z \in Z_k}$, cover E. For each $z \in Z_k$ consider Voronoi cell

$$V_z := \{x \in \mathbb{R}^{d+1} \colon |x - z| = \min_{z' \in Z_k} |x - z'|\} \,.$$

Then 1) $V_z \subset B(z, 2^{-4k})$, 2) $\{V_z\}_{z \in Z_k}$ cover E,
3) $\mathrm{dist}(z, \bigcup_{z' \in Z_k, z' \neq z} V_{z'}) \geq 2^{-4k-1}$. The last one because Z_k is 2^{-4k}-separated, the first one because Z_k is maximal such. Also
4) There are only finitely many $w \in Z_{k-1}$ such that $V_z \cap V_w \neq \emptyset$.
We say that $w \in Z_k$ is a descendant of $z \in Z_\ell$, $\ell \geq k$, if there is a chain $z_k = z, z_\ell = w$, $z_j \in Z_j$ such that $V_{z_j} \cap V_{z_{j+1}} \neq \emptyset$. $D(z)$ is the set f all descendants of z and

$$\widetilde{V}_z := \bigcup_{w \in D(z)} V_w \,.$$

Note that $\widetilde{V}_z$ contains V_z and is contained in the $2\sum_{\ell>k} 2^{-4\ell} = \frac{2}{15}2^{-4k}$-neighborhood of V_z. Thus,

$$\mathrm{dist}(z, \cup_{z'\in Z_k\setminus\{z\}} \widetilde{V}_{z'}) \geq 2^{-4k-1} - \frac{2}{15}2^{-4k} > 2^{-4k-2} . \qquad (4)$$

Nobility order. There exists a partial order $\prec$ on $\cup_k Z_k$ such that each Z_k is linearly ordered under $\prec$ and the ordering of Z_{k+1} is consistent with that of Z_k in the sense that if $z', z'' \in Z_{k+1}$ and $z' \prec z''$, then for every $w' \in Z_k$ such that $V_{w'} \cap V_{z'} \neq \varnothing$, there exists $w'' \in Z_k$ such that $V_{w''} \cap V_{z''} \neq \varnothing$ and $w' \preceq w''$. In other words, the ordering we are after is analogous to the classical "nobility order" in the society: comparing maximally "noble" ancestors one generation up defines "nobility".

Put now for each $z \in Z_k$

$$E_z := \widetilde{V}_z \setminus \bigcup_{z'\in Z_k, z\prec z'} \widetilde{V}_{z'} .$$

frame 18.

By (4) we have the left inclusion (the right one is clear too)

$$B(z, 2^{-4k-2}) \cap E \subset E_z \subset B(z, 2^{-4k+1})$$

for all $z \in Z_k$.
Next goal is to show the tiling: that for every $z \in Z_{k+1}$ there exists $w \in Z_k$ such that $E_z \subset E_w$. For a given $z \in Z_{k+1}$ choose w to be the largest in $\prec$ element of Z_k. Let $w' \in Z_k$ be such that $w \prec w'$. Let $z' \in Z_{k+1}, z' \in D(w')$. Automatically $z \prec z'$. And so $\{z' \in Z_{k+1} : z' \in D(w')\} \subset \{z' \in Z_{k+1} : z \prec z'\}$. On the other hand, by definition $V_{w'} \subset \bigcup_{z' \in Z_{k+1} :\, z' \in D(w')} V_{z'}$, and so

$$\widetilde{V}_{w'} = \bigcup_{z' \in Z_{k+1} :\, z' \in D(w')} \widetilde{V}_{z'} \subset \bigcup_{z' \in Z_{k+1}, z \prec z'} \widetilde{V}_{z'}$$

$$\widetilde{V}_z \setminus \bigcup_{z' \in Z_{k+1}, z \prec z'} \widetilde{V}_{z'} \subset \widetilde{V}_w \setminus \bigcup_{w' \in Z_k, w \prec w'} \widetilde{V}_{w'}$$

This is exactly $E_z \subset E_w$.

frame 19. Carleson families.

For us this will be the right notion of sparse, rare family of cells. From now on, we will fix a good AD regular in the entire space $\mathbb{R}^{d+1}$ measure μ and a David-Semmes lattice $\mathcal{D}$ associated with it.

Definition

A family $\mathcal{F} \subset \mathcal{D}$ is called Carleson with Carleson constant $C > 0$ if for every $P \in \mathcal{D}$, we have

$$\sum_{Q \in \mathcal{F}_P} \mu(Q) \leq C\mu(P)\,,$$

where

$$\mathcal{F}_P = \{Q \in \mathcal{F} : Q \subset P\}\,.$$

We will start with the definition of a δ-non-BAUP cell.

Definition

Let $\delta > 0$. We say that a cell $P \in \mathcal{D}$ is δ-non-BAUP if there exists a point $x \in P \cap \operatorname{supp}\mu$ such that for every hyperplane L passing through x, there exists a point $y \in B(x, \ell(P)) \cap L$ for which $B(y, \delta\ell(P)) \cap \operatorname{supp}\mu = \varnothing$.

Note that in this definition the plane L can go in any direction. In what follows, we will need only planes parallel to certain H but, since H is determined by the flatness direction of some unknown cell P, we cannot fix the direction of the plane L in the definition of non-BAUPness from the very beginning.

Theorem (David–Semmes)

Let μ be AD-regular. If for all $\delta > 0$ the family of δ-non-BAUP cells is a Carleson family, then μ is rectifiable.

frame 21. Main Theorem

Theorem

Let μ be an AD regular measure of dimension d in $\mathbb{R}^{d+1}$. If the associated d-dimensional Riesz transform operator

$$f \mapsto R * (f\mu), \quad \textit{where } R(x) = \frac{x}{|x|^{d+1}},$$

is bounded in $L^2(\mu)$, then the non-BAUP cells in the David-Semmes lattice associated with μ form a Carleson family.

Proposition 3.18 of David–Semmes 1993 (page 141) asserts that this condition "implies the WHIP and the WTP" and hence, by Theorem 3.9 (pages 137), the uniform rectifiability of μ.

Using the boundedness of R_μ in $L^2(\mu)$ we will establish **the abundance of flat cells**. On the other hand, if non-BAUP cells are not rare (not Carleson) they will be also abundant. Then we will be able to build intermitting layers of flat and non-BAUP cells. This will allow us to construct an analog of vector field ψ on non-BAUP scales. This is because non-BAUP cell has **holes** in $\mathrm{supp}\mu$ in it! Flat cells will play the role of the set E (which was totally flat). Then Riesz energy concentrated on each flat layer will be sufficiently large (the non-BAUP layer encompassing a flat layer and ψ of this non-BAUP layer ensures that). Then we will need that flat layers are **almost orthogonal**. Adding huge amount of not-so-small Riesz energies we get estimate from below on $\int |R_\mu \mathbf{1}|^2 \, d\mu$ as large as we wish. Contradiction.

frame 23. The flatness condition and its consequences

We shall fix a linear hyperplane $H \subset \mathbb{R}^{d+1}$. Let $z \in \mathbb{R}^{d+1}$, $A, \alpha, \ell > 0$ (we view A as a large number, α as a small number, and ℓ as a scale parameter). We want the measure μ to be close inside the ball $B(z, A\ell)$ to a multiple of the d-dimensional Lebesgue measure m_L on the hyperplane L containing z and parallel to H.

We say that a measure μ is geometrically (H, A, α)-flat at the point z on the scale ℓ if every point of $\mathrm{supp}\mu \cap B(z, A\ell)$ lies within distance $\alpha\ell$ from the affine hyperplane L containing z and parallel to H and every point of $L \cap B(z, A\ell)$ lies within distance $\alpha\ell$ from $\mathrm{supp}\mu$. We say that a measure μ is (H, A, α)-flat at the point z on the scale ℓ if it is geometrically (H, A, α)-flat at the point z on the scale ℓ and, in addition, for every Lipschitz function f supported on $B(z, A\ell)$ such that $\|f\|_{\mathrm{Lip}} \le \ell^{-1}$ and $\int f\, dm_L = 0$, we have

$$\left| \int f\, d\mu \right| \le \alpha \ell^d .$$

frame 24.

Note that the geometric (H, A, α)-flatness is a condition on $\mathrm{supp}\mu$ only. It doesn't tell one anything about the distribution of the measure μ on its support. The latter is primarily controlled by the second, analytic, condition in the full (H, A, α)-flatness. These two conditions are not completely independent: if, say, μ is AD regular, then the analytic condition implies the geometric one with slightly worse parameters. However, it will be convenient for us just to demand them separately.
The flatness means the possibility of mass transporting $\mu \,|\, B(z, A\ell)$ to $c \cdot m_L \,|\, B(z, A\ell)$ with small cost α.
Flatness allows to switch integration over μ to that over $c \cdot m_L$.
Below are technical but very useful lemmas estimating the error of such switching.

Lemma

Let μ be a nice measure (estimate from above). Assume that μ is (H, A, α)-flat at z on scale ℓ with some $A > 5$, $\alpha \in (0, 1)$. Let φ be any non-negative Lipschitz function supported on $B(z, 5\ell)$ with $\int \varphi \, dm_L > 0$. Put

$$a = \left(\int \varphi \, dm_L \right)^{-1} \int \varphi \, d\mu, \qquad \nu = a\varphi m_L \,.$$

Let Ψ be any function with $\|\Psi\|_{\mathrm{Lip}(\mathrm{supp}\varphi)} < +\infty$. Then

$$\left| \int \Psi \, d(\varphi\mu - \nu) \right| \leq C\alpha\ell^{d+2} \|\Psi\|_{\mathrm{Lip}(\mathrm{supp}\varphi)} \|\varphi\|_{\mathrm{Lip}} \,.$$

As a corollary, for every $p \geq 1$, we have $\left|\int |\Psi|^p \, d(\varphi\mu - \nu)\right| \leq C(p)\alpha\ell^{d+2} \|\Psi\|^{p-1}_{L^\infty(\mathrm{supp}\varphi)} \|\Psi\|_{\mathrm{Lip}(\mathrm{supp}\varphi)} \|\varphi\|_{\mathrm{Lip}}$.

Lemma

Assume in addition to the conditions of Lemma 9 that $\varphi \in C^2$, and that the ratio of integrals a is bounded from above by some known constant. Then

$$\left|\int \Psi\varphi[R^H(\varphi\mu - \nu)]\, d\mu\right|$$
$$\leq C\alpha^{\frac{1}{d+2}}\ell^{d+2}\left[\|\Psi\|_{L^\infty(\mathrm{supp}\varphi)} + \ell\|\Psi\|_{\mathrm{Lip}(\mathrm{supp}\varphi)}\right]\|\varphi\|^2_{\mathrm{Lip}}\,.$$

where $C > 0$ may, in addition to the dependence on d, which goes without mentioning, depend also on the growth constant of μ and the upper bound for a.

Disclaimer: Integral should be understood first. Split it as $\int \Psi\varphi[R^H(\varphi\mu)]\,d\mu - \int \Psi\varphi[R^H\nu]\,d\mu$. Then $R^H\nu = a\,R^H(\varphi\,dm_L)$ and so is a smooth function as φ is smooth. The first term should be understood as a form by using anti-symmetry of R^H.

The first lemma is just by definition. In the second Lemma choose $\delta = \alpha^{\frac{1}{d+2}}$ and split $R^H = R^H_{\delta\ell} + R^{H,\delta\ell}$. Then

$$\int \Psi\varphi[R^H_{\delta\ell}(\varphi\mu - \nu)]\,d\mu = -\int R^H_{\delta\ell}(\Psi\varphi\,d\mu)\,d(\varphi\mu - \nu)$$

is estimated by the first Lemma using $\|R^H_{\delta\ell}\|_{\mathrm{Lip}} \leq \delta^{-(d+1)}\ell^{-(d+1)}$ and $\|R^H_{\delta\ell}(\Psi\varphi\,d\mu)\|_{\mathrm{Lip}} \leq \|R^H_{\delta\ell}\|_{\mathrm{Lip}}\|\Psi\varphi\|_{L^1(\mu)}$.

The short range term $\int \Psi\varphi[R^{H,\delta\ell}(\varphi\mu - \nu)]\, d\mu$ essentially reduces to estimate:

$$\frac{1}{2}\Big| \iint_{|x-y|\le\delta\ell} R^H(x-y)(\Psi(x)-\Psi(y))\varphi(x)\varphi(y)\, d\mu(x)\, d\mu(y)\Big| \le$$

$$\le \frac{1}{2}\|\Psi\|_{\mathrm{Lip(supp}\varphi)}\|\varphi\|^2_{L^\infty} \iint_{x,y\in \mathrm{supp}\varphi, |x-y|<\delta\ell} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{d-1}} \le$$

$$\le C\delta\ell^{d+3}\|\Psi\|_{\mathrm{Lip(supp}\varphi)}\|\varphi\|^2_{\mathrm{Lip}}\,.$$

frame 29. Geometric Flattening Lemma

We are heading to the proof that the boundedness of R_μ in $L^2(\mu)$ implies flatness of abundant family of cells. The first step is the following analysis-to-geometry Lemma. Fix some continuous function $\psi_0 : [0, +\infty) \to [0, 1]$ such that $\psi_0 = 1$ on $[0, 1]$ and $\psi_0 = 0$ on $[2, +\infty)$. For $z \in \mathbb{R}^{d+1}$, $0 < r < R$, define

$$\psi_{z,r,R}(x) = \psi_0\left(\frac{|x - z|}{R}\right) - \psi_0\left(\frac{|x - z|}{r}\right) .$$

Lemma (Geometric Flattening Lemma)

Fix five positive parameters $A, \alpha, \beta, \widetilde{c}, \widetilde{C} > 0$. There exists $\rho > 0$ depending only on these parameters and the dimension d such that the following implication holds.
Suppose that μ is a $\widetilde{C}$-good measure on a ball $B(x, R)$ centered at a point $x \in \operatorname{supp}\mu$ that is AD regular in $B(x, R)$ with lower regularity constant $\widetilde{c}$. Suppose also that

$$|[R(\psi_{z,\delta R,\Delta R}\mu)](z)| \leq \beta$$

for all $\rho < \delta < \Delta < \frac{1}{2}$ and all $z \in B(x, (1-2\Delta)R)$ such that $\operatorname{dist}(z, \operatorname{supp}\mu) < \frac{\delta}{4}R$.
Then there exist a scale $\ell > \rho R$, a point $z \in B(x, R-(A+\alpha)\ell)$, and a linear hyperplane H such that μ is geometrically (H, A, α)-flat at z on the scale ℓ.

Replacing μ by $R^{-d}\mu(x+R\cdot)$ if necessary, we may assume without loss of generality that $x=0, R=1$.

The absence of geometric flatness and also **the boundedness** of $[R(\psi_{z,\delta,\Delta}\mu)](z)$ are inherited by weak limits. More precisely, let ν_k be a sequence of $\widetilde{C}$-good measures on $B(0,1)$ and AD-regular there with lower regularity constant $\widetilde{c}$. Assume that ν is another measure on $B(0,1)$ and $\nu_k \to \nu$ weakly in $B(0,1)$.

Lemma

- *If for some $A' > A$ and $0 < \alpha' < \alpha$, the measure ν is geometrically (H, A', α')-flat on the scale $\ell > 0$ at some point $z \in B(0, 1-(A'+\alpha)\ell)$, then for all sufficiently large k, the measure ν_k is geometrically (H, A, α)-flat at z on the scale ℓ.*
- *If for some $0 < \delta < \Delta < \frac{1}{2}$ and some $z \in B(0, 1-2\Delta)$ with $\mathrm{dist}(z, \mathrm{supp}\nu) < \frac{\delta}{4}$, we have $|[R(\psi_{z,\delta,\Delta}\nu)](z)| > \beta$, then for all sufficiently large k, we also have $\mathrm{dist}(z, \mathrm{supp}\nu_k) < \frac{\delta}{4}$ and $|[R(\psi_{z,\delta,\Delta}\nu_k)](z)| > \beta$.*

frame 32.

So suppose that with fixed 5 constants as above and with smaller and smaller ρ_k we still have μ_k's with **the absence** of geometric flatness and at the same time with **the boundedness** of $R[(\psi_{z,\delta,\Delta}\nu)](z)$, $0<\rho_k<\delta<\Delta<1/2$, for all $z\in B(0,1-2\Delta)$, $\mathrm{dist}(z,\mathrm{supp}\mu_k)<\frac{\delta}{4}$ by the same β. Then we can come to a weak limit, and get that this limit μ negates the following **Alternative**.

Alternative

If ν is any good measure on $B(0,1)$ that is AD regular there, then either for every $A,\alpha>0$ there exist a scale $\ell>0$, a point $z\in B(0,1-(A+\alpha)\ell)$ and a linear hyperplane H such that ν is geometrically (H,A,α)-flat at z on the scale ℓ, or

$$\sup_{\substack{0<\delta<\Delta<\frac{1}{2}\\ z\in B(0,1-2\Delta),\mathrm{dist}(z,\mathrm{supp}\nu)<\frac{\delta}{4}}} |[R(\psi_{z,\delta,\Delta}\nu)](z)| = +\infty\,.$$

We are left to prove the Alternative.

Sketching the proof of the Alternative

The negation of every of the two condition of the Alternative is inherited by all tangent measures of ν. Since ν is finite and AD regular in $B(0,1)$, its support is nowhere dense in $B(0,1)$. Take any point $z' \in B(0,\frac{1}{2}) \setminus \mathrm{supp}\nu$. Let z be a closest point to z' in $\mathrm{supp}\nu$. Note that since $0 \in \mathrm{supp}\nu$, we have $|z-z'| \le |z'|$, so $|z| \le 2|z'| < 1$. Also, the ball $B = B(z', |z-z'|)$ doesn't contain any point of $\mathrm{supp}\nu$. Let n be the outer unit normal to ∂B at z. Consider the blow-ups $\nu_{z,\lambda}$ of ν at z. As $\lambda \to 0$, the supports of $\nu_{z,\lambda}$ lie in a smaller and smaller neighborhood of the half-space $S = \{x \in \mathbb{R}^{d+1} : \langle x, n\rangle \ge 0\}$ bounded by the linear hyperplane $H = \{x \in \mathbb{R}^{d+1} : \langle x, n\rangle = 0\}$. So, every tangent measure of ν at z must have its support in half-space S. Thus, starting with any measure ν that gives a counterexample to the alternative we are trying to prove, we can modify it so that it is supported on a half-space. But this is impossible: either support is then on the boundary of S (then geometric flatness "almost" follows) or if otherwise, then the integral $\int_{B(0,\Delta)} \frac{\langle x,n\rangle}{|x|^{d+1}}\, d\nu(x)$ blows up.

Major step in the argument: from **geometric flatness** and the absence of large oscillation of $R^H\mu$ on $\mathrm{supp}\mu$ near some fixed point z on scales $\asymp \ell$ to the flatness of μ at z on scale ℓ.

Lemma

Fix four positive parameters $A, \alpha, \widetilde{c}, \widetilde{C}$. $\exists A', \alpha' > 0$ depending on $A, \alpha, \widetilde{c}, \widetilde{C}$ and d such that: if H is a linear hyperplane in $\mathbb{R}^{d+1}$, $z \in \mathbb{R}^{d+1}$, L is the affine hyperplane containing z and parallel to H, $\ell > 0$, and μ is a $\widetilde{C}$-good finite measure in $\mathbb{R}^{d+1}$ that is AD regular in $B(z, 5A'\ell)$ with the lower regularity constant $\widetilde{c}$. Assume that μ is geometrically $(H, 5A', \alpha')$-flat at z on the scale ℓ and, in addition, for every (vector-valued) Lipschitz function g with $\mathrm{supp} g \subset B(z, 5A'\ell)$, $\|g\|_{\mathrm{Lip}} \le \ell^{-1}$, and $\int g\, d\mu = 0$, one has

$$|\langle R^H_\mu 1, g\rangle_\mu| \le \alpha' \ell^d\,.$$

Then μ is (H, A, α)-flat at z on the scale ℓ.

The first step in proving the rectifiability of a measure is showing that its support is almost planar on many scales in the sense of the geometric $(H, 5A', \alpha')$-flatness in the assumptions of the Flattening Lemma. This step is not that hard and we will carry it next.
The second assumption involving the Riesz transform means, roughly speaking, that $R^H_\mu 1$ is almost constant on $\mathrm{supp}\mu \cap B(z, A'\ell)$ in the sense that its "wavelet coefficients" near z on the scale ℓ are small. There is no canonical smooth wavelet system in $L^2(\mu)$ when μ is an arbitrary measure but mean zero Lipschitz functions serve as a reasonable substitute. The boundedness of R^H_μ in $L^2(\mu)$ implies that $R^H_\mu 1 \in L^2(\mu)$ (because for finite measures μ, we have $1 \in L^2(\mu)$), so an appropriate version of the Bessel inequality can be used to show that large wavelet coefficients have to be rare and the balls satisfying the second assumption should also be viewed as typical.

frame 36.

Fix $A' > 1, \alpha' \in (0,1), \beta > 0$ to be chosen later. We want to show first that if $N > N_0(A', \alpha', \beta)$, then there exists a Carleson family $\mathcal{F}_1 \subset \mathcal{D}$ and a finite set $\mathcal{H}$ of linear hyperplanes such that every cell $P \in \mathcal{D} \setminus \mathcal{F}_1$ contains a **geometrically** $(H, 5A', \alpha')$-**flat** cell $Q \subset P$ at most N levels down from P for some linear hyperplane $H \in \mathcal{H}$ that may depend on P.

Let $R = \frac{1}{16}\ell(P)$. According to Geometric Flattening Lemma, we can choose $\rho > 0$ so that either
1) there is a scale $\ell > \rho R$ and a point $z \in B(z_P, R - 16[(5A'+5) + \frac{\alpha'}{3}]\ell) \subset P$ such that μ is geometrically $(H', 16(5A'+5), \frac{\alpha'}{3})$-flat at z on the scale ℓ for some linear hyperplane H',
2) or there exist $\Delta \in (0, \frac{1}{2})$, $\delta \in (\rho, \Delta)$ and a point $z \in B(z_P, (1-2\Delta)R)$ with $\mathrm{dist}(z, \mathrm{supp}\mu) < \frac{\delta}{4}R$ such that $|[R(\psi_{z,\delta R,\Delta R}\mu)](z)| > \beta$ where $\psi_{z,\delta R,\Delta R}$ is the function introduced on frame 29.

frame 37.

In the first case, take any point $z' \in \mathrm{supp}\mu$ such that $|z - z'| < \frac{\alpha'}{3}\ell$ and choose the cell Q with $\ell(Q) \in [\ell, 16\ell)$ that contains z'. Since $z' \subset B(z_P, R) \subset P$ and $\ell(Q) < \ell(P)$, we must have $Q \subset P$. Also, since $|z_Q - z'| \le 4\ell(Q)$, we have $|z - z_Q| < 4\ell(Q) + \frac{\alpha'}{3}\ell < 5\ell(Q)$. Note now that, if μ is geometrically $(H, 16A, \alpha)$-flat at z on the scale ℓ, then it is geometrically (H, A, α)-flat at z on every scale $\ell' \in [\ell, 16\ell)$.

Note also that the geometric flatness is a reasonably stable condition with respect to shifts of the point and rotations of the plane.

Applying these observations with $\ell' = \ell(Q)$, $z' = z_Q$, $\varepsilon = \frac{\alpha'}{3A}$, and choosing any finite ε-net Y on the unit sphere, we see that μ is geometrically $(H, 5A', \alpha')$-flat at z_Q on the scale $\ell(Q)$ with some H whose unit normal belongs to Y. Note also that the number of levels between P and Q in this case is $\log_{16} \frac{\ell(P)}{\ell(Q)} \le \log_{16} \rho^{-1} + C$.

Explanation of shifting and rotating

More precisely, if μ is geometrically $(H', A+5, \alpha)$-flat at z on the scale ℓ, then it is geometrically $(H, A, 2\alpha + A\varepsilon)$-flat at z' on the scale ℓ for every $z' \in B(z, 5\ell) \cap \mathrm{supp}\mu$ and every linear hyperplane H with unit normal vector n such that the angle between n and the unit normal vector n' to H' is less than ε. To see it, it is important to observe first that, despite the distance from z to z' may be quite large, the distance from z' to the affine hyperplane L' containing z and parallel to H' can be only $\alpha\ell$, so we do not need to shift L' by more than this amount to make it pass through z'. Combined with the inclusion $B(z', A\ell) \subset B(z, (A+5)\ell)$, this allows us to conclude that μ is $(H', A, 2\alpha)$-flat at z' on the scale ℓ. After this shift, we can rotate the plane L' around the $(d-1)$-dimensional affine plane containing z' and orthogonal to both n and n' by an angle less than ε to make it parallel to H. Again, no point of $L' \cap B(z, A\ell)$ will move by more than $A\varepsilon\ell$ and the desired conclusion follows.

In the second case of frame 36, there is a point $z \in B(z_P, (1-2\Delta)R)$ and a point $z' \in supp\,\mu$, such that

$$|[R(\psi_{z,\delta R,\Delta R}\mu)](z)| > \beta\,, |z - z'| < \frac{\delta}{4}R\,,$$

where

$$\psi_{z,r,R}(x) = \psi_0\left(\frac{|x-z|}{R}\right) - \psi_0\left(\frac{|x-z|}{r}\right)\,.$$

Let now Q and Q' be the largest cells containing z' under the restrictions that $\ell(Q) < \frac{\Delta}{32}R$ and $\ell(Q') < \frac{\delta}{32}R$. Since both bounds are less than $\ell(P)$ and the first one is greater than the second one, we have $Q' \subset Q \subset P$.

Now we want to show that the difference of averages of $R_\mu \chi_E$ over Q and Q' is at least $\beta - C$ in absolute value for every set $E \supset B(z, 2R)$ and, thereby, for every set $E \supset B(z_P, 5\ell(P))$. Here C depends only on the norm of operator R_μ in $L^2(\mu)$.

We can write $\chi_E = \psi_{z,\delta R,\Delta R} + f_1 + f_2$ where $|f_1|, |f_2| \le 1$ and $\mathrm{supp} f_1 \subset \bar{B}(z, 2\delta R)$, $\mathrm{supp} f_2 \cap B(z, \Delta R) = \varnothing$.

$$\int |R_\mu f_1|^2 \, d\mu \le C \int |f_1|^2 \, d\mu \le C(\delta R)^d \le C\ell(Q')^d \le C\mu(Q'),$$

Hence$|\langle R_\mu f_1 \rangle_Q|, |\langle R_\mu f_1 \rangle_{Q'}|$ are bounded by some C.
Note also that $Q \subset B(z', 8\ell(Q)) \subset B(z', \frac{\Delta}{4} R) \subset B(z, \frac{\Delta}{2} R)$, so the distance from Q to $\mathrm{supp} f_2$ is at least $\frac{\Delta}{2} R > \ell(Q)$. Thus,

$$\|R_\mu f_2\|_{\mathrm{Lip}(Q)} \le C\ell(Q)^{-1}$$

the difference of the averages of $R_\mu f_2$ over Q and Q' is bounded by some C.

frame 40.

We are left to to estimate the difference of averages $\langle R_\mu \psi_{z,\delta R,\Delta R}\rangle_Q - \langle R_\mu \psi_{z,\delta R,\Delta R}\rangle_{Q'}$. For this

$$\|R_\mu \psi_{z,\delta R,\Delta R}\|^2_{L^2(\mu)} \le C\|\psi_{z,\delta R,\Delta R}\|^2_{L^2(\mu)} \le C(\Delta R)^d \le C\ell(Q)^d \le C\mu(Q),$$

so the average over Q is bounded by a constant.
On the other hand,

$$Q' \subset B(z', 8\ell(Q')) \subset B(z', \frac{\delta}{4}R) \subset B(z, \frac{\delta}{2}R)\,.$$

Again $\mathrm{dist}(\mathrm{supp}\psi_{z,\delta R,\Delta R}, Q') \ge \frac{\delta}{2}R$. Therefore,

$$\|R(\psi_{z,\delta R,\Delta R}\mu)\|_{\mathrm{Lip}(B(z,\frac{\delta}{2}R))} \le C(\delta R)^{-1}\,.$$

But this means that $|\langle R_\mu \psi_{z,\delta R,\Delta R}\rangle_{Q'} - R_\mu \psi_{z,\delta R,\Delta R}(z)| \le C(\delta)$.
The quantity $|R_\mu \psi_{z,\delta R,\Delta R}(z)|$ is large! It is bigger than β.

We finally get

$$|\langle R_\mu \chi_E \rangle_Q - \langle R_\mu \chi_E \rangle_{Q'}| \geq \beta - 3C\,.$$

This conclusion can be rewritten as

$$\mu(P)^{-\frac{1}{2}} |\langle R_\mu \chi_E, \psi_P \rangle_\mu| \geq c \rho^{\frac{d}{2}} (\beta - C)$$

where

$$\psi_P = [\rho \ell(P)]^{\frac{d}{2}} \left(\frac{1}{\mu(Q)} \chi_Q - \frac{1}{\mu(Q')} \chi_{Q'} \right)$$

This is for any $E, B(z_P, 5\ell(P)) \subset E$.
Let us show that the set of cells satisfying the latter property can be only **rare**, that is **Carleson family**.

frame 42. Carlesonness of cells P satisfying the second case of frame 36.

Fix any cell P_0 and consider the cells P satisfying the second case of frame 36 (there exists a point z such that $|R_\mu \psi_{z,\delta R,\Delta R}(z)| > \beta$ with certain position of z inside P, $R \approx \ell(P)$, and large β). Then $B(z_P, 5\ell(P)) \subset B(z_{P_0}, 50\ell(P_0))$. Also we saw on the previous frame 41 that $\mu(P) \le C(\rho,\beta)|\langle R_\mu \chi_{B(z_{P_0},50\ell(P_0))}, \psi_P\rangle_\mu|^2$.

Here ψ_P form Haar system of depth $N \approx \log \frac{\ell(P)}{\ell(Q')}$,
$\ell(Q') \approx \delta\ell(P), \delta \in (\rho, 1/2)$, so $N \le c \log \frac{1}{\rho}$.
Any Haar system of depth N is **a Riesz system**. By the property of **Riesz system** (see frames 44–46 below) we get that

$$\sum_{P\subset P_0} \mu(P) \le C(\rho,\beta) \sum_{P\subset P_0} |\langle R_\mu \chi_{B(z_{P_0},50\ell(P_0))}, \psi_P\rangle_\mu|^2 \le$$

$$C\|R_\mu \chi_{B(z_{P_0},50\ell(P_0))}\|_\mu^2 .$$

The latter is smaller than $C\mu(B(z_{P_0}, 50\ell(P_0))) \le C'\mu(P_0)$, and we established Carleson property of P's as above.

frame 43. Abundance of geometrically flat cells is already obtained.

Fix $A, \alpha > 0$. We shall say that a cell $Q \in \mathcal{D}$ is (geometrically) (H, A, α)-flat if the measure μ is (geometrically) (H, A, α)-flat at z_Q on the scale $\ell(Q)$.
We have just shown (modulo estimates of Riesz system that follows) that there exists an integer N, a finite set $\mathcal{H}$ of linear hyperplanes in $\mathbb{R}^{d+1}$, and a Carleson family $\mathcal{F} \subset \mathcal{D}$ (depending on A, α) such that for every cell $P \in \mathcal{D} \setminus \mathcal{F}$, there exist $H \in \mathcal{H}$ and a **geometrically** (H, A, α)-**flat** cell $Q \subset P$ that is at most N levels down from P.

frame 44. Riesz systems: Haar system, Lipschitz wavelet system.

Let ψ_Q $(Q \in D)$ be a system of Borel $L^2(\mu)$ functions

Definition

The functions ψ_Q form a Riesz family with Riesz constant $C > 0$ if

$$\left\| \sum_{Q \in \mathcal{D}} a_Q \psi_Q \right\|^2_{L^2(\mu)} \leq C \sum a_Q^2$$

for any real coefficients a_Q.

Note that if the functions ψ_Q form a Riesz family with Riesz constant C, then for every $f \in L^2(\mu)$, we have

$$\sum_{Q \in \mathcal{D}} |\langle f, \psi_Q \rangle_\mu|^2 \leq C \|f\|^2_{L^2(\mu)} .$$

frame 45.

Assume next that for each cell $Q \in D$ we have a set Ψ_Q of $L^2(\mu)$ functions associated with Q.

Definition

The family Ψ_Q $(Q \in \mathcal{D})$ of sets of functions is a Riesz system with Riesz constant $C > 0$ if for every choice of functions $\psi_Q \in \Psi_Q$, the functions ψ_Q form a Riesz family with Riesz constant C.

Riesz systems are useful because of the following Lemma.

Lemma

Suppose that Ψ_Q is any Riesz system. Fix $A > 1$. For each $Q \in \mathcal{D}$, define

$$\xi(Q) = \inf_{E: B(z_Q, A\ell(Q)) \subset E, \mu(E) < +\infty} \sup_{\psi \in \Psi_Q} \mu(Q)^{-1/2} |\langle R_\mu \chi_E, \psi \rangle_\mu| .$$

Then $\forall \delta > 0, \mathcal{F}_\delta := \{Q \in \mathcal{D} : \xi(Q) \geq \delta\}$ is Carleson.

The one line proof of Lemma is on frame 42. But it is important that there are two natural classes of Riesz systems: **Haar systems of fixed depth** $\Psi^H(N)$, and **Lipschitz wavelet systems** $\Psi^L(A)$. Let now N be any positive integer. For each $Q \in \mathcal{D}$, define the set of Haar functions $\Psi^h_Q(N)$ of depth N as the set of all functions ψ that are supported on Q, are constant on every cell $Q' \in \mathcal{D}$ that is N levels down from Q, and satisfy $\int \psi\, d\mu = 0$, $\int \psi^2\, d\mu \le C$. The Riesz property follows immediately from the fact that $\mathcal{D}$ can be represented as a finite union of the sets $\mathcal{D}^{(j)} = \cup_{k:k\equiv j \mod N} \mathcal{D}_k$ ($j = 0, \dots, N-1$) and that for every choice of $\psi_Q \in \Psi^h_Q(N)$, the functions ψ_Q corresponding to the cells Q from a fixed $\mathcal{D}^{(j)}$ form a bounded orthogonal family.

Our $\psi_P = [\rho\ell(P)]^{\frac{d}{2}} \left(\frac{1}{\mu(Q)}\chi_Q - \frac{1}{\mu(Q')}\chi_{Q'}\right)$ from frame 41 are obviously from $\Psi^H(N)$ with $N \le c \log \frac{1}{\rho}$, so the main inequality of frame 42 is done, and YES the abundance of geometrically flat cells is completely established.

frame 47. Lipschitz wavelet systems.

We will need them to establish the abundance of (H, A, α) flat (not just geometrically flat) cells.

In the Lipschitz wavelet system, the set $\Psi^{\ell}_{Q}(A)$ consists of all Lipschitz functions ψ supported on $B(z_Q, A\ell(Q))$ such that $\int \psi \, d\mu = 0$ and $\|\psi\|_{\mathrm{Lip}} \le C\ell(Q)^{-\frac{d}{2}-1}$. Since μ is nice, we automatically have $\int |\psi|^2 \, d\mu \le C(A)\ell(Q)^{-d}\mu(Q) \le C(A)$ in this case.

If $Q, Q' \in \mathcal{D}$ and $\ell(Q') \le \ell(Q)$, then, for any two functions $\psi_Q \in \Psi^{\ell}_{Q}(A)$ and $\psi_{Q'} \in \Psi^{\ell}_{Q'}(A)$, we can have $\langle \psi_Q, \psi_{Q'} \rangle_\mu \ne 0$ only if $B(z_Q, A\ell(Q)) \cap B(z_{Q'}, A\ell(Q')) \ne \varnothing$, in which case,

$$|\langle \psi_Q, \psi_{Q'} \rangle_\mu| \le \|\psi_Q\|_{\mathrm{Lip}} \mathrm{diam}(Q') \|\psi_{Q'}\|_{L^1(\mu)} \le C(A) \left[\frac{\ell(Q')}{\ell(Q)} \right]^{\frac{d}{2}+1} .$$

THEN:

$$\left\| \sum_{Q\in\mathcal{D}} a_Q \psi_Q \right\|_{L^2(\mu)}^2 \le 2 \sum_{Q,Q'\in\mathcal{D},\, \ell(Q')\le\ell(Q)} |a_Q|\cdot|a_{Q'}|\cdot|\langle\psi_Q,\psi_{Q'}\rangle_\mu|$$

$$\le C(A) \sum_{\substack{Q,Q'\in\mathcal{D},\, \ell(Q')\le\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\ne\varnothing}} \left[\frac{\ell(Q')}{\ell(Q)}\right]^{\frac{d}{2}+1} |a_Q|\cdot|a_{Q'}|$$

$$\le C(A) \sum_{\substack{Q,Q'\in\mathcal{D},\, \ell(Q')\le\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\ne\varnothing}} \left\{\left[\frac{\ell(Q')}{\ell(Q)}\right]^{d+1} |a_Q|^2 + \frac{\ell(Q')}{\ell(Q)}|a_{Q'}|^2\right\}.$$

$$\sum_{\substack{Q'\in\mathcal{D}:\, \ell(Q')\le\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\ne\varnothing}} \left[\frac{\ell(Q')}{\ell(Q)}\right]^{d+1} \le C, \quad \sum_{\substack{Q\in\mathcal{D}:\, \ell(Q')\le\ell(Q)\\ B(z_Q,A\ell(Q))\cap B(z_{Q'},A\ell(Q'))\ne\varnothing}} \le C.$$

are bounded by some constants independent of Q and Q'.

frame 49.

We recall Flattening Lemma from frame 34, that says how to get flat cell if it is already geometrically flat.

Lemma

Fix four positive parameters $A, \alpha, \widetilde{c}, \widetilde{C}$. $\exists A', \alpha' > 0$ depending on $A, \alpha, \widetilde{c}, \widetilde{C}$ and d such that: if H is a linear hyperplane in $\mathbb{R}^{d+1}$, $z \in \mathbb{R}^{d+1}$, L is the affine hyperplane containing z and parallel to H, $\ell > 0$, and μ is a $\widetilde{C}$-good finite measure in $\mathbb{R}^{d+1}$ that is AD regular in $B(z, 5A'\ell)$ with the lower regularity constant $\widetilde{c}$. Assume that μ is geometrically $(H, 5A', \alpha')$-flat at z on the scale ℓ and, in addition, for every (vector-valued) Lipschitz function g with $\mathrm{supp} g \subset B(z, 5A'\ell)$, $\|g\|_{\mathrm{Lip}} \le \ell^{-1}$, and $\int g\, d\mu = 0$, one has

$$|\langle R_\mu^H 1, g\rangle_\mu| \le \alpha' \ell^d\,.$$

Then μ is (H, A, α)-flat at z on the scale ℓ.

frame 50.

We already saw that all cells P (except for a rare (Carleson) family $\mathcal{F}_1$) are such that not more than N generation down inside P a cell Q lies, which is (H, A', α')-geometrically flat, where A', α' depend on A, α as Flattening Lemma requires, and $H = H_Q$ belongs to $\mathcal{H}$, a finite family (cardinality of it depends on A, α too).

Given P, we find such Q, and Flattening Lemma applied to any $\mu := \mu \cdot 1_E, E \supset B(z_Q, 100A'\ell(Q))$, shows that either Q is (H, A, α)-flat, or for each such E there exists a function $g = g_E$ such that it is supported on $B(z_Q, 5A'\ell(Q))$, $\int g\, d\mu = 0$, Lipschitz with norm at most $1/\ell(Q) \le C(N)/\ell(P)$ and

$$\langle R^H_\mu 1_E, g\rangle \ge \alpha'\ell(Q)^d = c(N)\alpha'\ell(P)^d\,.$$

Consider $\psi_P = \psi_{P,E} = g/\ell(P)^{\frac{d}{2}}$. They form a Lipschitz wavelet system $\Psi^L(C)$, as on frame 47. Therefore,

$$\xi(P) = \mu(P)^{-\frac{1}{2}} \inf_{E:\ E \supset B(z_p, C\ell(P))} \sup_{\psi \in \Psi^L(C)} |\langle R_\mu 1_E, \psi\rangle| \ge C(N)\alpha'\,.$$

We know that such P can form only a rare (Carleson) family if R_μ is a bounded operator in $L^2(\mu)$. Call it $\mathcal{F}_2$. So by the exception of two rare families $\mathcal{F}_1, \mathcal{F}_2$, any other cell $P \in \mathcal{D}$ will have inside it and not more than N (fixed number depending on A, α) generations down a sub-cell Q, which is (H, A, α)-flat. Here H will be chosen from a finite family $\mathcal{H}$ of hyperplanes (having fixed cardinality depending on A, α).

The abundance of flat cells is completely proved.

Recall that our goal is to prove that the family of all non-BAUP cells $P \in \mathcal{D}$ is Carleson. In view of the just proved **abundance of flat cells in several fixed directions**, it will suffice to show that we can choose $A, \alpha > 0$ such that for every fixed linear hyperplane H and for every integer N, the corresponding family $\mathcal{F} = \mathcal{F}(A, \alpha, H, N)$ of all non-BAUP cells $P \in D$ containing an (H, A, α)-flat cell Q at most N levels down from P is Carleson.
The idea. Suppose this is not the case. Then there will be P from $\mathcal{F}$ =family of non-BAUP cells containing a flat cell in a fixed direction at most N generations down **such that it can be tiled (up to tiny measure) by arbitrarily large number of layers of non-BAUP cells.** Use now the abundance of flat cells. We can also tile the same cell P by layers of (H, A, α)-flat cells Q (up to tiny measure) also with as many layers as we wish.
Moreover we can alternate layers. Namely:

frame 53. Alternating non-BAUP and flat layers.

Lemma

If $\mathcal{F}$ is not Carleson, then for every positive integer K and every $\eta > 0$, there exist a cell $P \in \mathcal{F}$ and $K+1$ alternating pairs of finite layers $\mathfrak{P}_k, \mathfrak{Q}_k \subset \mathcal{D}$ $(k = 0, \dots, K)$ such that

- $\mathfrak{P}_0 = \{P\}$.
- $\mathfrak{P}_k \subset \mathcal{F}_P$ *for all* $k = 0, \dots, K$.
- *All layers* $\mathfrak{Q}_k$ *consist of* (H, A, α)*-flat cells only.*
- *Each individual layer (either* $\mathfrak{P}_k$*, or* $\mathfrak{Q}_k$*) consists of pairwise disjoint cells.*
- *If* $Q \in \mathfrak{Q}_k$*, then there exists* $P' \in \mathfrak{P}_k$ *such that* $Q \subset P'$ $(k = 0, \dots, K)$.
- *If* $P' \in \mathfrak{P}_{k+1}$*, then there exists* $Q \in \mathfrak{Q}_k$ *such that* $P' \subset Q$ $(k = 0, \dots, K-1)$.
- $\sum_{Q \in \mathfrak{Q}_K} \mu(Q) \geq (1-\eta)\mu(P)$.

Suppose $\mathcal{F}$ is not Carleson. For every $\eta' > 0$ and every positive integer M, we can find a cell $P \in \mathcal{F}$ and $M+1$ layers $\mathcal{L}_0, \dots, \mathcal{L}_M \subset \mathcal{F}_P$ that have the desired Cantor-type hierarchy and satisfy $\sum_{P' \in \mathcal{L}_M} \mu(P') \geq (1-\eta')\mu(P)$.

We will go now from the layer $\mathcal{L}_m$ to $\mathcal{L}_{m+SN}$, where $S = S(N)$ will be large and $M \approx KSN$, where K is from the previous frame. We take $P' \in \mathcal{L}_m$ and choose $Q(P')$ less than N generations down, which is flat. Those $P'' \in \mathcal{L}_{M+N}$ that are inside such $Q(P')$ we color white, the collection of $Q(P')$ we color blue. Notice that at this moment the mass of all non-colored $P'' \in \mathcal{L}_{m+N}$ is $\leq (1 - c4^{-4dN})\mu(P)$.

In those $P'' \in \mathcal{L}_{m+N}$ that are not colored again we will have $Q(P'')$ less than N generations down that are flat, color them blue, color white those $P''' \in \mathcal{L}_{m+2N}$ that are in some of $Q(P'')$. Notice that at this moment the mass of all non-colored $P''' \in \mathcal{L}_{m+2N}$ is $\leq (1 - c4^{-4dN})^2\mu(P)$.

frame 55.

Non-colored follow non-colored, and in S steps (if $S = S(N)$ is sufficiently large) the portion of $\mu(P)$ of non-colored cells become very small. Then we stop and put $m_{new} := m + SN$, we consider only the part of layer $\mathcal{L}_{m+SN}$, namely those cells of it that lie in some white colored cells. Call it $\mathcal{L}'_{m_{new}}$.
So if $\mathcal{L}_m$ was $\mathfrak{P}_k$, then $\mathcal{L}'_{m_{new}}$ will be our $\mathfrak{P}_{k+1}$.
Consider all blue cells we have on the road. Take the family of maximal blue cells out of those which we just constructed. This will be layer of disjoint (H, A, α)-flat cells and this will be our layer $\mathfrak{Q}_k$.
Given K, we choose S very large, η' very small. Then we make the error in tiling $\mu(P)$ only of the order $(K+1)[\eta' + (1 - c4^{-4dN})^S]$, which is as small as we wish. Lemma of frame 53 is proved.

Fix K. Choose $\varepsilon > 0$, $A, \alpha > 0$, $\eta > 0$ in this order. Construct layers as on frame 53. Consider flat layers $\mathfrak{Q}_k$ ignoring the non-BAUP layers $\mathfrak{P}_k$ almost entirely.
For a cell $Q \in \mathcal{D}$ and $t > 0$, define

$$Q_t = \{x \in Q : \mathrm{dist}(x, \mathbb{R}^{d+1} \setminus Q) \geq t\ell(Q)\}\,.$$

Note that $\mu(Q \setminus Q_t) \leq Ct^{\gamma}\mu(Q)$ for some fixed $\gamma > 0$. This is stated on frame 15. Let φ_0 be any C^{∞} function supported on $B(0,1)$ and such that $\int \varphi_0\, dm = 1$ where m is the Lebesgue measure in $\mathbb{R}^{d+1}$. Put

$$\varphi_Q = \chi_{Q_{2\varepsilon}} * \frac{1}{(\varepsilon\ell(Q))^d}\,\varphi_0\Big(\frac{\cdot}{\varepsilon\ell(Q)}\Big)\,.$$

Then $\varphi_Q = 1$ on $Q_{3\varepsilon}$ and $\mathrm{supp}\varphi_Q \subset Q_{\varepsilon}$. In particular, the diameter of $\mathrm{supp}\varphi_Q$ is at most $8\ell(Q)$.

In addition,

$$\|\varphi_Q\|_{L^\infty} \le 1, \quad \|\nabla\varphi_Q\|_{L^\infty} \le \frac{C}{\varepsilon\ell(Q)}, \quad \|\nabla^2\varphi_Q\|_{L^\infty} \le \frac{C}{\varepsilon^2\ell(Q)^2}.$$

From now on, we will be interested only in the cells Q from the flat layers $\mathfrak{Q}_k$. With each such cell Q we will associate the corresponding approximating plane $L(Q)$ containing z_Q and parallel to H and the approximating measure $\nu_Q = a_Q\varphi_Q m_{L(Q)}$ where a_Q is chosen so that

$$\nu_Q(\mathbb{R}^{d+1}) = \int \varphi_Q \, d\mu \,.$$

Both integrals $\int \varphi_Q \, dm_{L(Q)}$ and $\int \varphi_Q \, d\mu$ are comparable to $\ell(Q)^d$, provided that $\varepsilon < \frac{1}{48}$, say. In particular, in this case, the normalizing factors a_Q are bounded by some constant.

frame 58.

Define

$$G_k = \sum_{Q \in \mathfrak{Q}_k} \varphi_Q R^H[\varphi_Q \mu - \nu_Q], \quad k = 0, \dots, K.$$

Now put

$$F_k = G_k - G_{k+1} \text{ when } k = 0, \dots, K-1, \qquad F_K = G_K.$$

Note that

$$\sum_{m=k}^{K} F_m = G_k.$$

frame 59. "Orthogonality" of telescopic layers.

This is almost orthogonality of "errors" (errors between genuine and flat situations).

Lemma

Assuming that $\varepsilon < \frac{1}{48}$, $A > 5$, *and* $\alpha < \varepsilon^8$, *we have*

$$|\langle F_k, G_{k+1}\rangle| \leq \sigma(\varepsilon, \alpha)\mu(P)$$

for all $k = 0, \ldots, K-1$, *where* $\sigma(\varepsilon, \alpha)$ *is some positive function such that*

$$\lim_{\varepsilon \to 0+} [\lim_{\alpha \to 0+} \sigma(\varepsilon, \alpha)] = 0\,.$$

Long and difficult. Frames 25, 26 are constantly used. And the boundedness of R_μ in $L^2(\mu)$ is used.

frame 59A. A flavor of the reasoning of almost orthogonality.

This is a typical block in the proof of almost orthogonality. It is clear that it uses frame 26 and smallness of α in flatness.

Lemma

Suppose that $Q \in \mathfrak{Q}_k$. Then

$$\sum_{Q' \in \mathfrak{Q}_{k+1}, Q' \subset Q} |\langle R^H(\chi_{Q \setminus Q'} \mu), \varphi_{Q'} R^H(\varphi_{Q'} \mu - \nu_{Q'}) \rangle_\mu| \le C \alpha^{\frac{1}{d+2}} \varepsilon^{-3} \mu(Q)\,.$$

frame 60.

Now notice that $G_0 = \sum_{Q\in\mathfrak{Q}_0} \varphi_Q R^H[\varphi_Q\mu - \nu_Q]$. We want to see first that

$$\|G_0\|_\mu^2 \le C\mu(P)\,.$$

As the summands have pairwise disjoint supports, it will suffice to prove the inequality

$$\|\varphi_Q R^H(\varphi_Q\mu - \nu_Q)\|^2_{L^2(\mu)} \le C\mu(Q)$$

for each individual $Q \in \mathfrak{Q}_0$ and then observe that $\sum_{Q\in\mathfrak{Q}_k} \mu(Q) \le \mu(P)$. Of course $\|\varphi_Q R^H(\varphi_Q\mu\|^2_{L^2(\mu)} \le C\mu(Q)$ by the boundedness of R_μ. But the estimate $\|\varphi_Q R^H(\nu_Q)\|^2_{L^2(\mu)} \le C\mu(Q)$ is not so trivial because we start with flat measure $\nu_Q := a_Q\varphi_Q m_L$ but we send it by R^H into $L^2(\mu)$. Such an estimate can be obtained however by using an error estimate of frame 26.

At this point, we need to know that the non-BAUPness condition depends on a positive parameter δ. We will fix that δ from now on in addition to fixing the measure μ. Note that despite the fact that we need to prove that the family of non-BAUP cells is Carleson for every $\delta > 0$, the David-Semmes uniform rectifiability criterion does not require any particular rate of growth of the corresponding Carleson constant as a function of δ.

frame 62.

We have the identity

$$\|G_0\|^2_{L^2(\mu)} = \left\|\sum_{k=0}^{K} F_k\right\|^2_{L^2(\mu)} = \sum_{k=0}^{K} \|F_k\|^2_{L^2(\mu)} + 2\sum_{k=0}^{K-1} \langle F_k, G_{k+1}\rangle_\mu .$$

As we have seen on a couple of previous frames, $\|G_0\|^2_{L^2(\mu)} \le C\mu(P)$, and the scalar products can be made arbitrarily small by first choosing $\varepsilon > 0$ small enough and then taking a sufficiently small $\alpha > 0$ depending on ε. So we will get a contradiction if we are able to bound $\|F_k\|^2_{L^2(\mu)}$ for $k = 0, \dots, K-1$ from below by $\tau^2\mu(P)$, with some $\tau = \tau(\delta) > 0$ (as usual, the dependence on the dimension d and the regularity constants of μ is suppressed).

We choose very large K, then we choose $A > A_0(\delta), \varepsilon < \varepsilon_0(\delta), \eta < \eta_0(\varepsilon), \alpha < \alpha_0(\varepsilon, \delta)$..

frame 63. Densely packed cells.

Fix $k \in \{0, 1, \ldots, K-1\}$. We can write the function F_k as

$$F_k = \sum_{Q \in \mathfrak{Q}_k} F^Q$$

where

$$F^Q = \varphi_Q R^H(\varphi_Q \mu - \nu_Q) - \sum_{Q' \in \mathfrak{Q}_{k+1}, Q' \subset Q} \varphi_{Q'} R^H(\varphi_{Q'} \mu - \nu_{Q'}) .$$

We shall call a cell $Q \in \mathfrak{Q}_k$ densely packed if $\sum_{Q' \in \mathfrak{Q}_{k+1}, Q' \subset Q} \mu(Q') \geq (1-\varepsilon)\mu(Q)$. Otherwise we shall call the cell Q loosely packed. The main claim of this section is that the loosely packed cells constitute a tiny minority of all cells in $\mathfrak{Q}_k$ if $\eta \leq \varepsilon^2$.

frame 64.

Indeed, we have

$$\sum_{\substack{Q\in\mathfrak{Q}_k\\ Q \text{ is packed loosely}}} \mu(Q) \le \varepsilon^{-1} \sum_{Q\in\mathfrak{Q}_k} \mu\left(Q \setminus \left(\bigcup_{Q'\in\mathfrak{Q}_{k+1}, Q'\subset Q} Q'\right)\right)$$

$$= \varepsilon^{-1}\left[\sum_{Q\in\mathfrak{Q}_k}\mu(Q) - \sum_{Q'\in\mathfrak{Q}_{k+1}}\mu(Q')\right]$$

$$\le \varepsilon^{-1}\left[\mu(P) - \sum_{Q'\in\mathfrak{Q}_{k+1}}\mu(Q')\right] \le \frac{\eta}{\varepsilon}\mu(P) \le \varepsilon\mu(P)\,.$$

We can immediately conclude from here that

$$\sum_{\substack{Q\in\mathfrak{Q}_k\\ Q \text{ is densely packed}}} \mu(Q) = \sum_{Q\in\mathfrak{Q}_k} \mu(Q) - \sum_{\substack{Q\in\mathfrak{Q}_k\\ Q \text{ is loosely packed}}} \mu(Q)$$

$$\geq (1-\eta)\mu(P) - \varepsilon\mu(P) \geq (1-2\varepsilon)\mu(P)\,.$$

From now on, we will fix the choice $\eta = \varepsilon^2$.

We claim now that to estimate $\|F_k\|^2_{L^2(\mu)}$ from below by $\tau^2\mu(P)$, it suffices to show that for every densely packed cell $Q \in \mathfrak{Q}_k$, we have $\|F^Q\|^2_{L^2(\mu)} \geq 2\tau^2\mu(Q)$. To see it, just write

$$\|F_k\|^2_{L^2(\mu)} = \sum_{Q\in\mathfrak{Q}_k} \|F^Q\|^2_{L^2(\mu)} \geq \sum_{\substack{Q\in\mathfrak{Q}_k \\ Q \text{ is densely packed}}} \|F^Q\|^2_{L^2(\mu)}$$

$$\geq \sum_{\substack{Q\in\mathfrak{Q}_k \\ Q \text{ is densely packed}}} 2\tau^2\mu(Q) \geq 2(1-2\varepsilon)\tau^2\mu(P) \geq \tau^2\mu(P)\,,$$

provided that $\varepsilon < \frac{1}{4}$.

From now on, we will fix $k \in \{0, \dots, K-1\}$ and a densely packed cell $Q \in \mathfrak{Q}_k$. We denote by $\mathfrak{Q}$ the set of all cells $Q' \in \mathfrak{Q}_{k+1}$ that are contained in the cell Q.

Lemma

The goal of this section is to show that there exists a subset $\mathfrak{Q}'$ of $\mathfrak{Q}$ such that $\sum_{Q' \in \mathfrak{Q}'} \mu(Q') \geq (1 - C\varepsilon)\mu(Q)$ and

$$\|F^Q\|_{L^2(\mu)} \geq \frac{1}{2}\|R^H(\nu - \nu_Q)\|_{L^2(\nu)} - \sigma(\varepsilon, \alpha)\sqrt{\mu(Q)},$$

where $\nu = \sum_{Q' \in \mathfrak{Q}'} \nu_{Q'}$ and $\sigma(\varepsilon, \alpha)$ is some positive function such that $\lim_{\varepsilon \to 0+}[\lim_{\alpha \to 0+} \sigma(\varepsilon, \alpha)] = 0$.

The proof is long and technical, but looking at

$$F^Q = \varphi_Q R^H(\varphi_Q \mu - \nu_Q) - \sum_{Q' \in \mathfrak{Q}_{k+1}, Q' \subset Q} \varphi_{Q'} R^H(\varphi_{Q'} \mu - \nu_{Q'})$$

we see that the claim is at least natural, as it says that μ "cancels" out inside F_Q, and we also replace μ outside in $\|F^Q\|_{L^2(\mu)}$ by the measure ν consisting of flat pieces parallel and close to flat ν_Q.

The statement of the lemma holds with

$$\sigma(\varepsilon, \alpha) = C[\varepsilon^{\frac{\gamma}{4}} + \alpha^{\frac{1}{2}} \varepsilon^{-\frac{2d+3}{2}} + \alpha \varepsilon^{-d-3}] .$$

frame 69. Next measure modification. Reflection trick.

Fix a hyperplane L parallel to H at the distance $2\Delta\ell(Q)$ from $\mathrm{supp}\mu \cap Q$. Number Δ is small compared to ε and large compared to α. Let S be the (closed) half-space bounded by L that contains $\mathrm{supp}\mu \cap Q$. For $x \in S$, denote by x^* the reflection of x about L. Define the kernels

$$\widetilde{R}^H(x,y) = R^H(x-y) - R^H(x^*-y), \qquad x, y \in S$$

and denote by $\widetilde{R}^H$ the corresponding operator. We will assume that $\alpha << \Delta$, so the approximating hyperplanes $L(Q')$ $(Q' \in \mathfrak{Q}')$ and $L(Q)$, which lie within the distance $\alpha\ell(Q)$ from $\mathrm{supp}\mu \cap Q$ are contained in S and lie at the distance $\Delta\ell(Q)$ or greater from the boundary hyperplane L.

Lemma

The goal of this section is to show that, for some appropriately chosen $\Delta = \Delta(\alpha, \varepsilon) > 0$, and under our usual assumptions about ε, A, and α, we have

$$\|R^H(\nu - \nu_Q)\|_{L^2(\nu)} \geq \|\widetilde{R}^H \nu\|_{L^2(\nu)} - \sigma(\varepsilon, \alpha)\sqrt{\mu(Q)}$$

where, again, $\sigma(\varepsilon, \alpha)$ is some positive function such that

$$\lim_{\varepsilon \to 0+} [\lim_{\alpha \to 0+} \sigma(\varepsilon, \alpha)] = 0\,.$$

Let T be operator with kernel $R^H(x^* - y)$, $\widetilde{R}^H$ with $R^H(x - y) - R^H(x^* - y)$. To compare $\|R^H(\nu - \nu_Q)\|$ with $\|\widetilde{R}^H \nu\|$ one needs to estimate $\|R^H \nu_Q - T\nu\|$, so one needs two smallnesses: 1) $\|R^H \nu_Q - T\nu_Q\|$; 2) $\|T(\nu - \nu_Q)\|$ (all norms in $L^2(\nu)$).

frame 71. Explanation of $\|R^H\nu_Q - T\nu_Q\|$ smallness.

Elementary estimates:

a) $\|R^H\nu_Q\|_{\mathrm{Lip}} \le \frac{C}{\varepsilon^2\ell(Q)}$ (smoothness of φ_Q is needed),

therefore $b) \, |R^H\nu_Q(x) - T\nu_Q(x)| \le \|R^H\nu_Q\|_{\mathrm{Lip}} |\Delta\ell(Q)| \le \dfrac{C\Delta}{\varepsilon^2}$.

Thus, $\|R^H\nu_Q - T\nu_Q\|_{L^2(\nu)} \le \frac{C\Delta}{\varepsilon^2}\sqrt{\mu(Q)}$.

frame 72. Explanation of $\|T(\nu-\nu_Q)\|$ smallness.

Obviously

$$\|R^H(\cdot^* - y)\|_{L^\infty(S)} \le \frac{1}{\Delta^d \ell(Q)^d} \text{ and } \|R^H(\cdot^* - y)\|_{\mathrm{Lip}(S)} \le \frac{C}{\Delta^{d+1}\ell(Q)^{d+1}}$$

Hence,

$$\begin{aligned}
&\|T(\nu_Q-\nu)\|_{\mathrm{Lip}(S)} \\
&\quad\le \sup_{y\in(\mathrm{supp}\nu\cup\mathrm{supp}\nu_Q)} \|R^H(\cdot^* - y)\|_{\mathrm{Lip}(S)}(\|\nu\|+\|\nu_Q\|)) \\
&\quad\le \frac{C}{\Delta^{d+1}\ell(Q)^{d+1}}\mu(Q) \le \frac{C}{\Delta^{d+1}\ell(Q)}\,.
\end{aligned}$$

Similarly, $\|T(\nu_Q-\nu)\|_{L^\infty(S)} \le \frac{C}{\Delta^d}$. Thus, by frame 25

$$\begin{aligned}
\left|\int |T(\nu_Q-\nu)|^2\, d(\varphi_{Q'}\mu - \nu_{Q'})\right| &\le C\alpha\ell(Q')^{d+2}\frac{1}{\Delta^d}\frac{1}{\Delta^{d+1}\ell(Q)}\frac{1}{\varepsilon\ell(Q')} \\
&\le C\alpha\Delta^{-2d-1}\varepsilon^{-1}\ell(Q')^d \le C\alpha\Delta^{-2d-1}\varepsilon^{-1}\mu(Q')\,.
\end{aligned}$$

Now we want to sum up over $Q'\in\mathfrak{Q}'$, where the last set was mentioned on frame 67: for our goals now it can be considered as the whole collection of $Q'\in\mathfrak{Q}_{k+1}$ lying inside our $Q\in\mathfrak{Q}_k$. Let $\Phi:=\sum_{Q'\in\mathfrak{Q}'}\varphi_{Q'}$. Summing over $Q'\in\mathfrak{Q}'$, we get

$$\int|T(\nu_Q-\nu)|^2\,d\nu\le\int|T(\nu_Q-\nu)|^2\,d(\Phi\mu)+C\alpha\Delta^{-2d-1}\varepsilon^{-1}\mu(Q)\,,$$

The last integrand we write as

$$T(\nu_Q-\nu)=(T\nu_Q-T\varphi_Q\mu)+(T\varphi_Q\mu-T\Phi\mu)+(T\Phi\mu-T\nu)\,.$$

We estimate the smallness of each term in $L^2(\Phi\mu)$ (or, which is practically the same, $L^2(\mu_Q)$) on the next 3 slides.

frame 74. Explanation of $\|T(\varphi_Q\mu - \nu_Q)\|_{L^2(\Phi\mu)}$ smallness.

On the other hand, applying frame 25 again, we see that for every $x \in \mathrm{supp}\mu_Q$,

$$
\begin{aligned}
\left|[T(\varphi_Q\mu - \nu_Q)](x)\right| &= \left|\int R^H(x^* - \cdot)d(\varphi_Q\mu - \nu_Q)\right| \\
&\le C\alpha\ell(Q)^{d+2}\|R^H(x^* - \cdot)\|_{\mathrm{Lip}(S)}\|\varphi_Q\|_{\mathrm{Lip}} \\
&\le C\alpha\ell(Q)^{d+2}\frac{1}{\Delta^{d+1}\ell(Q)^{d+1}}\frac{1}{\varepsilon\ell(Q)} \le C\alpha\Delta^{-d-1}\varepsilon^{-1}
\end{aligned}
$$

as an obvious estimate (from frame 72) is used with exchanged x, y: $|R^H(x^* - \cdot)\|_{\mathrm{Lip}(S)} \le \frac{C}{\Delta^{d+1}\ell(Q)^{d+1}}$ Hence, $\|T(\varphi_Q\mu - \nu_Q)\|_{L^2(\Phi\mu)} \le C\alpha\Delta^{-d-1}\varepsilon^{-1}\sqrt{\mu(Q)}$.

frame 75. Explanation of $\|T(\Phi\mu-\nu)\|_{L^2(\Phi\mu)}$ smallness.

Similarly to the previous slide, for every $Q'\in\mathfrak{Q}'$, we have

$$\left|[T(\varphi_{Q'}\mu-\nu_{Q'})](x)\right| = \left|\int R^H(x^*-\cdot)d(\varphi_{Q'}\mu-\nu_{Q'})\right|$$
$$\le C\alpha\ell(Q')^{d+2}\|R^H(x^*-\cdot)\|_{\mathrm{Lip}(S)}\|\varphi_{Q'}\|_{\mathrm{Lip}}$$
$$\le C\alpha\ell(Q')^{d+2}\frac{1}{\Delta^{d+1}\ell(Q)^{d+1}}\frac{1}{\varepsilon\ell(Q')}$$
$$\le C\alpha\Delta^{-d-1}\varepsilon^{-1}\frac{\ell(Q')^d}{\ell(Q)^d}\le C\alpha\Delta^{-d-1}\varepsilon^{-1}\frac{\mu(Q')}{\mu(Q)}\,.$$

Summing these inequalities over $Q'\in\mathfrak{Q}'$, we get

$$|[T(\Phi\mu-\nu)](x)|\le C\alpha\Delta^{-d-1}\varepsilon^{-1}\;\forall x\in\mathrm{supp}\mu_Q\,,$$

therefore $\|T\Phi\mu-T\nu\|_{L^2(\Phi\mu)}\le C\alpha\Delta^{-d-1}\varepsilon^{-1}\sqrt{\mu(Q)}$.

frame 76. Explanation of $\|T(\Phi\mu - \varphi_Q\mu)\|_{L^2(\Phi\mu)}$ smallness.

Since the operator norm of T in $L^2(\mu_Q)$ is bounded by a constant, (this is because

$$R^H(x^* - y) = R^H_{\Delta\ell(Q)}(x-y) + [R^H(x^*-y) - R^H_{\Delta\ell(Q)}(x-y)]$$

and the last kernel has Poisson estimate in absolute value) we have

$$\|T((\varphi_Q - \Phi)\mu)\|_{L^2(\mu_Q)} \le \|\varphi_Q - \Phi\|_{L^2(\mu)} \le C\varepsilon^{\frac{\gamma}{2}}\sqrt{\mu(Q)}$$

by the fact that the union of $Q' \in \mathfrak{Q}'$ take $(1 - C\varepsilon)$-portion of measure μ of cell Q (see Lemma on frame 67). Thus, we finally get

$$\|R^H(\nu - \nu_Q)\|_{L^2(\nu)}$$
$$\ge \|\widetilde{R}^H\nu\|_{L^2(\nu)} - C\left[\varepsilon^{\frac{\gamma}{2}} + \Delta\varepsilon^{-2} + \alpha^{\frac{1}{2}}\Delta^{-\frac{2d+1}{2}}\varepsilon^{-\frac{1}{2}} + \alpha\Delta^{-d-1}\varepsilon^{-1}\right]\sqrt{\mu(Q)}\,.$$

frame 77.

Now we choose $\Delta = \varepsilon^3$ and $\alpha = \varepsilon^C$ with large C. We come to the point that we need to estimate from below **the Riesz Energy**

$$\|\widetilde{R}^H \nu\|_{L^2(\nu)} ,$$

where $\nu := \sum_{Q' \in \mathfrak{Q}', Q' \subset Q} \nu_{Q'}$. It is **truly desirable** to have $\|\widetilde{R}^H \nu\|_{L^2(\nu)} \geq \ldots$ using another than ε constant. **To give a δ-breath.** The subset $\mathfrak{Q}'$ of the set $\{Q' \colon Q' \subset Q, Q' \in \mathfrak{Q}_{k+1}\}$ is chosen in Lemma on frame 67. In fact, it is almost the whole $\{Q' \colon Q' \subset Q, Q' \in \mathfrak{Q}_{k+1}\}$, the difference being the use of a certain Marcinkiewicz function to choose $\mathfrak{Q}'$.
To estimate Riesz Energy we need function $\psi, \widetilde{R}^H \psi = 1$ on ν and such that: see frame 88. For that we need first non-BAUP layer $\mathfrak{P}_{k+1}$ tiling Q over $\mathfrak{Q}_{k+1}$ and special family of cells in it.

frame 78. A collection of P's (inside Q) of non-BAUP layer $\mathfrak{P}_{k+1}$.

One can construct (under our usual assumptions of ε is sufficiently small in terms of δ, A is sufficiently large in terms of δ, α is sufficiently small in terms of ε and δ), a family $\mathfrak{P}' \subset \mathfrak{P}_{k+1}$ such that

- Every cell $P' \subset \mathfrak{P}'$ is contained in Q_ε and satisfies $\ell(P') \leq 2\alpha\delta^{-1}\ell(Q)$.
- $\sum_{P' \in \mathfrak{P}'} \mu(P') \geq c\mu(Q)$.
- The balls $B(z_{P'}, 10\ell(P'))$, $P' \in \mathfrak{P}'$ are pairwise disjoint.
- The function

$$h(x) = \sum_{P' \in \mathfrak{P}'} \left[\frac{\ell(P')}{\ell(P') + \operatorname{dist}(x, P')} \right]^{d+1}$$

satisfies $\|h\|_{L^\infty} \leq C$.

79. Figure.

We start with showing that every δ-non-BAUP cell P' contained in Q has much smaller size than Q. Indeed, we know that $\mathrm{supp}\mu \cap B(z_Q, A\ell(Q))$ is contained in the $\alpha\ell(Q)$-neighborhood of $L(Q)$ and that $B(y, \alpha\ell(Q)) \cap \mathrm{supp}\mu \neq \varnothing$ for every $y \in B(z_Q, A\ell(Q)) \cap L(Q)$. Suppose that $P' \subset Q$ is δ-non-BAUP. If $A > 5$, then

$$B(x_{P'}, \ell(P')) \subset B(z_Q, 5\ell(Q)) \subset B(z_Q, A\ell(Q)) .$$

Moreover, since $y_{P'} - x_{P'} \in H$, we have

$$\mathrm{dist}(y_{P'}, L(Q)) = \mathrm{dist}(x_{P'}, L(Q)) \leq \alpha\ell(Q) .$$

Let $y^*_{P'}$ be the projection of $y_{P'}$ to $L(Q)$. Then $|y^*_{P'} - y_{P'}| \leq \alpha\ell(Q)$ and $|y^*_{P'} - z_Q| \leq |y_{P'} - z_Q| < A\ell(Q)$. Thus, the ball $B(y_{P'}, 2\alpha\ell(Q)) \supset B(y^*_{P'}, \alpha\ell(Q))$ intersects $\mathrm{supp}\mu$, so $\delta\ell(P') < 2\alpha\ell(Q)$, i.e., $\ell(P') \leq 2\alpha\delta^{-1}\ell(Q)$.

Let now $\mathfrak{P} = \{P' \in \mathfrak{P}_{k+1} : P' \subset Q\}$. Consider the Marcinkiewicz function

$$g(P') = \sum_{P'' \in \mathfrak{P}} \left[\frac{\ell(P'')}{D(P', P'')} \right]^{d+1}$$

The standard argument with integration it over Q shows that

$$\sum_{P' \in \mathfrak{P}} g(P') \mu(P') \leq C_1 \mu(Q)$$

for some $C_1 > 0$ depending on the dimension d and the goodness parameters of μ only. Define

$$\mathfrak{P}^* = \{P' \in \mathfrak{P} : P' \subset Q_\varepsilon, g(P') \leq 3C_1\}.$$

Note that

$$\sum_{P'\in\mathfrak{P}^*}\mu(P')\ge\sum_{P'\in\mathfrak{P}}\mu(P')-\sum_{P'\in\mathfrak{P}:P'\not\subset Q_\varepsilon}\mu(P')-\sum_{P'\in\mathfrak{P}:g(P')>3C_1}\mu(P')\,.$$

However,

$$\sum_{P'\in\mathfrak{P}}\mu(P')\ge\sum_{Q'\in\mathfrak{Q}}\mu(Q')\ge(1-\varepsilon)\mu(Q)\,.$$

Further, since the diameter of each $P'\in\mathfrak{P}$ is at most $8\ell(P')\le 8\alpha\delta^{-1}\ell(Q)$, every cell $P'\in\mathfrak{P}$ that is not contained in Q_ε is contained in $Q\setminus Q_{2\varepsilon}$, provided that $\alpha<\frac{1}{8}\varepsilon\delta$. Thus, under this restriction,

$$\sum_{P'\in\mathfrak{P}:P'\not\subset Q_\varepsilon}\mu(P')\le\mu(Q\setminus Q_{2\varepsilon})\le C\varepsilon^\gamma\mu(Q)\,.$$

Further, since the diameter of each $P' \in \mathfrak{P}$ is at most $8\ell(P') \leq 8\alpha\delta^{-1}\ell(Q)$, every cell $P' \in \mathfrak{P}$ that is not contained in Q_ε is contained in $Q \setminus Q_{2\varepsilon}$, provided that $\alpha < \frac{1}{8}\varepsilon\delta$. Thus, under this restriction,

$$\sum_{P' \in \mathfrak{P}: P' \not\subset Q_\varepsilon} \mu(P') \leq \mu(Q \setminus Q_{2\varepsilon}) \leq C\varepsilon^\gamma \mu(Q)\,.$$

Finally, by Chebyshev's inequality,

$$\sum_{P' \in \mathfrak{P}: g(P') > 3C_1} \mu(P') \leq \frac{\mu(Q)}{3}\,.$$

Bringing these three estimates together, we get the inequality $\sum_{P' \in \mathfrak{P}^*} \mu(P') \geq \frac{1}{2}\mu(Q)$, provided that A, ε, α satisfy some restrictions of the admissible type.

Now we will rarefy the family $\mathfrak{P}^*$ a little bit more. Consider the balls $B(z_{P'}, 10\ell(P'))$, $P' \in \mathfrak{P}^*$. By the classical Vitali covering theorem, we can choose some subfamily $\mathfrak{P}' \subset \mathfrak{P}^*$ such that the balls $B(z_{P'}, 10\ell(P'))$, $P' \in \mathfrak{P}'$ are pairwise disjoint but

$$\bigcup_{P' \in \mathfrak{P}'} B(z_{P'}, 30\ell(P')) \supset \bigcup_{P' \in \mathfrak{P}^*} B(z_{P'}, 10\ell(P')) \supset \bigcup_{P' \in \mathfrak{P}^*} P' .$$

Then we will still have

$$\sum_{P' \in \mathfrak{P}'} \mu(P') \geq c \sum_{P' \in \mathfrak{P}'} \ell(P')^d$$
$$\geq c \sum_{P' \in \mathfrak{P}'} \mu(B(z_{P'}, 30\ell(P'))) \geq c \sum_{P' \in \mathfrak{P}^*} \mu(P') \geq c\mu(Q) .$$

The estimate on $h = \sum_{P' \in \mathfrak{P}'} \left[\frac{\ell(P')}{\ell(P') + \mathrm{dist}(x, P')}\right]^{d+1}$ follows from the Marcinkiewicz choice of $\mathfrak{P}^*$.

frame 84. Die Zubereitung für ψ, η: $\psi = \Delta \int^x \eta$.

Fix the non-BAUPness parameter $\delta \in (0,1)$. Fix any C^∞ radial function η_0 supported in $B(0,1)$ such that $0 \le \eta_0 \le 1$ and $\eta_0 = 1$ on $B(0, \frac{1}{2})$. For every $P' \in \mathfrak{P}'$, define

$$\eta_{P'}(x) = \eta_0 \left(\frac{1}{\delta \ell(P')} (x - x_{P'}) \right) - \eta_0 \left(\frac{1}{\delta \ell(P')} (x - y_{P'}) \right) .$$

Note that $\eta_{P'}$ is supported on the ball $B(z_{P'}, 6\ell(P'))$. This ball is contained in Q, provided that $12\alpha\delta^{-1} < \varepsilon$ (recall that $\ell(P') \le 2\alpha\delta^{-1}\ell(Q)$ and $P' \subset Q_\varepsilon$). Also $\eta_{P'} \ge 1$ on $B(x_{P'}, \frac{\delta}{2}\ell(P'))$ and the support of the negative part of $\eta_{P'}$ is disjoint with $\mathrm{supp}\mu$. Put

$$\eta = \sum_{P' \in \mathfrak{P}'} \eta_{P'} .$$

Since even the balls $B(z_{P'}, 10\ell(P'))$ corresponding to different $P' \in \mathfrak{P}'$ are disjoint, we have $-1 \le \eta \le 1$.
We want to show that $\int \eta \, d\nu \ge c(\delta)\mu(Q)$ with some $c(\delta) > 0$.

frame 85.

Obviously, $\int \eta \, d\mu \geq c(\delta)\mu(Q)$ with some $c(\delta) > 0$. This is because of the choice of $\mathfrak{P}'$ and because, where η is negative does not carry any mass μ.

Moreover,

$$\int \eta \, \Phi \, d\mu \geq \int \eta_+ \, d\mu - \Big| \int (\chi_Q - \Phi) \, d\mu \Big| \geq c(\delta)\mu(Q) - \varepsilon^\gamma \mu(Q) \geq \frac{c}{2}\mu(Q) \, .$$

So we need to estimate as a small thing $\int \eta \, (d\Phi\mu - \nu)$, which is the sum over $Q' \in \mathfrak{Q}'$ of $\int \eta \, (d\varphi_{Q'}\mu - \nu_{Q'})$. By frame 25 we have

$$|\int \eta \, (d\varphi_{Q'}\mu - \nu_{Q'})| \leq C\alpha\ell(Q')^{d+2}\|\varphi_{Q'}\|_{\mathrm{Lip}}\|\eta\|_{\mathrm{Lip(supp}\varphi_{Q'})} \leq$$

$$C\alpha\varepsilon^{-1}\ell(Q')^{d+1}\|\eta\|_{\mathrm{Lip(supp}\varphi_{Q'})} \leq C\alpha\varepsilon^{-1}\mu(Q') \sup_{P' : \, B(z_{P'}, 6\ell(P')) \cap Q'_\varepsilon \neq \emptyset} \frac{\ell(Q')}{\delta\ell(P')} \, .$$

For $Q' \subset P'$, fine. Otherwise $Q' \cap P' = \emptyset$, $B(z_{P'}, 6\ell(P')) \cap Q'_\varepsilon \neq \emptyset$ give $C\ell(P') \geq \varepsilon\ell(Q')$. And again smallness of α kills all $\varepsilon^{-2}\delta^{-1}$.

Fix $P' \in \mathfrak{P}'$. Let $e_{P'}$ be the unit vector in the direction $y_{P'} - x_{P'}$. Put

$$u_{P'}(x) = \int_{-\infty}^{0} \eta_{P'}(x + te_{P'})\, dt\,.$$

Let us think that H is parallel to $x_{d+1} = 0$ and that $e_1 = e_{P'}$ (this is without loss of generality). Then $\partial_1 u = \eta$. But $R^H = (\partial_1, \dots, \partial_d)\frac{1}{|x|^{d-1}}$. Therefore,

$$R^H \Delta u = R^H \Delta \int \eta = (\partial_1, \dots, \partial_d)\frac{1}{|x|^{d-1}} \star \Delta \int \eta = (\partial_1, \dots, \partial_d) \int \eta\,.$$

We showed that

$$R^{H,1} \Delta u = \partial_1 \int \eta = \eta\,.$$

Since the restriction of $\eta_{P'}$ to any line parallel to $e_{P'}$ consists of two opposite bumps, the support of $u_{P'}$ is contained in the convex hull of $B(x_{P'}, \delta\ell(P'))$ and $B(y_{P'}, \delta\ell(P'))$. Also, since $\|\nabla^j \eta_{P'}\|_{L^\infty} \le C(j)[\delta\ell(P')]^{-j}$ and since $\mathrm{supp}\eta_{P'}$ intersects any line parallel to $e_{P'}$ over two intervals of total length $4\delta\ell(P')$ or less, we have

$$|\nabla^j u_{P'}(x)| \le \int_{-\infty}^{0} |(\nabla^j \eta_{P'})(x + te_{P'})|\, dt \le \frac{C(j)}{[\delta\ell(P')]^{j-1}}$$

for all $j \ge 0$. Define the vector fields

$$\psi_{P'} = (\Delta u_{P'})e_{P'}, \qquad \psi = \sum_{P' \in \mathfrak{P}'} \psi_{P'} .$$

Then, clearly, $(R^H)^*(\psi m) = \eta$ and all below are satisfied ($m := m_{d+1}$):

- $\psi = \sum_{P' \in \mathfrak{P}'} \psi_{P'}$, $\mathrm{supp}\psi \subset S$, $\mathrm{dist}(\mathrm{supp}\psi, L) \geq \Delta \ell(Q) = \varepsilon^3 \ell(Q)$.
- $\psi_{P'}$ is supported in the $2\ell(P')$-neighborhood of P' and satisfies

$$\int \psi_{P'} = 0, \quad \|\psi_{P'}\|_{L^\infty} \leq \frac{C}{\delta \ell(P')}, \quad \|\psi_{P'}\|_{\mathrm{Lip}} \leq \frac{C}{\delta^2 \ell(P')^2}\,.$$

- $\int |\psi|\, dm \leq C\delta^{-1}\mu(Q)$.
- $(R^H)^*(\psi m) = \eta$.
- $\|T^*(\psi m)\|_{L^\infty(\mathrm{supp}\nu)} \leq C\alpha\delta^{-2}\varepsilon^{-3d-3}$.
- $\|\widetilde{R}^H(|\psi| m)\|_{L^2(\nu)} \leq C\delta^{-1}\sqrt{\mu(Q)}\,$.

In fact,

$$\int |\psi|\, dm = \sum_{P'\in\mathfrak{P}'} \int |\psi_{P'}|\, dm \le C \sum_{P'\in\mathfrak{P}'} [\delta\ell(P')]^{-1} m(B(z_{P'}, 6\ell(P')))$$
$$\le C\delta^{-1} \sum_{P'\in\mathfrak{P}'} \ell(P')^d \le C\delta^{-1} \sum_{P'\in\mathfrak{P}'} \mu(P') \le C\delta^{-1}\mu(Q)\,.$$

To get the uniform estimate for $T^*(\psi m)$, note that

$$|[T^*(\psi_{P'} m)](x)| = \left| \int \langle R^H(x^* - \cdot), \psi_{P'} \rangle\, dm \right| \le C\delta^{-1} \| R^H(x^* - \cdot)\|_{\mathrm{Lip}(s}$$
$$\cdot\, \ell(P')^{d+1} \le C\delta^{-1}\Delta^{-d-1} \frac{\ell(P')^{d+1}}{\ell(Q)^{d+1}} \le C\alpha\delta^{-2}\Delta^{-d-1} \frac{\mu(P')}{\mu(Q)}$$

for every $x \in \mathrm{supp}\nu$ (we remind the reader that $\ell(P') \le 2\alpha\delta^{-1}\ell(Q)$). Adding up and recalling our choice $\Delta = \varepsilon^3$:

$$\| T^*\psi\|_{L^\infty(\mathrm{supp}\nu)} \le C\alpha\delta^{-2}\varepsilon^{-3d-3} \sum_{P'\in\mathfrak{P}'} \frac{\mu(P')}{\mu(Q)} \le C\alpha\delta^{-2}\varepsilon^{-3d-3}\,.$$

frame 90. The bound of $\|\widetilde{R}^H(|\psi|\, dm)\|_{L^2(\nu)}$.

First we estimate $\|\widetilde{R}^H(|\psi|\, dm)\|_{L^2(\mu_Q)}$. And then use our transfer estimates midificating the measure as it has been already done many time before.

Recall that for every $P' \in \mathfrak{P}'$, we have $\int |\psi_{P'}|\, dm \le C\delta^{-1}\ell(P')^d$.

Hence, we can choose constants $b_{P'} \in (0, C\delta^{-1})$ so that $|\psi_{P'}|m - b_{P'}\chi_{P'}\mu$ is a balanced signed measure, i.e.,

$$\int |\psi_{P'}|\, dm = b_{P'} \int \chi_{P'}\, d\mu\,.$$

Let

$$f = \sum_{P' \in \mathfrak{P}'} b_{P'}\chi_{P'}\,.$$

Our goal is to prove first

$$|\widetilde{R}^H(|\psi| m)| \le C\delta^{-1} + |\widetilde{R}^H(f\mu)| + \sum_{P' \in \mathfrak{P}'} \chi_{V(P')} |\widetilde{R}^H(b_{P'}\chi_{P'}\mu)|\,,$$

where for each $P' \in \mathfrak{P}'$, denote by $V(P')$ the set of all points $x \in \mathbb{R}^{d+1}$ such that $\mathrm{dist}(x, P') < \mathrm{dist}(x, P'')$ for all $P'' \in \mathfrak{P}'$.

This estimate of $|\widetilde{R}^H(|\psi|m)|$ from the previous slide converts to converts into

$$\|\widetilde{R}^H(|\psi|m)\|^2_{L^2(\mu_Q)}$$
$$\leq C\left[\delta^{-2}\mu(Q) + \|f\|^2_{L^2(\mu)} + \sum_{P'\in\mathfrak{P}'} \|b_{P'}\chi_{P'}\|^2_{L^2(\mu)}\right] \leq C\delta^{-2}\mu(Q)\,,$$

which we wanted. To get the pointwise estimate of frame 90 we write for $x \in V(P')$.

$$[\widetilde{R}^H(|\psi|m - f\mu)](x) = [\widetilde{R}^H(|\psi_{P'}|m)](x) - [\widetilde{R}^H(b_{P'}\chi_{P'}\mu)](x)$$
$$+ \sum_{P''\in\mathfrak{P}', P''\neq P'} [\widetilde{R}^H(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)](x)\,.$$

If $x \in V(P')$ and cells are Vitali disjoint, then $\mathrm{dist}(x, P'') \geq c\ell(P'')$ and so

$$\left|R^H(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)](x)\right| = \left|\int K^H(x - \cdot)\, d(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)\right.$$

$$= \left|\int [K^H(x - \cdot) - K^H(x - z_{P''})]\, d(|\psi_{P''}|m - b_{P''}\chi_{P''}\mu)\right|$$

$$\leq 2\|K^H(x - \cdot) - K^H(x - z_{P''})\|_{L^\infty(P'')} \int |\psi_{P''}|\, dm$$

$$\leq \frac{C\ell(P'')}{\mathrm{dist}(x, P'')^{d+1}}\delta^{-1}\ell(P'')^d \leq C\delta^{-1}\left[\frac{\ell(P'')}{\ell(P'') + \mathrm{dist}(x, P'')}\right]^{d+1},$$

and the same for $R^H(x^* - y)$. Hence all this huge sum on the previous slide is $\leq \delta^{-1}h(x) \leq C/\delta$ by the Marcinkiewicz choice of $\mathfrak{P}'$, see slide 78.

Note also that

$$\|\widetilde{R}^H(|\psi_{P'}|m)\|_{L^\infty} \leq C\delta^{-1}$$

(this is just the trivial bound $C\ell(P')$ for the integral of the absolute value of the kernel over a set of diameter $12\ell(P')$ multiplied by the bound $\frac{C}{\delta\ell(P')}$ for the maximum of $|\psi_{P'}|$). Therefore,

$$\|\widetilde{R}^H(|\psi|m)\|^2_{L^2(\mu_Q)} \leq C\delta^{-2}\mu(Q)$$

is proved, and then we (non-trivially, but habitually) transfer this into

$$\|\widetilde{R}^H(|\psi|m)\|^2_{L^2(\nu)} \leq C\delta^{-2}\mu(Q)\,.$$

frame 94. Smearing of the measure ν

Exactly as in the beginning of the lectures we replace the measure ν by a compactly supported measure $\widetilde{\nu}$ that has a bounded density with respect to the $(d+1)$-dimensional Lebesgue measure m in $\mathbb{R}^{d+1}$. More precisely, for every $\varkappa > 0$, we will construct a measure $\widetilde{\nu}$ with the following properties:

- $\widetilde{\nu}$ is absolutely continuous and has bounded density with respect to m.
- $\mathrm{supp}\widetilde{\nu} \subset S$ and $\mathrm{dist}(\mathrm{supp}\widetilde{\nu}, L) \geq \Delta\ell(Q)$.
- $\widetilde{\nu}(S) = \nu(S) \leq \mu(Q)$.
- $\int \eta\, d\widetilde{\nu} \geq \int \eta\, d\nu - \varkappa$.
- $\int |\widetilde{R}^H(|\psi|m)|^2\, d\widetilde{\nu} \leq \int |\widetilde{R}^H(|\psi|m)|^2\, d\nu + \varkappa$.
- $\int |\widetilde{R}^H\widetilde{\nu}|^2\, d\widetilde{\nu} \leq \int |\widetilde{R}^H\nu|^2\, d\nu + \varkappa$.

frame 95. Suppose $\|\widetilde{R}^H\nu\|_{L^2(\nu)} < \lambda\mu(Q)$ with tiny λ.

Then, choosing sufficiently small smearing parameter we get very small $\varkappa > 0$ and we can ensure that the measure $\widetilde{\nu}$ constructed in the previous section, satisfies

$$\int |\widetilde{R}^H\widetilde{\nu}|^2\, d\widetilde{\nu} < \lambda\mu(Q)\,, \int \eta\, d\widetilde{\nu} \geq \theta\mu(Q)\,, \int |\widetilde{R}^H(|\psi|m)|^2\, d\widetilde{\nu} \leq \Theta\mu(Q)$$

where $\theta, \Theta > 0$ are two quantities depending only on δ (plus, of course, the dimension d and the goodness and AD-regularity constants of μ).

Our aim is to show that if $\lambda = \lambda(\delta) > 0$ is chosen small enough, then these three conditions are incompatible.

Then of course $\|\widetilde{R}^H\nu\|_{L^2(\nu)} \geq \lambda\mu(Q)$ with not-so-tiny λ, and almost orthogonality finishes the contradiction.

frame 96. Extremal problem.

For non-negative $a \in L^\infty(m)$, define $\widetilde{\nu}_a = a\widetilde{\nu}$ and consider the extremal problem

$$\Xi(a) = \lambda\mu(Q)\|a\|_{L^\infty(m)} + \int |\widetilde{R}^H \widetilde{\nu}_a|^2 d\widetilde{\nu}_a \to \min$$

under the restriction $\int \eta\, d\widetilde{\nu}_a \geq \theta\mu(Q)$. Note that since $\widetilde{\nu}$ is absolutely continuous and has bounded density with respect to m, the measure $\widetilde{\nu}_a$ is well defined and has the same properties.
The first goal is to show that the minimum is attained and for every minimizer a, we have $\|a\|_{L^\infty(m)} \leq 2$ and

$$|\widetilde{R}^H \widetilde{\nu}_a|^2 + 2(\widetilde{R}^H)^*[(\widetilde{R}^H \widetilde{\nu}_a)\widetilde{\nu}_a] \leq 6\lambda\theta^{-1}$$

everywhere in S. This is done **precisely** as in the beginning of the lectures. Review.

frame 97. Contradiction: why this smallness is impossible?

Integrate the last inequality against $|\psi|\, dm$, where ψ is the vector field constructed recently on frames 88–93. We get

$$\int |\widetilde{R}^H \widetilde{\nu}_a|^2 \cdot |\psi|\, dm + 2\int \Big[(\widetilde{R}^H)^*[(\widetilde{R}^H \widetilde{\nu}_a)\widetilde{\nu}_a]\Big] \cdot |\psi|\, dm$$
$$\leq 6\lambda\theta^{-1} \int |\psi| dm \leq C\lambda\theta^{-1}\delta^{-1}\mu(Q)\,.$$

Rewrite the second integral on the left as

$$\int \Big\langle \widetilde{R}^H \widetilde{\nu}_a, \widetilde{R}^H(|\psi| m)\Big\rangle\, d\widetilde{\nu}_a\,.$$

frame 98.

Then, by the Cauchy inequality,

$$\int \left[(\widetilde{R}^H)^*[(\widetilde{R}^H\widetilde{\nu}_a)\widetilde{\nu}_a]\right] \cdot |\psi|\, dm$$

$$\leq \left[\int |\widetilde{R}^H\widetilde{\nu}_a|^2\, d\widetilde{\nu}_a\right]^{\frac{1}{2}} \left[\int |\widetilde{R}^H(|\psi|m)|^2\, d\widetilde{\nu}_a\right]^{\frac{1}{2}}$$

$$\leq \Xi(a)^{\frac{1}{2}} \left[\int |\widetilde{R}^H(|\psi|m)|^2\, d\widetilde{\nu}_a\right]^{\frac{1}{2}} .$$

Recall that $\|a\|_{L^\infty(m)} \leq 2$, so we can replace $\widetilde{\nu}_a$ by $\widetilde{\nu}$ in the last integral losing at most a factor of 2. Taking into account that

$$\int |\widetilde{R}^H(|\psi|m)|^2\, d\widetilde{\nu} \leq \Theta\mu(Q)\,,$$

we get

$$\left|\int \left[(\widetilde{R}^H)^*[(\widetilde{R}^H\widetilde{\nu}_a)\widetilde{\nu}_a]\right] \cdot |\psi|\, dm\right| \leq C\,[\lambda\Theta]^{\frac{1}{2}}\,\mu(Q)\,.$$

frame 99.

Thus,

$$\int |\widetilde{R}^H\widetilde{\nu}_a|\cdot|\psi|\,dm \le \Big(\int |\widetilde{R}^H\widetilde{\nu}_a|^2\cdot|\psi|\,dm\Big)^{1/2}\Big(\int |\psi|\,dm\Big)^{1/2} \le C(\delta)\lambda^{1/4}\mu(Q)\,.$$

In particular, $\int \langle \widetilde{R}^H\widetilde{\nu}_a, \psi\rangle\,dm \le C(\delta)\lambda^{\frac{1}{4}}\mu(Q)$. However,

$$\int \langle \widetilde{R}^H\widetilde{\nu}_a, \psi\rangle\,dm = \int [(\widetilde{R}^H)^*(\psi m)]\,d\widetilde{\nu}_a$$

$$= \int [(R^H)^*(\psi m)]\,d\widetilde{\nu}_a - \int [T^*(\psi m)]\,d\widetilde{\nu}_a \ge \int \eta\,d\widetilde{\nu}_a - \sigma(\varepsilon,\alpha)\widetilde{\nu}_a(S)$$

This yields

$$\int [(\widetilde{R}^H)^*(\psi m)]\,d\widetilde{\nu}_a \ge \theta\mu(Q) - \sigma(\varepsilon,\alpha)\widetilde{\nu}_a(S) \ge [\theta - 2\sigma(\varepsilon,\alpha)]\mu(Q) \ge \frac{\theta}{2}\mu(Q)$$

if ε and α are chosen small enough (in this order). Thus, if λ has been chosen smaller than a certain constant depending on δ only, we get a contradiction.