Spectral Theory of Orthogonal Polynomials

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Lecture 9: Fuchsian Groups and Finite Gaps, I

- Lecture 7: Periodic OPRL
- Lecture 8: Finite Gap Isospectral Torus
- Lecture 9: Fuchsian Groups and Finite Gaps, I
- Lecture 10: Fuchsian Groups and Finite Gaps, II

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

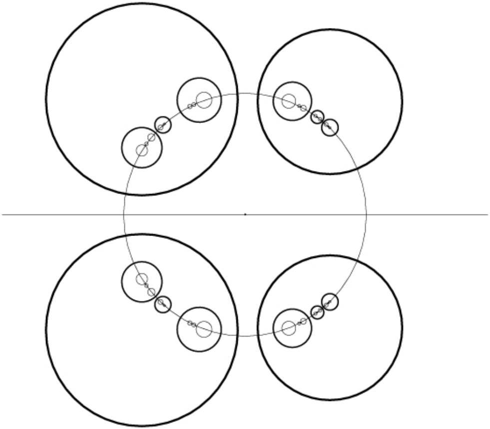
Some History

The Fuchsian group approach to finite gap problems is due to Sodin–Yuditskii [J. Geom. Anal. **7** (1997), 387–435] and developed to get Szegő asymptotics by Peherstorfer–Yuditskii [J. Anal. Math. **89** (2003), 113-154]. This work was extended and explicated in a series of papers by Christiansen–Simon–Zinchenko [Const. Approx. **32** (2010), 1–65; **33**, (2011), 365–403; **35** (2012) 259–272].

Much earlier Widom [Adv. Math **3** (1969), 127-232] discussed OPs on unions of smooth curves and found the almost periodicity we'll see in Lecture 10.

A Pretty Graphic

The poster for the lecture series included the following graphic:



A Pretty Graphic

Clearly a pretty pattern, but what does it have to do with OPs? This lecture will answer that. Let's begin by analyzing some of its features.

It has one (faint) large circle that represents $\partial\mathbb{D}$, the boundary of the unit disk. All the other circles are "orthocircles," i.e., cross $\partial\mathbb{D}$ orthogonally.

This is no coincidence. They are geodesics in the hyperbolic metric or rather the part within $\mathbb{D}$ are geodesics in the Poincaré metric.

A Pretty Graphic

They come in nested circles, three inside each earlier "generation."Indeed, this nesting really goes on indefinitely but we only show three generations.

There is an additional orthocircle showing—the straight line $(-1, 1)$.

The fact that there appear to be bigger and smaller, even really tiny, circles is an artifact of the Euclidean view we make so that the "circle at ∞" ($\partial\mathbb{D}$) is visible. For any circle, even the really tiny ones, there is a Möbius transformation which is an automorphism of $\mathbb{D}$ and isometry in the hyperbolic metric mapping that circle to $\mathbb{R}$ and the part inside $\mathbb{D}$ to $(-1, 1)$.

Universal Cover of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$

Recall in analyzing Szegő asymptotics and the Shohat–Nevai theorem on $[-2, 2]$, we mapped from $\mathbb{D}$ (via $z \mapsto x = z + z^{-1}$) and considered $\log(M(z)/zB(z))$ relating its Taylor coefficients at $z = 0$ to its boundary values and Taylor coefficients of $B(z)$.

We could take logs because $\mathbb{C} \cup \{\infty\} \setminus [-2, 2]$ was simply connected and so an image of $\mathbb{D}$.

Universal Cover of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$

If $\ell \geq 1$, $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ is no longer simply connected. To get logs, we'll need to lift the function to the universal cover of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$.

Any Riemann surface has a universal cover which is also a Riemann surface since the local analytic structure "below" lifts. The uniformization theorem says that this universal cover is $\mathbb{D}$ except for a few special cases of the underlying surface: $\mathbb{C} \cup \{\infty\}$, $\mathbb{C}$, a torus, $\mathbb{C} \setminus \{0\}$.

Universal Cover of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$

So there is a covering map $\mathbf{x}(z)$ from $\mathbb{D}$ to $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ which is many to one.

As with any covering map, there is a discrete group of transformations which in this case preserve the complex structure so are Möbius transformations of $\mathbb{D}$ to $\mathbb{D}$.

Thus, there is a discrete group of Möbius transformations (aka Fuchsian group), Γ, so that $\mathbf{x}(\gamma(z)) = \mathbf{x}(z)$. Indeed, $\mathbf{x}(z) = \mathbf{x}(w) \Leftrightarrow \exists\, \gamma \in \Gamma$ with $\gamma(z) = w$.

Finite Gap Fundamental Domains

If $\mathbf{x}$ is a map of the required type and $g : \mathbb{D} \to \mathbb{D}$ is a Möbius automorphism, then $\mathbf{x} \circ g$ is also a covering map although the Fuchsian group is now $g^{-1}\Gamma g$.

We normalize $\mathbf{x}$ by demanding $\mathbf{x}(0) = \infty$ and $\lim_{z\to 0, z\neq 0} z\,\mathbf{x}(z) > 0$. This implies $\mathbf{x}$ maps the region in $\mathbb{D} \cap \mathbb{C}_+$ just above $\mathbb{R}$ to $\mathbb{C}_-$.

The Dirichlet domain of Γ is defined to be ($\rho =$ Poincaré metric $\tanh[\rho(w,z)] = |z-w|/|1-\bar{z}w|$)

$$D(\Gamma) = \{w \in \mathbb{D} \mid \rho(w,0) = \inf_{\gamma\in\Gamma} \rho(w, \gamma(0))\}$$

Finite Gap Fundamental Domains

$$\mathring{D}(\Gamma) = \{w \in \mathbb{D} \mid \rho(w, 0) < \inf_{\gamma \neq e} \rho(w, \gamma(0))\}$$

$\mathring{D}(\Gamma)$ is the interior of $D(\Gamma)$ and $D(\Gamma)$ is the closure of $\mathring{D}(\Gamma)$.

D and $\mathring{D}$ are fundamental domains for $\mathbf{x}$ in that $\mathbf{x}$ is 1–1 on $\mathring{D}$, and in our case, 2–1 on $D \setminus \mathring{D}$. It will turn out that $\mathbf{x}[\mathring{D}] = \mathbb{C} \cup \{\infty\} \setminus [\alpha_1, \beta_{\ell+1}]$ and $\mathbf{x}$ is 1–1 on $D \setminus \mathring{D} \cap \mathbb{C}_+$.

Finite Gap Fundamental Domains

By normalization, $\mathbf{x}(z) \sim C/z$, $C > 0$ near $z = 0$, so z running from 0 to -1, has $\mathbf{x}(z)$ going from $-\infty \in \mathbb{R}$ up to α_1. Why α_1? Because $z \to \partial\mathbb{D}$ means $\mathbf{x}(z)$ must approach a point of $\cup_{j=1}^{\ell+1}[\alpha_j, \beta_j]$.

We have thus proven $\mathbf{x} : (-1, 1)$ to $\mathbb{R} \cup \{\infty\} \setminus [\alpha_1, \beta_{\ell+1}]$.

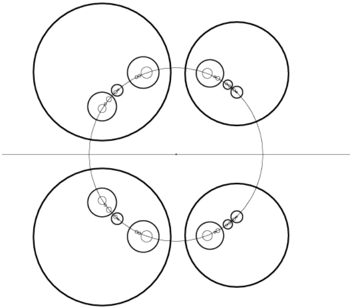
If we go slightly above $[\alpha_1, \beta_1]$ or below, $\mathbf{x}^{-1}$ maps onto a piece almost on $\partial\mathbb{D}$ in $\mathbb{C}_-$ (or $\mathbb{C}_+$).

Finite Gap Fundamental Domains

If we now reach (β_1, α_2), $\mathbf{x}^{-1}$ must map in $\mathbb{D}$ along a curve. If we had normalized, so that $\widetilde{\mathbf{x}}(0) = \frac{1}{2}(\beta_1 + \alpha_2)$, by the same analysis (β_1, α_2) would be the image of $(-1, 1)$. Since $\widetilde{\mathbf{x}} = \mathbf{x} \circ g$, we see the curve must be an image of $(-1, 1)$ under a Möbius transformation, that is an orthocircle.

Finite Gap Fundamental Domains

We can now understand part of the figure.



We have 2 gaps and 3 bands and we can understand the fundamental domain.

Finite Gap Fuchsian Group

If $\mathfrak{e}$ is a subset of $\mathbb{R}$ with ℓ gaps, the fundamental domain is $\mathbb{D}$ with the "inside" of 2ℓ disjoint orthocircles removed, ℓ in the upper half-plane and their ℓ conjugates.

Let $Cz = \bar{z}$ and let R_j be reflection in the jth orthocircle in the upper half-plane, explicitly if the circle is $|z - z_j| = r_j$, then

$$R_j z = z_j + \frac{r_j^2}{\bar{z} - \bar{z}_j}$$

which is a conjugate Möbius transform with $R_j \infty = z_j$; R_j leaves the orthocircle pointwise fixed.

Finite Gap Fuchsian Group

Let $\gamma_j = R_j C$ which is a Móbius transformation.

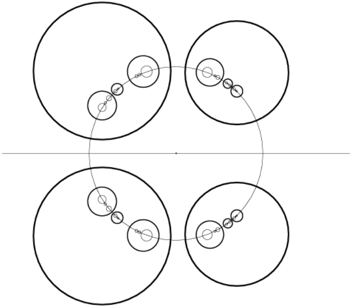
Since $\mathbf{x}$ is real on $(-1, 1)$, $\mathbf{x}(\bar{z}) = \overline{\mathbf{x}(z)}$.

Since $\mathbf{x}$ is real on orthocircle associated to R_j, $\mathbf{x}(R_j z) = \overline{\mathbf{x}(z)}$.

Thus, $\mathbf{x}(\gamma_j z) = \mathbf{x}(z)$, i.e., $\gamma_j \in \Gamma$.

It is not hard to show that Γ is generated by the γ_j's.

We now return to our example



The second generation circles inside one of the first generation circles are exactly the image of the three other first generation circles, etc.

Finite Gap Fuchsian Group

$$\Lambda = \{\text{limit points of } \{\gamma(0) \mid \gamma \in \Gamma\}\}$$

is easily seen to be nowhere dense and we'll shortly see that it is of Hausdorff dimension strictly less than 1.

$\mathbf{x}$ has an analytic continuation to $\partial\mathbb{D} \setminus \Lambda$ since it has boundary values (mapping to $\cup_{j=1}^{\ell+1}[\alpha_j, \beta_j]$) and we can use the Schwarz reflection principle.

Indeed, $\mathbf{x}$ has a meromorphic continuation to $\mathbb{C} \cup \{\infty\} \setminus \Lambda$. By mapping $\mathbb{C} \setminus \bar{\mathbb{D}}$ to $\mathcal{S}_-$, one sees this extended $\mathbf{x}$ is essentially a covering map of $\mathcal{S}$.

Beardon's Theorem

A. F. Beardon (Acta Math. **127** (1971), 221–258) proved an important result about certain finitely generated Fuchsian groups that include the ones associated to finite gap sets.

It has all of the following consequences:

- The set of limit points of the orbit $\{\gamma(0) \mid \gamma \in \Gamma\}$ (which is the same as the limit points of $\{\gamma(z) \mid \gamma \in \Gamma\}$ for any $z \in \mathbb{D}$) has Hausdorff dimension strictly less than 1.
- If $\mathcal{R}_k$ is the union of the interiors of all $2\ell(2\ell-1)^{k-1}$ orthocircles at generation k, and $\partial\mathcal{R}_k = \partial\mathbb{D} \cap \mathcal{R}_k$ and $|\cdot|$ is $d\theta/2\pi$ measure, then $|\partial\mathcal{R}_k| \leq C_0 e^{-C_1 k}$.
- For some $s < 1$, we have

$$\sum_{\gamma\in\Gamma} (1 - |\gamma(z)|)^s < \infty \quad \text{for all } z \in \mathbb{D}$$

so, in particular, this holds for $s = 1$.

Since $\sum|1-\gamma(z_0)| < \infty$, we can form the Blaschke products

$$B(z, z_0) = \prod_{\gamma\in\Gamma} b_{\gamma(z_0)}(z)$$

$b_{\gamma_0(z_j)}(z)$ and $b_{z_0}(\gamma_0^{-1}(z))$ have the same zeros and poles and so the ratio is a constant, which is magnitude 1 on $\partial\mathbb{D}$, so a phase factor. Since $\{\gamma\gamma_0 \mid \gamma \in \Gamma\} = \{\gamma \in \Gamma\}$, we see that for each z_0, there is $C_{z_0}(\gamma)$ a map of Γ to $\partial\mathbb{D}$ so that

Blaschke Products

$B(\gamma(z), z_0) = C_{z_0}(\gamma) B(z, z_0)$

Such a function is called character automorphic.

Thus, $-\log|B(z, z_0)|$ defines a function on $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e} \cup \{\mathbf{x}(z_0)\}$, is harmonic on that set and goes to zero as one approaches $\mathfrak{e}$. (since $|B(z, z_0)| \to 1$ as $z \in \partial\mathbb{D}$ in the "bands").

Blaschke Products

Since $-\log|B(z, z_0)|$ has a log singularity as $\mathbf{x}(z) \to \mathbf{x}(z_0)$ we see it is a potential theorist's Green's function with charge at $\mathbf{x}(z_0)$.

In particular, if $B(z) \equiv B(z, 0)$, we see that

$$|B(z)| = \exp(-G_{\mathfrak{e}}(\mathbf{x}(z)))$$

If x_∞ is defined by $\mathbf{x}(z) = \frac{x_\infty}{z} + O(1)$

$$B(z) = \frac{C(\mathfrak{e})}{x_\infty} z + O(z^2)$$

$\rho_{\mathfrak{e}}([\alpha_j, \beta_j])$ is related to change of $\arg B$ over a piece of $\partial\mathbb{D}$.

Blaschke Products

A further result we'll need is that if $\{z_j\}_{j=1}^{\infty} \subset D(\Gamma)$ and

$$\sum_j (1 - |z_j|) < \infty$$

then $\prod_{j=1}^{\infty} B(z, z_j)$ is absolutely convergent and defines a function vanishing exactly at $\{\gamma(z_j) \mid j = 1, \ldots, \infty; \gamma \in \Gamma\}$.

The restriction that the z_j lie in $D(\Gamma)$ is critical because otherwise $\prod_{\gamma \in \Gamma} B(z, \gamma(0))$ would be absolutely convergent which it is not.