Spectral Theory of Orthogonal Polynomials

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Lecture 7: Periodic OPRL

- Lecture 5: Killip–Simon Theorem on $[-2, 2]$
- Lecture 6: Szegő Asymptotics and Shohat-Nevai for $[-2, 2]$
- Lecture 7: Periodic OPRL
- Lecture 8: Finite Gap Isospectral Torus

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

Floquet Solutions

The lecture title is a bit of a misnomer in that we'll mainly discuss whole line periodic Jacobi matrices although the half-line objects will enter a lot in future lectures.

So $\{a_n, b_n\}_{n=-\infty}^{\infty}$ are two-sided sequences with some $p > 0$ in $\mathbb{Z}$ so that

$$a_{n+p} = a_n \quad b_{n+p} = b_n$$

For $z \in \mathbb{C}$ fixed, we are interested in solutions $\{u_n\}_{n=0}^{\infty}$ of

$$a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = z u_n$$

Floquet Solutions

that also obey for some $\lambda \in \mathbb{C}$ $(\lambda = e^{i\theta}, \theta \in \mathbb{C})$

$$u_{n+p} = \lambda u_n$$

Such solutions are called Floquet soluations as they are analogs of solutions of ODE, especially Hill's equation $-u'' + Vu = zu$, $V(x+p) = V(x)$.

The analysis of such solutions is a delightful amalgam of three tools, the first of which is just the fact that the set of all solutions of the difference equation is two-dimensional.

Thus, there are, for z fixed, at most two different λ's for which there is a solution. If λ_1, λ_2 are two such λ's, their Wronskian is non-zero so constancy of the Wronskian implies $\lambda_1 \lambda_2 = 1$.

Periodic B.C. Jacobi Matrices

The (twisted) periodic boundary condition Jacobi matrix $J^{\mathrm{per},\lambda}$ is $p \times p$. It is the finite Jacobi matrix with $1p$ and $p1$ matrix elements added:

$$J_{jj} = b_j, \quad J_{j\,j+1} = a_j, \quad J_{j\,j-1} = a_{j-1}$$

$$J_{1p} = a_p \lambda^{-1}, \quad J_{p1} = a_p \lambda$$

If $\{u_n\}_{n=-\infty}^{\infty}$ is a Floquet solution, $u_0 = \lambda^{-1} u_p$, $u_{p+1} = \lambda u_1$ so $\widetilde{u} = \{u_n\}_{n=1}^{p}$ has $J^{\mathrm{per},\lambda}\widetilde{u} = z\widetilde{u}$.

Conversely, if $\widetilde{u}$ solves this, the unique u with $u_{n+p} = \lambda u_n$ and $\widetilde{u} = \{u_n\}_{n=1}^{\infty}$ is a Floquet solution.

Periodic B.C. Jacobi Matrices

This implies

- For any λ, there are at most p z's which have a Floquet solution for that λ. (We'll see soon that if $\lambda \neq \pm 1$, there are exactly p.)
- If $\lambda = e^{i\theta}$, $\theta \in \mathbb{R}$, $\lambda \neq \pm 1$, there are precisely p distinct z's all real, for which there are Floquet solutions with that λ.

The reality comes from hermicity of $J^{\text{per},\lambda}$.

If $\lambda \neq \pm 1$, $\bar{\lambda} \neq \lambda$. If u is a Floquet solution for λ, since z is real, $\bar{u}$ is a Floquet solution for $\bar{\lambda}$ so there is a unique solution for that z. Thus, for $\lambda \in \partial\mathbb{D} \setminus \{\pm 1\}$, $J^{\text{per},\lambda}$ has p eigenvalues and each simple.

The Discriminant

The third tool concerns the p-step transfer matrix.

$T_p(z)\binom{u_1}{a_0u_0} = \lambda\binom{u_1}{a_0u_0}$ is equivalent to $\binom{u_1}{a_0u_0}$ generating a Floquet solution ! (Note: a_0 may not be 1.)

In terms of the OP's for $\{a_n, b_n\}_{n=1}^{\infty}$,

$$T_p(z) = \begin{pmatrix} p_p(z) & -q_p(z) \\ a_p p_{p-1}(z) & -a_p q_{p-1}(z) \end{pmatrix}$$

The discriminant, $\Delta(z)$, is defined by

$$\Delta(z) = \mathrm{Tr}\big(T_p(z)\big) = p_p(z) - a_p q_{p-1}(z)$$

is a (real) polynomial of degree exactly p.

The Discriminant

Since $\det\bigl(T_p(z)\bigr) = 1$, it has algebraic eigenvalues λ and λ^{-1} where

$$\Delta(z) = \lambda + \lambda^{-1}; \quad \Delta(z) = 2\cos\theta \text{ if } \lambda = e^{i\theta}.$$

Floquet solutions correspond to geometric eigenvalues for $T_p(z)$. If $\lambda \neq \pm 1$, it has multiplicity one, so is geomtric. $\lambda = \pm 1$ has multiplicity 2, so there can be one or two Floquet solutions.

An important consequence of the fact that $\Delta(z) \in (-2, 2)$ implies all z's are real is $\Delta^{-1}\bigl[(-2,2)\bigr] \subset \mathbb{R}$.

The Discriminant

A basic fact of analytic functions is that if $f(z)$ is real (i.e., $f(\bar{z}) = \overline{f(z)}$), $x_0 \in \mathbb{R}$ with $f'(x_0) = 0$, there are non-real z's near x_0 with $f(z)$ real and near $f(x_0)$.

Thus, $\Delta^{-1}\big[(-2,2)\big] \subset \mathbb{R} \Rightarrow \Delta'(x_0) \neq 0$ if $\Delta(x_0) \in (-2,2)$.

Thus, $\Delta^{-1}\big[(-2,2)\big] = (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) \cup \ldots \cup (\alpha_p, \beta_p)$

where $\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \alpha_3 < \ldots < \beta_p$

with Δ a smooth bijection of (α_j, β_j) to $(-2,2)$.

Could be orientation reversing or not.

The Discriminant

Since $\Delta(x) \to \infty$ as $x \to \infty$, we must have $\Delta(\beta_p) = 2$.

It follows that $\Delta(\alpha_p) = -2$, $\Delta(\beta_{p-1}) = -2$, $\Delta(\alpha_{p-1}) = 2 \ldots$

i.e., $\Delta(\beta_j) = (-1)^{p-j}2, \quad \Delta(\alpha_j) = (-1)^{p-j-1}2$

If the α's and β's are all distinct, we have p points where $\Delta(x) = 2$ and p where $\Delta(x) = -2$.

Since $\deg \Delta = p$, these are all the points.

If $\beta_{j-1} = \alpha_j$, there is one less point where $\Delta(x) = (-1)^{p-j-1}2$, but $\Delta'(\alpha_j) = 0$ since $\Delta - (-1)^{p-j-1}2$ has the same sign on both sides of α_j. It follows that

Opens and Closed Gaps

Theorem. $\Delta^{-1}([-2,2]) = \cup_{j=1}^{p}[\alpha_j, \beta_j]$ *and*

$\Delta^{-1}(\{-2,2\}) = \{\alpha_j, \beta_j\}_{j=1}^{p}$ *and*

$\Delta'(\alpha_j) = 0 \Leftrightarrow \alpha_j = \beta_{j-1}$, $\Delta'(\beta_j) = 0 \Leftrightarrow \beta_j = \alpha_{j+1}$

and in that case, Δ'' *is not zero at that point.*

The $[\alpha_j, \beta_j]$ are called the bands and (β_j, α_{j+1}) the gaps.

If $\beta_j < \alpha_{j+1}$, we say that gap j is open.

If $\beta_j = \alpha_{j+1}$, we say gap j is closed.

Opens and Closed Gaps

Furthur analysis shows at a closed gap (with $\Delta(\alpha) = 2$ for simplicity) there are two periodic (Floquet) solutions, while at each of the edges of an open gap there is only one periodic (Floquet) solution. The transfer matrix has a Jordan anomaly, i.e., $\det = 1$, $\mathrm{Tr} = 2$, but $T \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Each of the gaps where $\Delta(x) \geq 2$ has two periodic solutions—either two at $\beta_j = \alpha_{j+1}$ or one each at β_j and α_{j+1} so there are p periodic Floquet solutions, as there must be from the J^{per} analysis.

Spectrum and Spectral Types

If z is such that $\Delta(z) \notin [-2,2]$, then the roots of $\lambda + \lambda^{-1} = \Delta(z)$ have $|\lambda| > 1$, $|\lambda^{-1}| < 1$. It follows that there are different solutions $u_\pm$ decaying exponentially at $\pm\infty$ so their Wronskian is not zero. By the earlier analysis,

$$G_{nm}(z) = u^+_{\max(n,m)}(z) u^-_{\min(m.n)}(z)/W(z)$$

is the matrix for $\big(J - z\big)^{-1}$, i.e., $z \notin \sigma(J)$.

If $\Delta(z) \in [-2,2]$, there is a bounded Floquet solution (since $|\lambda| = 1$). Then $\|(J-z)[u\chi_{[-N,N]}]\|$ is bounded, but since $\sum_{j=1}^p |u_{m+j}|^2$ is constant, $\|u\chi_{[-N,N]}\| \to \infty$ so $z \in \sigma(J)$. Thus

Theorem. $\sigma(J) = \cup_{j=1}^p [\alpha_j, \beta_j]$.

Spectrum and Spectral Types

If $\Delta(z) \in (-2, 2)$, we get that all solutions are bounded at $\pm\infty$ and then by a Wronskian argument, $|u_n|^2 + |u_{n+1}|^2$ is bounded from below. So by a Carmona-type formula, one should expect purely a.c. spectrum. But this is whole line, not half line !

Here is a replacement: Away from the bands, $G_{nn} = u_n^+ u_n^- / W$ as we've seen. By continuity of eigenfunctions of transfer matrix in z, $u_n^\pm$ has a limit at $z = x + i\varepsilon$ with $\varepsilon \downarrow 0$ which are Floquet solutions. This is true at least at interiors of bands where the transfer matrix has distinct eigenvalues.

Spectrum and Spectral Types

W is non-vanishing on each (α_j, β_j) since u^+ and u^- are distinct Floquet solutions ($e^{\pm i\theta}$). Thus, $G_{nn}(z)$ is continuous from $\mathbb{C}_+$ to $\mathbb{C}_+ \cup \mathbb{R} \setminus \{\alpha_j, \beta_j\}_{j=1}^p$.

But if $\mu^{(n)}$ is the spectral measure of δ_n:

$$G_{nn}(z) = \int \frac{d\mu^{(n)}(x)}{x - z}$$

The continuity implies $d\mu^{(n)}$ is purely a.c., so we have proven

Theorem. *A periodic two-sided Jacobi matrix has purely absolutely continuous spectrum.*

One can write out an explicit spectral representation with Floquet solutions with $z \in (\alpha_j, \beta_j)$ as continuum eigenfunctions.

Potential Theory

We start with a puzzle. Δ determines $\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \dots$ as the roots of $\Delta^2 - 4$.

Conversely, given $\beta_p, \alpha_{p-1}, \beta_{p-2}, \dots$, $\Delta - 2$ is determined up to a constant since we know its zeros.

That constant is determined by α_p when Δ is -2. Thus, $\beta_p, \alpha_{p-1}, \beta_{p-2}$ plus α_p determine the remaining $p-1$ α's and β's. Why this rigidity? Why can't we have $2p$ arbitrary α's and β's?

The answer will lie in potential theory.

Potential Theory

For any $z \in \mathbb{C}$, there are two Floquet indices, $\lambda_{\pm}$, solving $\lambda + \lambda^{-1} = \Delta(z)$. If $|\lambda_+| \geq 1$, we see that

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \log \|T_n(\lambda)\| = \frac{1}{p} \log |\lambda_+(z)|$$

Solving the quadratic equation for λ

$$\gamma(z) = \frac{1}{p}\left[\log\left|\frac{\Delta(z)}{2} + \sqrt{\left(\frac{\Delta(z)}{2}\right)^2 - 1}\right|\right]$$

On $\mathfrak{e} = \cup_{j=1}^{p}[\alpha_j, \beta_j]$, $|\ldots| = 1$, so $\gamma(z) \geq 0$, $\gamma(z) = 0$ on $\mathfrak{e}$. $\gamma(z)$ is harmonic on $\mathbb{C} \setminus \mathfrak{e}$ since $\frac{\Delta}{2} + \sqrt{(\frac{\Delta}{2}) - 1}$ is analytic and non-vanishing there and $\gamma(z) = \log(|z|) + O(1)$ at ∞, since $\Delta(z)$ is a degree p polynomial.

Potential Theory

Thus $\gamma(z) = G_{\mathfrak{e}}(z)$ is the potential theorists' Green's function. Thus,

Theorem. $\gamma(z)$ *as given above is the potential theorists' Green's function and periodic Jacobi parameters are associated to regular measures (in the Stahl–Totik sense).*

Corollary. $C(\mathfrak{e}) = (a_1 \cdots a_p)^{1/p}$

Potential Theory

By general principles, if $G_{\mathfrak{e}}$ is smooth up to $\mathfrak{e}$ on $\mathfrak{e}^{\text{int}}$, the equilibrium measure $d\rho_{\mathfrak{e}}(x) = f_{\mathfrak{e}}(x)dx$ where

$$f_{\mathfrak{e}}(x) = \frac{1}{\pi}\frac{\partial}{\partial y}G_{\mathfrak{e}}(x+iy)\ |_{y=0}$$

Thus, the equilibrium measure is

$$f_{\mathfrak{e}}(z) = \frac{1}{p\pi}\frac{|\Delta'(x)|}{\sqrt{4-\Delta^2(x)}} = \frac{1}{p\pi}\left|\frac{d}{dx}\arccos\Big(\frac{\Delta(x)}{2}\Big)\right|$$

In each, band $\Delta(\lambda)$ goes from -2 to 2, so $\arccos(\frac{\Delta}{2})$ from π to 0. Thus,

Theorem. $\rho_{\mathfrak{e}}\big([\alpha_j, \beta_j]\big) = \frac{1}{p}$.

This explains the puzzle mentioned earlier.

This is also a density of zeros way of understanding why the above $f_{\mathfrak{e}}$ is the DOS. For the periodic eigenfunctions with a box of size kp are the Floquet solutions with $\lambda = e^{2\pi ij/k}$, $j = 0, 1, 2, \ldots, k-1$.