Spectral Theory of Orthogonal Polynomials

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Lecture 6: Szegő Asymptotics and Shohat-Nevai for $[-2, 2]$

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- Lecture 5: Killip–Simon Theorem on $[-2, 2]$
- Lecture 6: Szegő Asymptotics and Shohat-Nevai for $[-2, 2]$
- Lecture 7: Periodic OPRL

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

Case 0 Sum Rule

Recall C_0 step-by-step says

$$-\log(a_1) = Z(\mu) - Z(\mu_1) + \mathcal{E}_0(J) - \mathcal{E}_0(J_1)$$

where $\mathcal{E}_0(J) = \sum_{j,\pm} \log|\beta^{\pm}(J)|, \quad E = \beta + \beta^{-1}, \ |\beta| > 1$

and $Z(\mu) = -\frac{1}{2} S(\mu^{(0)} \mid \mu) - \frac{1}{2}\log 2$

(Note $S(\mu^{(0)} \mid \mu_0) = -\log 2$)

$$= \frac{1}{4\pi} \int \log\left(\frac{\sin\theta}{\operatorname{Im} M(e^{i\theta})}\right) d\theta.$$

$d\mu^{(0)} = \mathrm{Sz}\left(\frac{d\theta}{2\pi}\right)$. Recall $d\mu_0 = \mathrm{Sz}(\sin^2\theta \, \frac{d\theta}{\pi})$ is free half-line.

So formally, C_0 is

$$-\log\left(\sum_{j=1}^{\infty} a_j\right) = Z(\mu) - \mathcal{E}_0(J)$$

Unlike P_2, not all terms positive.

(Extended) Shohat–Nevai Theorem

Theorem (Extended Shohat–Nevai Theorem). *Let* $d\mu = f(x)\,dx + d\mu_s$. $\sigma_{\text{ess}}(J) = [-2,2]$. *Suppose that*

$$\sum_{n,\pm} \left(|E_n^{\pm}| - 2\right)^{\frac{1}{2}} < \infty$$

Then

$$\int_{-2}^{2} (4 - x^2)^{-\frac{1}{2}} \log\, f(x) > -\infty \Leftrightarrow \overline{\lim}\, a_1 \cdots a_n > 0$$

In that case

$$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty$$

$\prod_{n=1}^{N} a_n$, $\sum_{n=1}^{N} (a_n - 1)$, $\sum_{n=1}^{N} b_n$ *all have limits* (*in* $(0,\infty)$, *resp.*, $(-\infty,\infty)$)

(Extended) Shohat–Nevai Theorem

Remarks. 1. $\sum(|E_n^\pm|-2)^{\frac{1}{2}}<\infty$ is called Blaschke condition for reasons we'll see below.

2. One variant of this theorem is that among the three conditions:

(i) $\sum(|E_n^\pm|-2)^{\frac{1}{2}}<\infty$;

(ii) Szegő integral $>-\infty$,

(iii) $\lim\,(a_1\cdots a_n)$ exists in $(0,\infty)$ (not just $\overline{\lim}$),

any two imply the third.

3. Recall Lieb–Thirring (proven for Jacobi by Hundertmark–Simon)

$$\sum(|E_n^\pm|-2)^p\leq\sum(|a_n-1|^{p+\frac{1}{2}}+|b_n|^{p+\frac{1}{2}})\text{ for }p\geq\tfrac{1}{2}.$$

(Extended) Shohat–Nevai Theorem

Thus $J - J_0 \in \ell_1$ (trace class), equivalent to $\sum |a_n - 1| + |b_n| < \infty$ implies both Blaschke condition and $\lim\,(a_1 \cdots a_n)$ exists so we have

Nevai Conjecture. $\sum |a_n - 1| + |b_n| < \infty \Rightarrow$ Szegő condition.

We refer you to [SzTh], Section 3.8 for the proof of the extended Shohat–Nevai Theorem.The idea is to use the C_0 step-by-step sum rule and lsc of Z much like we did for Killip–Simon.

Szegő Asymptotics—Results

Theorem (Damanik–Simon [Inv. Math **165** (2006), 1–50]). *Let the Jacobi parameters obey*

(a) $\sum_{n=1}^{\infty}(a_n - 1)^2 + b_n^2 < \infty$

(b) $\lim_{n\to\infty} \prod_{j=1}^{n} a_j$ *exists in* $(0, \infty)$

(c) $\lim_{n\to\infty} \sum_{j=1}^{n} b_j$ *exists in* $\mathbb{R}$.

Then, for all $z \in \mathbb{D} \setminus \{0\}$ *with* $z + z^{-1} \notin \sigma(J)$, $\lim_{n\to\infty} z^n p_n(z + z^{-1})$ *exists uniformly on compacts and is non-zero.*

Conversely, if that limit exists uniformly and is non-zero for $\{z \mid |z| = r\}$ *for all* $r \in (0, \varepsilon)$, *then (a)–(c) hold.*

Szegő Asymptotics—Results

Corollary (Peherstorfer-Yuditskii [Proc. AMS **129** (2001) 3213–3220]). *If* $\sum_{n,\pm}(|E_n^\pm| - 2)^{\frac{1}{2}} < \infty$ *and Szegő condition holds, then* $\lim z^n p_n(z + z^{-1})$ *exists, etc.*

For by Shohat–Nevai, we get all the required conditions for above theorem.

For each $\frac{1}{2} \le p < \frac{3}{2}$, Damanik–Simon construct examples with $a_n \equiv 1$, $\sum_{n=1}^\infty b_n^2 < \infty$, $\lim_{n\to\infty} \sum_{j=1}^n b_j$ exists but $\sum(|E_n^\pm| - 2)^p = \infty$.

For such examples, the Szegő condition fails, but you still get Szegő asymptotics ! This came as a surprise to many. Of course, if an ℓ^2 condition holds, then the sum is finite for $p = 3/2$ by Killip–Simon.

Szegő Asymptotics—Results

Here is the idea of the construction, at least if $p < 1$.

For whole line $a_n \equiv 1$, $b_n = 0$ for $n \neq 0$, $b_0 = \pm\varepsilon$ has a single eigenvalue of size $\pm 2 \pm C\varepsilon^2 + O(\varepsilon^3)$.
(I think $C = \frac{1}{4}$?)

Fix a sequence of numbers β_j of alternating sign, $|\beta_j| \to 0$, $\beta_1 > 0$, and integers, $0 < m_1 < m_2 < \dots$. Take $a_n \equiv 1$, $b_n = 0$ if $n \notin \{m_j\}$. $b_{m_j} = \beta_j$. As $m_1, m_{j+1} - m_j, \dots$ all get very large, J has eigenvalues very close to $(-1)^{j+1}\left[2 + C\beta_j^2\right]$, at least for j large.

Take $\beta_j = k^{-\beta}$, $j = 2k-1$; $\beta_j = -k^{-\beta}$, $j = 2k$.

Trivially, $\sum_{n=1}^{N} \beta_n$ converges to 0 and if $\beta > \frac{1}{2}$, $\sum_{n=1}^{\infty} b_n^2 < \infty$.

$|E_j^{\pm}| - 2 \sim Cj^{-2\beta}$. If $2\beta\, p < 1$, $\sum(|E_j^{\pm}| - 2)^p = \infty$.

Szegő Asymptotics—Results

That Szegő asymptotics implies the conditions on the a's and b's is not hard. For each n, for z near 0,

$$z^n p_n(z + \frac{1}{z}) = \frac{1}{a_1 \cdots a_n}\big(1 + z(\sum_{j=1}^{n} b_j) + O(z^2)\big)$$

so $z^n p_n(z + \frac{1}{z})$ is analytic near $z = 0$ and Szegő asymptotics implies convergence of the Taylor coefficients.

The first two coefficients give convergence of $\prod_1^n a_j$ and $\sum_1^n b_j$ by the above and the third coefficient yields the conditional convergence of $\sum_1^n (a_j - 1)^2 + b_j^2$ but since the sum of positive numbers, conditional convergence implies absolute convergence.

Jost Asymptotics

In the last lecture, we defined the Weyl solution, $g_n(x)$, $x \in \mathbb{C} \setminus \sigma(J)$

We say we have Jost asymptotics at z_0 if and only if

$$\lim_{n\to\infty} -z_0^{-(n+1)}\, g_n(z_0 + \frac{1}{z_0}) \equiv \frac{1}{u(z_0)}$$

exists and is non-zero. In that case, u is called the Jost function, the Jost solution is defined to be

$u_n(z) = -u(z)\, g_{n-1}(z + \frac{1}{z})$

so $u_n(z) \sim z^n$.

Jost Asymptotics

Define $x_n = -z_0^{-(n+1)}\, g_n(z_0 + \frac{1}{z_0})$, $y_n = z_0^n\, p_n(z_0 + \frac{1}{z_0})$.

Theorem (Damanik–Simon). *Suppose* $a_n \to 1$, $b_n \to 0$. *Fix* z_0. *Then* $\lim x_n = x_\infty$ *if and only if* $\lim y_n = y_\infty$ *and then*

$$x_\infty\, y_\infty = (1 - z^2)^{-1}$$

.

Proof (Christensen–Simon–Zinchenko [Const. Approx **33** (2011), 365–403]). The Wronskian of p_n and q_n is 1, so the Wronskian of p_n and $g_n = q_n + m\, p_n$ is 1 also. Thus, with $G_{nn} = \langle \delta_n, (J - (z + z^{-1}))^{-1} \delta_n \rangle$

$$G_{nn} = p_{n-1}(z + z^{-1})\, g_{n-1}(z + z^{-1})$$

Jost Asymptotics

Let J_0 be the whole line free $(a_n \equiv 1, b_n \equiv 0)$ Jacobi matrix.

Then, $G_{nn}^{(0)}(z) \equiv \langle \delta_n, (J_0 - (z + z^{-1}))^{-1} \delta_n \rangle$

$$= -(z - z^{-1})^{-1}$$

(by computing Wronskian of z^{-n} and z^n)

and $a_n \to 1$, $b_n \to 0 \Rightarrow \lim G_{nn}(z) = G_{00}^{(0)}(z)$.

Thus, $y_{n-1}\, x_{n-1} \to (1 - z^2)^{-1} \Rightarrow$ result.

Peherstorfer-Yuditskii Approach

Rather than prove Jost asymptotics in the Damanik–Simon generality, we suppose we have a Szegő condition and a Blaschke condition and sketch how to get Szegő asymptotics directly but still using the Jost function. (Our approach follows Peherstorfer-Yuditskii.)

The condition $\sum\big(|E_j^\pm| - 2\big)^{\frac{1}{2}} < \infty$ is equivalent to $\sum(1 - |\beta_j^\pm|) < \infty$ where $E_j^\pm = \beta_j^\pm + (\beta_j^\pm)^{-1}$, $|\beta_j| < 1$.

Thus, $B(z) = \prod b_{\beta_j^\pm}(z)$ exists (hence Blaschke condition) and one defines

$$u(z) = B(z) \exp\Big(\int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\Big(\frac{\sin\theta}{\operatorname{Im} M(e^{i\theta})}\Big) \frac{d\theta}{2\pi}\Big)$$

where $\operatorname{Im} M(e^{i\theta}) = \pi f(2\cos\theta)$ (if $0 < \theta < \pi$).

Peherstorfer-Yuditskii Approach

By the Szegő condition, the integral defines a function $E(z)$ with $(1-z^2)\,E(z)^{-1} \in H^2$.

A calculation reminiscent of Szegő's yields

$$\int \left| p_n(x) - \frac{\operatorname{Im}\left[\bar{u}(e^{i\theta(x)})e^{i(n+1)\theta(x)}\right]}{\sin(\theta(x))} \right|^2 f(x)\,dx +$$

$\displaystyle\int |p_n(x)|^2 \, d\mu_s(x)$ goes to zero.

This implies Szegő and Jost asymptotics and that u as defined above is the Jost function.