Spectral Theory of Orthogonal Polynomials

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Lecture 6: Szegő Asymptotics and Shohat-Nevai for \([-2, 2]\)
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Case 0 Sum Rule

Recall $C_0$ step-by-step says

$$-\log(a_1) = Z(\mu) - Z(\mu_1) + \mathcal{E}_0(J) - \mathcal{E}_0(J_1)$$

where $\mathcal{E}_0(J) = \sum_{j,\pm} \log |\beta^{\pm}(J)|$, $E = \beta + \beta^{-1}$, $|\beta| > 1$

and $Z(\mu) = -\frac{1}{2} S(\mu^{(0)} | \mu) - \frac{1}{2} \log 2$

(Note $S(\mu^{(0)} | \mu_0) = -\log 2$)

$$= \frac{1}{4\pi} \int \log \left( \frac{\sin \theta}{\text{Im} M(e^{i\theta})} \right) d\theta.$$  

$d\mu^{(0)} = Sz\left( \frac{d\theta}{2\pi} \right)$. Recall $d\mu_0 = Sz(\sin^2 \theta \frac{d\theta}{\pi})$ is free half-line.

So formally, $C_0$ is

$$-\log \left( \sum_{j=1}^{\infty} a_j \right) = Z(\mu) - \mathcal{E}_0(J)$$

Unlike $P_2$, not all terms positive.
(Extended) Shohat–Nevai Theorem

**Theorem** (Extended Shohat–Nevai Theorem). Let 
\[ d\mu = f(x) \, dx + d\mu_s. \] 
\[ \sigma_{\text{ess}}(J) = [-2, 2]. \] Suppose that

\[ \sum_{n, \pm} \left( |E_n^{\pm}| - 2 \right)^{\frac{1}{2}} < \infty \]

Then

\[ \int_{-2}^{2} (4 - x^2)^{-\frac{1}{2}} \log f(x) > -\infty \iff \lim a_1 \cdots a_n > 0 \]

In that case

\[ \sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty \]

\[ \prod_{n=1}^{N} a_n, \sum_{n=1}^{N} (a_n - 1), \sum_{n=1}^{N} b_n \text{ all have limits} \]

(in \((0, \infty), \text{ resp.}, (-\infty, \infty)\))
(Extended) Shohat–Nevai Theorem

**Remarks.** 1. \( \sum (|E_{n}^{\pm}| - 2)^{1/2} < \infty \) is called Blaschke condition for reasons we’ll see below.

2. One variant of this theorem is that among the three conditions:
   (i) \( \sum (|E_{n}^{\pm}| - 2)^{1/2} < \infty \);
   (ii) Szegő integral \( > -\infty \),
   (iii) \( \lim (a_{1} \cdots a_{n}) \) exists in \((0, \infty)\) (not just \( \lim \)), any two imply the third.

3. Recall Lieb–Thirring (proven for Jacobi by Hundertmark–Simon)
   \[ \sum (|E_{n}^{\pm}| - 2)^{p} \leq \sum (|a_{n} - 1|^{p+1/2} + |b_{n}|^{p+1/2}) \text{ for } p \geq \frac{1}{2}. \]
Thus $J - J_0 \in \ell_1$ (trace class), equivalent to
$\sum |a_n - 1| + |b_n| < \infty$ implies both Blaschke condition
and $\lim (a_1 \cdots a_n)$ exists so we have

**Nevai Conjecture.** $\sum |a_n - 1| + |b_n| < \infty \Rightarrow$ Szegő
condition.

We refer you to [SzTh], Section 3.8 for the proof of the
extended Shohat–Nevai Theorem. The idea is to use the $C_0$
step-by-step sum rule and Isc of $Z$ much like we did for
Killip–Simon.
Let the Jacobi parameters obey

(a) $\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty$

(b) $\lim_{n \to \infty} \prod_{j=1}^{n} a_j$ exists in $(0, \infty)$

(c) $\lim_{n \to \infty} \sum_{j=1}^{n} b_j$ exists in $\mathbb{R}$.

Then, for all $z \in \mathbb{D} \setminus \{0\}$ with $z + z^{-1} \notin \sigma(J)$,

$$\lim_{n \to \infty} z^n p_n(z + z^{-1})$$

exists uniformly on compacts and is non-zero.

Conversely, if that limit exists uniformly and is non-zero for

$$\{z \mid |z| = r\} \text{ for all } r \in (0, \varepsilon),$$

then (a)–(c) hold.
Corollary (Peherstorfer-Yuditskii [Proc. AMS 129 (2001) 3213–3220]). If \( \sum_{n,\pm} (|E_n^\pm| - 2)^{1/2} < \infty \) and Szeg\'o condition holds, then \( \lim_{z \to \infty} z^n p_n(z + z^{-1}) \) exists, etc.

For by Shohat–Nevai, we get all the required conditions for above theorem.

For each \( \frac{1}{2} \leq p < \frac{3}{2} \), Damanik–Simon construct examples with \( a_n \equiv 1, \sum_{n=1}^{\infty} b_n^2 < \infty, \lim_{n \to \infty} \sum_{j=1}^{n} b_j \) exists but \( \sum (|E_n^\pm| - 2)^p = \infty \).

For such examples, the Szeg\'o condition fails, but you still get Szeg\'o asymptotics! This came as a surprise to many. Of course, if an \( \ell^2 \) condition holds, then the sum is finite for \( p = 3/2 \) by Killip–Simon.
Here is the idea of the construction, at least if $p < 1$.

For whole line $a_n \equiv 1$, $b_n = 0$ for $n \neq 0$, $b_0 = \pm \varepsilon$ has a single eigenvalue of size $\pm 2 \pm C \varepsilon^2 + O(\varepsilon^3)$.

(I think $C = \frac{1}{4}$?)

Fix a sequence of numbers $\beta_j$ of alternating sign, $|\beta_j| \to 0$, $\beta_1 > 0$, and integers, $0 < m_1 < m_2 < \ldots$. Take $a_n \equiv 1$, $b_n = 0$ if $n \notin \{m_j\}$. $b_{m_j} = \beta_j$. As $m_1, m_{j+1} - m_j, \ldots$ all get very large, $J$ has eigenvalues very close to $(-1)^{j+1} [2 + C \beta_j^2]$, at least for $j$ large.

Take $\beta_j = k^{-\beta}$, $j = 2k - 1$; $\beta_j = -k^{-\beta}$, $j = 2k$.

Trivially, $\sum_{n=1}^{N} \beta_n$ converges to 0 and if $\beta > \frac{1}{2}$, $\sum_{n=1}^{\infty} b_n^2 < \infty$.

$|E_j^{\pm}| - 2 \sim C j^{-2\beta}$. If $2 \beta p < 1$, $\sum(|E_j^{\pm}| - 2)^p = \infty$. 

That Szegő asymptotics implies the conditions on the $a$’s and $b$’s is not hard. For each $n$, for $z$ near 0,

$$z^n p_n(z + \frac{1}{z}) = \frac{1}{a_1 \cdots a_n} (1 + z(\sum_{j=1}^{n} b_j) + O(z^2))$$

so $z^n p_n(z + \frac{1}{z})$ is analytic near $z = 0$ and Szegő asymptotics implies convergence of the Taylor coefficients.

The first two coefficients give convergence of $\prod_{1}^{n} a_j$ and $\sum_{1}^{n} b_j$ by the above and the third coefficient yields the conditional convergence of $\sum_{1}^{n} (a_j - 1)^2 + b_j^2$ but since the sum of positive numbers, conditional convergence implies absolute convergence.
Jost Asymptotics

In the last lecture, we defined the Weyl solution, $g_n(x)$, $x \in \mathbb{C} \setminus \sigma(J)$

We say we have Jost asymptotics at $z_0$ if and only if

$$\lim_{n \to \infty} -z_0^{-(n+1)} g_n(z_0 + \frac{1}{z_0}) \equiv \frac{1}{u(z_0)}$$

exists and is non-zero. In that case, $u$ is called the Jost function, the Jost solution is defined to be

$$u_n(z) = -u(z) g_{n-1}(z + \frac{1}{z})$$

so $u_n(z) \sim z^n$. 
Define \( x_n = -z_0^{-(n+1)} g_n(z_0 + \frac{1}{z_0}) \), \( y_n = z_0^n p_n(z_0 + \frac{1}{z_0}) \).

**Theorem** (Damanik–Simon). Suppose \( a_n \to 1 \), \( b_n \to 0 \). Fix \( z_0 \). Then \( \lim x_n = x_\infty \) if and only if \( \lim y_n = y_\infty \) and then

\[
x_\infty y_\infty = (1 - z^2)^{-1}
\]

**Proof** (Christensen–Simon–Zinchenko [Const. Approx 33 (2011), 365–403]). The Wronskian of \( p_n \) and \( q_n \) is 1, so the Wronskian of \( p_n \) and \( g_n = q_n + m p_n \) is 1 also. Thus, with \( G_{nn} = \langle \delta_n, (J - (z + z^{-1}))^{-1} \delta_n \rangle \)

\[
G_{nn} = p_{n-1}(z + z^{-1}) g_{n-1}(z + z^{-1})
\]
Let $J_0$ be the whole line free ($a_n \equiv 1, b_n \equiv 0$) Jacobi matrix.

Then, $G_{nn}^{(0)}(z) \equiv \langle \delta_n, (J_0 - (z + z^{-1}))^{-1} \delta_n \rangle$

$$= - (z - z^{-1})^{-1}$$

(by computing Wronskian of $z^{-n}$ and $z^n$)

and $a_n \rightarrow 1, b_n \rightarrow 0 \Rightarrow \lim G_{nn}(z) = G_{00}^{(0)}(z)$.

Thus, $y_{n-1} x_{n-1} \rightarrow (1 - z^2)^{-1} \Rightarrow \text{result.}$
Rather than prove Jost asymptotics in the Damanik–Simon generality, we suppose we have a Szegő condition and a Blaschke condition and sketch how to get Szegő asymptotics directly but still using the Jost function. (Our approach follows Peherstorfer-Yuditskii.)

The condition \( \sum (|E_j^±| - 2)^{\frac{1}{2}} < \infty \) is equivalent to \( \sum (1 - |\beta_j^±|) < \infty \) where \( E_j^± = \beta_j^± + (\beta_j^±)^{-1}, |\beta_j| < 1 \).

Thus, \( B(z) = \prod b_{\beta_j^±}(z) \) exists (hence Blaschke condition) and one defines

\[
u(z) = B(z) \exp \left( \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \left( \frac{\sin \theta}{\text{Im} M(e^{i\theta})} \right) \frac{d\theta}{2\pi} \right)\]

where \( \text{Im} M(e^{i\theta}) = \pi f(2 \cos \theta) \) (if \( 0 < \theta < \pi \)).
By the Szegő condition, the integral defines a function $E(z)$ with $(1 - z^2) E(z)^{-1} \in H^2$.

A calculation reminiscent of Szegő's yields

$$
\int \left| p_n(x) - \frac{\text{Im}[\bar{u}(e^{i\theta(x)}) e^{i(n+1)\theta(x)}]}{\sin(\theta(x))} \right|^2 f(x) \, dx + 
\int |p_n(x)|^2 \, d\mu_s(x) \text{ goes to zero.}
$$

This implies Szegő and Jost asymptotics and that $u$ as defined above is the Jost function.